Additional material on V-geometrical ergodicity of Markov kernels via finite-rank approximations

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Abstract

Under the standard drift/minorization and strong aperiodicity assumptions, this paper provides an original and quite direct approach of the V-geometrical ergodicity of a general Markov kernel P, which is by now a classical framework in Markov modelling. This is based on an explicit approximation of the iterates of P by positive finite-rank operators, combined with the Krein-Rutman theorem in its version on topological dual spaces. Moreover this allows us to get a new bound on the spectral gap of the transition kernel. This new approach is expected to shed new light on the role and on the interest of the above mentioned drift/minorization and strong aperiodicity assumptions in V-geometrical ergodicity.

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1 Introduction

Throughout the paper P is a Markov kernel on a measurable space $(\mathbb{X}, \mathcal{X})$. For any positive measure μ on $\mathbb{X}$ and any μ-integrable function $f : \mathbb{X} \to \mathbb{C}$, $\mu(f)$ denotes the integral $\int f d\mu$. When P admits a unique invariant distribution denoted by π, an important question in the theory of Markov chains is to find condition for the $n-$th iterate P^n of P to converge to π when $n \to +\infty$, and to control $\|P^n - \pi(\cdot)1_{\mathbb{X}}\|$ for some functional norm. In this paper we consider the standard $V-$weighted norm $\|\cdot\|_V$ associated with some $[1, +\infty)$-valued function V on $\mathbb{X}$. Then the property

$$\|P^n - \pi(\cdot)1_{\mathbb{X}}\|_V := \sup_{|f|\le V} \sup_{x\in\mathbb{X}} \frac{\big|(P^n f)(x) - \pi(f)\big|}{V(x)} \longrightarrow 0 \quad \text{when} \quad n \to +\infty$$

implies that there exists $\rho \in (0, 1)$ such that $\|P^n - \pi(\cdot)1_{\mathbb{X}}\|_V = \mathrm{O}(\rho^n)$: this corresponds to the so-called V-geometrical ergodicity property, see [MT93, RR04]. The infimum of all the real numbers ρ such that the previous property holds true is the so-called spectral gap of P, denoted by $\rho_V(P)$.

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Since the classical work by Meyn and Tweedie [MT93, MT94], it is well known that P is V-geometrically ergodic provided that usual irreducibility/aperiodicity assumptions hold true and that the following drift/minorization conditions are fulfilled: there exist $S \in \mathcal{X}$, called a small set, and a positive measure ν on $(\mathbb{X}, \mathcal{X})$ such that

$$\exists \delta \in (0,1),\ \exists L > 0, \quad PV \leq \delta\, V + L\, 1_S, \tag{$\mathbf{D}$}$$

$$\forall x \in \mathbb{X},\ \forall A \in \mathcal{X}, \quad P(x, A) \geq \nu(1_A)\, 1_S(x). \tag{$\mathbf{M}$}$$

Condition ($\mathbf{M}$) when the small set S is the entire state space $\mathbb{X}$ is the so-called Doeblin condition. The proofs in [MT93, MT94, Bax05] are based on renewal theory involving the study of the return times to the small set S and Kendall's theorem. Actually the renewal theory easily applies to the atomic case (i.e. when S is an atom), and it has to be applied to the split chain in the general case.

In this paper, under Assumptions ($\mathbf{D}$)-($\mathbf{M}$) and the (strong) aperiodicity condition as in [Bax05]

$$\nu(1_S) > 0, \tag{$\mathbf{SA}$}$$

we revisit the V-geometrical ergodicity property of P thanks to a simple constructive approach based on an explicit approximation of the iterates of P by positive finite-rank operators, combined with Krein-Rutman theorem [KR50]. This theorem can be thought of as an abstract dual Perron-Frobenius statement. It is stated at the end of this section in our specific case of positive operators acting on a weighted-suppremum norm space.

Specifically in Section 2, the following sequence $(\beta_k)_{k\geq 1}$ of positive measures on $(\mathbb{X}, \mathcal{X})$ is recursively defined from the positive measure ν and the small S in Condition ($\mathbf{M}$):

$$\beta_1(\cdot) := \nu(\cdot) \quad \text{and} \quad \forall n \geq 2, \quad \beta_n(\cdot) := \nu\big(P^{n-1}\,\cdot\,\big) - \sum_{k=1}^{n-1} \nu\big(P^{n-k-1} 1_S\big)\, \beta_k(\cdot).$$

Then, under Conditions ($\mathbf{D}$)-($\mathbf{M}$), the following assertions are obtained:

(i) $\forall n \geq 1$, $P^n - T_n = (P-T)^n$ with $T_n := \sum_{k=1}^n \beta_k(\cdot) P^{n-k} 1_S$ satisfying $0 \leq T_n \leq P^n$;

(ii) $r := \lim_n \big(\|P^n - T_n\|_V\big)^{1/n} < 1$, thus $\forall \gamma \in (r, 1)$, $\|P^n - T_n\|_V = \mathrm{O}(\gamma^n)$;

(iii) $r \leq (\delta \nu(1_{\mathbb{X}}) + \tau)/(\nu(1_{\mathbb{X}}) + \tau) < 1$, with $\tau := \max(0, L - \nu(V))$.

In Section 3, under Conditions ($\mathbf{D}$)-($\mathbf{M}$), the unique invariant distribution π of P is obtained from the explicit series

$$\pi = \pi(1_S) \sum_{k=1}^{+\infty} \beta_k,$$

which extends a well-known formula when P satisfies the Doeblin condition, see [LC14], or when P is irreducible and recurrent positive according to [Num84, p 74]. More important, as a result of the above assertion (ii), we easily derive the rate $\beta_n(V) = \mathrm{O}(\gamma^n)$ as well as an approximation of π by an explicit sequence of probability measures with the same convergence rate. In Sections 4 and 5, under the additional assumption ($\mathbf{SA}$), an original proof of the V-geometrical ergodicity is derived from the results of Sections 2-3. More precisely, setting

$$\varrho_S := \limsup_{n \to +\infty} \left(\sup_{x \in \mathbb{X}} \frac{\big|(P^n(x, S) - \pi(1_S)\big|}{V(x)} \right)^{\frac{1}{n}}$$

the V-geometrical ergodicity follows from the following bounds of the spectral gap of P

$$\rho_V(P) \le \max\left(r, \varrho_S\right) \le \left(\min\left\{|z| : 1 < |z| < 1/r,\ \sum_{k=1}^{+\infty} \beta_k(1_{\mathbb{X}})\, z^k = 0\right\}\right)^{-1} < 1 \qquad (2)$$

with the convention that the above minimum equals to $1/r$ if the related set is empty (in this case $\rho_V(P) \le r$).

Although the results of Section 5 seem sound like those in [Bax05], who also introduced a real number similar to ϱ_S, it is worth noticing that they differ completely from their content and their proofs. Indeed, on the one hand the renewal theory is not used here, on the other hand no intermediate Markov kernel is required in our work, in particular we do not use the split chain. Our method is mainly based on the Krein-Rutman theorem. Recall that the classical Perron-Frobenius theorem is a useful result for obtaining positive eigenvectors belonging to the maximal positive eigenvalue of a finite non-negative matrix. Here the Krein-Rutman theorem plays the same role (on the dual side). The following four stages outline our approach. First, the minorization condition (**M**) provides the positive finite-rank operator T_n in the above assertion (i). Mention that such an approach has been used in [HL21] to study inhomogeneous products of Markov kernels satisfying the Doeblin condition. Second, the geometric rate of $\|(P-T)^n\|_V$ is obtained under Conditions (**D**)-(**M**) thanks to the Krein-Rutman theorem. Third, the existence and uniqueness of the invariant distribution π is deduced from the Krein-Rutman theorem too. Four, standard arguments on power series are used to prove Inequalities (2) under the additional assumption (**SA**).

As mentioned in [Bax05] (see also the references therein), the bounds of $\rho_V(P)$ obtained in the literature may be still quite far off $\rho_V(P)$, and we do not presume to give here a better bound of $\rho_V(P)$. Actually this new approach is expected to shed new light, as for instance in [HM11], on the role of Assumptions (**D**)-(**M**)-(**SA**) in the study of the V-geometrical ergodicity. For the sake of completeness, an alternative proof of the V-geometrical ergodicity of P under Assumptions (**D**)-(**M**)-(**SA**), as well as a more precise estimate of $\rho_V(P)$, are addressed in Appendix A by using more sophisticated spectral arguments due to quasi-compactness.

Notations and basic material

Let $V : \mathbb{X} \to [1, +\infty)$ be a measurable function such that $V(x_0) = 1$ for some $x_0 \in \mathbb{X}$. Let $(\mathcal{B}_V, \|\cdot\|_V)$ denote the weighted-supremum Banach space

$$\mathcal{B}_V := \left\{\ f : \mathbb{X} \to \mathbb{C},\ \text{measurable}\ : \|f\|_V := \sup_{x\in\mathbb{X}} \frac{|f(x)|}{V(x)} < \infty\ \right\}.$$

If Q is a bounded linear operator on $\mathcal{B}_V$, its operator norm $\|Q\|_V$ is defined by

$$\|Q\|_V := \sup_{f\in\mathcal{B}_V, \|f\|_V \le 1} \|Qf\|_V.$$

If Q_1 and Q_2 are bounded linear operators on $\mathcal{B}_V$, we write $Q_1 \le Q_2$ when the following property holds: $\forall f \in \mathcal{B}_V,\ f \ge 0,\ Q_1 f \le Q_2 f$. Under Assumption (**D**), the following functional action of P

$$\forall f \in \mathcal{B}_V,\ \forall x \in \mathbb{X}, \quad (Pf)(x) := \int_{\mathbb{X}} f(y) P(x, dy)$$

is well-defined and provides a bounded linear operator on $\mathcal{B}_V$. Recall that P is said to be V-geometrically ergodic if there exists a P-invariant probability measure π on $(\mathbb{X}, \mathcal{X})$ such that $\pi(V) < \infty$ and if there exist some rate $\rho \in (0,1)$ and constant $C_\rho > 0$ such that

$$\forall n \geq 0, \qquad \sup_{f \in \mathcal{B}_V, \|f\|_V \leq 1} \|P^n f - \pi(f) 1_{\mathbb{X}}\|_V \leq C_\rho \, \rho^n. \tag{3}$$

Denoting by Π the rank-one operator $f \mapsto \pi(f) 1_{\mathbb{X}}$ on $\mathcal{B}_V$, Property (3) rewrites as

$$\forall n \geq 0, \quad \|(P - \Pi)^n\|_V = \|P^n - \Pi\|_V \leq C_\rho \, \rho^n. \tag{4}$$

The spectral gap of P, denoted by $\rho_V(P)$, is defined as the spectral radius $r(P - \Pi)$ of the operator $P - \Pi$, that is

$$\rho_V(P) = \lim_{n \to +\infty} \big(\|(P - \Pi)^n\|_V\big)^{\frac{1}{n}} = \lim_{n \to +\infty} \big(\|P^n - \Pi\|_V\big)^{\frac{1}{n}}. \tag{5}$$

Equivalently $\rho_V(P)$ is the infimum of all the real numbers ρ such that (3) holds true for some positive constant C_ρ. Finally $\mathcal{B}'_V$ denotes the topological dual space of $\mathcal{B}_V$, that is the Banach space composed of all the continuous linear forms on $\mathcal{B}_V$, equipped with its usual norm:

$$\forall \eta \in \mathcal{B}'_V, \quad \|\eta\|'_V = \sup_{f \in \mathcal{B}_V, \|f\|_V \leq 1} |\eta(f)|.$$

Note that, if $\eta \in \mathcal{B}'_V$ is non-negative (i.e. $\forall f \in \mathcal{B}_V : f \geq 0 \Rightarrow \eta(f) \geq 0$), then $\|\eta\|'_V = \eta(V)$.

Finally, for the sake of simplicity, let us state the Krein-Rutman theorem for the positive operators on $\mathcal{B}_V$. In such a context, a proof can be directly obtained from [MN91, Th 4.1.5, p 251] using $E := \mathcal{B}_V$ and $\|\cdot\|_e := \|\cdot\|_V$.

Krein-Rutman theorem *If L is a positive bounded linear operator on $\mathcal{B}_V$ such that its spectral radius $r(L) = \lim_n \|L^n\|_V^{1/n} > 0$, then there exists a non-trivial non-negative $\eta \in \mathcal{B}'_V$ such that $\eta \circ L = r(L)\,\eta$.*

2 Approximation of P^n by a positive finite-rank operator

Let P be a Markov kernel satisfying Conditions (**D**)-(**M**). We set $\beta_1(\cdot) := \nu(\cdot)$, and for every $n \geq 2$, the element $\beta_n(\cdot)$ of $\mathcal{B}'_V$ is defined by the following recursive formula :

$$\forall f \in \mathcal{B}_V, \quad \beta_n(f) := \nu\big(P^{n-1} f\big) - \sum_{k=1}^{n-1} \nu\big(P^{n-k-1} 1_S\big)\, \beta_k(f). \tag{6}$$

Note that $\beta_1(\cdot) = \nu(\cdot)$ is defined as a positive measure on $(\mathbb{X}, \mathcal{X})$ and that $\beta_1(V) = \nu(V) < \infty$ from (**D**)-(**M**). Thus $\beta_1(\cdot)$ defines a non-negative element of $\mathcal{B}'_V$. It follows from induction that, for every $n \geq 1$, $\beta_n(\cdot)$ is well defined as an element of $\mathcal{B}'_V$. Actually the next proposition shows that, for every $n \geq 1$, $\beta_n(\cdot)$ can be defined as a positive measure on $(\mathbb{X}, \mathcal{X})$ such that $\beta_n(V) < \infty$. Let T be the rank-one operator on $\mathcal{B}_V$ defined by :

$$\forall f \in \mathcal{B}_V, \quad Tf := \nu(f)\, 1_S = \beta_1(f)\, 1_S.$$

It follows from $T \geq 0$ and (**M**) that $0 \leq T \leq P$.

Proposition 2.1 *Assume that P satisfies Assumptions (**D**)-(**M**). Then*

$$\forall n \geq 1, \quad T_n := P^n - (P-T)^n = \sum_{k=1}^{n} \beta_k(\cdot) P^{n-k} 1_S \qquad \textit{and} \qquad 0 \leq T_n \leq P^n. \tag{7}$$

Moreover, for every $n \geq 1$, β_n is a positive measure on $(X, \mathbb{X})$ such that $\beta_n(V) < \infty$, that is: there exists a positive measure on $(X, \mathbb{X})$ (still denoted by β_n) such that $\int_{\mathbb{X}} V \, d\beta_n < \infty$ and: $\forall f \in \mathcal{B}_V, \ \beta_n(f) = \int_{\mathbb{X}} f \, d\beta_n$.

Proof. The first equality in (7) is just the definition of T_n. That $0 \leq T_n \leq P^n$ follows from $0 \leq T \leq P$. The second equality in (7) for $n = 1$ is obvious from the definition of T. Now assume that this second equality holds true for some $n \geq 1$. Then

$$P^{n+1} - T_{n+1} := (P-T)^{n+1} \quad = \quad (P-T)(P^n - T_n) = P^{n+1} - PT_n - TP^n + TT_n$$

from which we deduce that, for every $f \in \mathcal{B}_V$

$$\begin{aligned} T_{n+1} f &= PT_n f + TP^n f - TT_n f \\ &= \sum_{k=1}^{n} \beta_k(f) P^{n-k+1} 1_S + \left(\beta_1(P^n f) - \sum_{k=1}^{n} \beta_k(f) \nu(P^{n-k} 1_S) \right) 1_S \\ &= \sum_{k=1}^{n} \beta_k(f) P^{n+1-k} 1_S + \beta_{n+1}(f) 1_S \end{aligned} \tag{8}$$

with $\beta_{n+1}(\cdot)$ defined in (6). This provides the second equality in (7) by induction.

As already mentioned $\beta_1(\cdot) = \nu(\cdot)$ is defined as a positive measure on $(\mathbb{X}, \mathcal{X})$ such that $\beta_1(V) < \infty$. Next, for every $n \geq 1$, the element $\beta_n(\cdot)$ is defined as an element of $\mathcal{B}'_V$ and for every $f \in \mathcal{B}_V$, we have from (6) and then from (7)

$$\beta_n(f) = \nu\big(P^{n-1} f\big) - \sum_{k=1}^{n-1} \beta_k(f) \, \nu\big(P^{n-k-1} 1_S\big) = \nu\big(P^{n-1} f - T_{n-1} f\big). \tag{9}$$

It follows that $\beta_n(\cdot)$ is a non-negative element of $\mathcal{B}'_V$ since $P^{n-1} \geq T_{n-1}$. To complete the proof, let us prove by induction that, for every $n \geq 1$, β_n is a positive measure on $(X, \mathbb{X})$ such that $\beta_n(V) < \infty$. Assume that, for some $n \geq 2$, the following property holds: for every $1 \leq k \leq n-1$, $\beta_k(\cdot)$ is a positive measure on $(X, \mathbb{X})$ such that $\beta_k(V) < \infty$. That is: for every $1 \leq k \leq n-1$ there exists a positive measure on $(X, \mathbb{X})$ (still denoted by β_k) such that $\int_{\mathbb{X}} V \, d\beta_k < \infty$ and $\forall f \in \mathcal{B}_V, \ \beta_k(f) = \int_{\mathbb{X}} f \, d\beta_k$. Then $\beta_n(\cdot)$ in (6) is a finite linear combination of positive measures on $(X, \mathbb{X})$. It follows that $\beta_n(\cdot)$ is itself a positive measure on $(X, \mathbb{X})$ since we have proved that β_n is non-negative. □

Under Assumptions (**D**)-(**M**), let us introduce the spectral radius $r := r(P-T)$ of $P-T$ on $\mathcal{B}_V$:

$$r := \lim_{n \to +\infty} \big(\|(P-T)^n\|_V \big)^{\frac{1}{n}} = \lim_{n \to +\infty} \big(\|P^n - T_n\|_V \big)^{\frac{1}{n}}. \tag{10}$$

Theorem 2.1 *Assume that P satisfies Conditions (**D**)-(**M**). Then*

$$r \leq \frac{\delta \, \nu(1_{\mathbb{X}}) + \tau}{\nu(1_{\mathbb{X}}) + \tau} < 1 \quad \textit{where} \quad \tau := \max(0, L - \nu(V)). \tag{11}$$

Inequality (11) has already been established to prove [HL14a, Th. 5.2] in another purpose. Here a short proof of (11) is given to highlight the use of the Krein-Rutman theorem.

Proof. Condition (**D**) implies that $PV \leq \delta V + L\, 1_{\mathbb{X}}$, thus: $\forall n \geq 1,\ \|P^n\|_V = \|P^n V\|_V \leq (1-\delta+L)/(1-\delta)$. Then the spectral radius $r(P)$ of P is one from $P1_{\mathbb{X}} = 1_{\mathbb{X}}$ and $1_{\mathbb{X}} \in \mathcal{B}_V$. Recall that $T := \nu(\cdot)\, 1_S$. Set $R := P - T$ with spectral radius $r := r(R)$. We know that $0 \leq R \leq P$, thus $r \leq r(P) = 1$. If $r = 0$, then (11) is obvious. Now assume that $r \in (0,1]$. Then there exists $\eta \in \mathcal{B}'_V$, $\eta \geq 0$, $\eta \neq 0$ such that $\eta \circ R = r\,\eta$ from the Krein-Rutman theorem. Since $P = T + R$, we have $\eta \circ P = \eta \circ T + r\,\eta$, so that $\eta(P1_{\mathbb{X}}) = \eta(1_{\mathbb{X}}) = \eta(T1_{\mathbb{X}}) + r\,\eta(1_{\mathbb{X}})$. Hence $\eta(T1_{\mathbb{X}}) = (1-r)\eta(1_{\mathbb{X}})$. Observing that $T1_{\mathbb{X}} = \nu(1_{\mathbb{X}})\, 1_S$ and $\nu(1_{\mathbb{X}}) > 0$, and that $\eta \geq 0$ and $1_{\mathbb{X}} \leq V$, it follows that

$$\eta(1_S) = \frac{(1-r)\eta(1_{\mathbb{X}})}{\nu(1_{\mathbb{X}})} \leq \frac{(1-r)\eta(V)}{\nu(1_{\mathbb{X}})}.$$

We have $RV = PV - \nu(V)1_S \leq \delta V + (L - \nu(V))\, 1_S$ from (**D**). Hence

$$r\,\eta(V) = \eta(RV) \leq \delta\,\eta(V) + \tau\,\eta(1_S) \leq \delta\,\eta(V) + \tau\,\frac{(1-r)\eta(V)}{\nu(1_{\mathbb{X}})}.$$

Since $\eta \neq 0$, we have $\eta(V) = \|\eta\|'_V \neq 0$, and (11) follows from the last inequality. □

Note that, for every $n \geq 1$, the operator T_n defined in Proposition 2.1 is positive and finite-rank, more precisely $\mathrm{Im}(T_n)$ is contained in the $n-$dimensional subspace of $\mathcal{B}_V$ generated by the functions $1_S, P1_S, \ldots, P^{n-1}1_S$. The following corollary is a direct consequence of Proposition 2.1 and Theorem 2.1.

Corollary 2.1 *Assume that P satisfies (**D**)-(**M**). Then, for every $\gamma \in (r,1)$, there exists $C_\gamma > 0$ such that*

$$\forall n \geq 1,\ \forall f \in \mathcal{B}_V, \quad \|P^n f - T_n f\|_V = \left\| P^n f - \sum_{k=1}^{n} \beta_k(f) P^{n-k} 1_S \right\|_V \leq C_\gamma\, \gamma^n\, \|f\|_V. \tag{12}$$

Under Conditions (**D**)-(**M**), Inequality (12) provides a geometric convergence rate for the difference between the n-th iterate of P and the positive finite-rank operator T_n. This will be a central preliminary property for obtaining the results of Sections 3, 4 and 5.

3 Existence and approximation of π

Let us introduce

$$\forall n \geq 1, \quad \mu_n := \sum_{k=1}^{n} \beta_k \tag{13}$$

with the β_k's defined in (6). It follows from Proposition 2.1 that μ_n is a positive measure on $(\mathbb{X}, \mathcal{X})$ such that $\mu_n(V) < \infty$. We provide a very short proof that P has a unique invariant probability π with a simple representation from the β_k's.

Theorem 3.1 *Assume that P satisfies (**D**)-(**M**). Then P has a unique P-invariant distribution π. Moreover π satisfies:*

$$\pi = \pi(1_S) \sum_{k=1}^{+\infty} \beta_k, \tag{14}$$

where the series $\sum_{k=1}^{+\infty} \beta_k$ is absolutely convergent in $\mathcal{B}'_V$ given that

$$\forall \gamma \in (r,1),\ \forall n \geq 1, \quad \|\beta_n\|'_V \leq \nu(V) C_\gamma\, \gamma^{n-1} \tag{15}$$

where C_γ is given in Corollary 2.1. Moreover $\pi(V) < \infty$.

Proof. Under Condition (**D**), we know from the proof of Theorem 2.1 that the spectral radius $r(P)$ of P is one. Next, we know from the Krein-Rutman theorem that there exists a non zero and non-negative element $\phi \in \mathcal{B}'_V$ such that $\phi \circ P = \phi$. We obtain using the P-invariance of ϕ and (12) that

$$\forall n \geq 1, \qquad \|\phi - \phi(1_S) \sum_{k=1}^{n} \beta_k\|'_V \leq C_\gamma\, \gamma^n \tag{16}$$

where $\gamma \in (r,1)$ and C_γ are given in Corollary 2.1. It follows that $\phi = \phi(1_S) \sum_{k=1}^{+\infty} \beta_k$ in $\mathcal{B}'_V$. Actually this series absolutely converges in $\mathcal{B}'_V$ since we have for every $n \geq 2$

$$\|\beta_n\|'_V \leq \|\nu\|'_V\, \|P^{n-1} - T_{n-1}\|_V \leq \nu(V) C_\gamma\, \gamma^{n-1}$$

from (9) and Corollary 2.1, and from $\|\nu\|'_V = \nu(V)$. Next $\mu := \sum_{k=1}^{+\infty} \beta_k$ defines a sigma-additive measure. Since $\beta_k(1_{\mathbb{X}}) \leq \beta_k(V) = \|\beta_k\|'_V$ for any $k \geq 1$, we have $\mu(1_{\mathbb{X}}) \leq \mu(V) < +\infty$ and μ is a bounded positive measure. Thus ϕ is a bounded positive measure and ϕ is a P-invariant probability up to a normalization factor. □

The following theorem states that the P-invariant probability π may be approximated by a sequence of probability measures defined from the β_k's. Indeed, $\mu_n(1_{\mathbb{X}}) \geq \beta_1(1_{\mathbb{X}}) = \nu(1_{\mathbb{X}}) > 0$ for every $n \geq 1$. Thus, we can define from (13) the following probability measure $\widetilde{\mu}_n(\cdot)$ on $(\mathbb{X}, \mathcal{X})$ such that $\widetilde{\mu}_n(V) < \infty$:

$$\forall n \geq 1, \quad \widetilde{\mu}_n(\cdot) = \frac{1}{\mu_n(1_{\mathbb{X}})}\, \mu_n(\cdot). \tag{17}$$

Theorem 3.2 *Assume that P satisfies (**D**)-(**M**). Let γ be such that $r < \gamma < 1$, and let n_0 be the smallest integer number such that $C_\gamma\, \gamma^{n_0} < 1$, with C_γ given in Corollary 2.1. Then the following assertion holds for the P-invariant probability π:*

$$\forall n \geq n_0,\ \forall f \in \mathcal{B}_V, \quad \big|\pi(f) - \widetilde{\mu}_n(f)\big| \leq \frac{L}{1-\delta}\left(1 + \frac{L}{(1-\delta)(1 - C_\gamma \gamma^n)}\right) \|f\|_V\, C_\gamma\, \gamma^n. \tag{18}$$

Proof. Using the notations of Proposition 2.1 and Corollary 2.1, we deduce from the $P-$invariance of π that $\pi \circ T_n = \pi(1_S)\mu_n$. It follows from (12) that

$$\forall n \geq 1,\ \forall f \in \mathcal{B}_V, \quad \big|\pi(f) - \pi(1_S)\mu_n(f)\big| \leq C_\gamma\, \gamma^n\, \pi(V)\, \|f\|_V. \tag{19}$$

Lemma 3.1 *We have $\pi(V) \leq L/(1-\delta)$ and*

$$\forall n \geq 1, \quad \mu_n(V) \leq \frac{L}{1-\delta} \quad \textit{and} \quad \forall n \geq n_0, \quad \left|\frac{1}{\mu_n(1_{\mathbb{X}})} - \pi(1_S)\right| \leq \frac{C_\gamma\, \gamma^n\, \pi(V)}{1 - C_\gamma \gamma^n}.$$

Proof. We deduce from (**D**) that $\pi(V) \le \delta\,\pi(V) + L\,\pi(1_S)$. Thus $\pi(1_S) > 0$ since $\delta < 1$ and $\pi(V) > 0$. This gives $\pi(V) \le L\pi(1_S)/(1-\delta) \le L/(1-\delta)$. We have $\pi(1_S)\mu_n = \pi \circ T_n \le \pi \circ P^n = \pi$ from Proposition 2.1, so that $\pi(1_S)\mu_n(V) \le \pi(V)$. Therefore $\mu_n(V) \le L/(1-\delta)$. Now Property (19) with $f := 1_{\mathbb{X}}$ gives

$$\forall n \ge 1, \quad \left|1 - \pi(1_S)\mu_n(1_{\mathbb{X}})\right| \le C_\gamma\, \gamma^n\, \pi(V). \tag{20}$$

Let $n \ge n_0$. Then $\pi(1_S)\mu_n(1_{\mathbb{X}}) \ge 1 - C_\gamma\gamma^n$, thus $\mu_n(1_{\mathbb{X}}) \ge \pi(1_S)\mu_n(1_{\mathbb{X}}) \ge 1 - C_\gamma\gamma^n > 0$ from $\pi(1_S) \le 1$ and the definition of n_0. It follows from (20) and from the last inequality that

$$\forall n \ge n_0, \quad \left|\frac{1}{\mu_n(1_{\mathbb{X}})} - \pi(1_S)\right| \le \frac{C_\gamma\, \gamma^n\, \pi(V)}{\mu_n(1_{\mathbb{X}})} \le \frac{C_\gamma\, \gamma^n\, \pi(V)}{1 - C_\gamma\gamma^n}.$$

The proof of Lemma 3.1 is complete. □

Let $n \ge n_0$ and let $f \in \mathcal{B}_V$. Note that $|\mu_n(f)| \le \mu_n(V)\|f\|_V$. We obtain that

$$\begin{aligned}
\left|\pi(f) - \widetilde{\mu}_n(f)\right| &\le \left|\pi(f) - \pi(1_S)\mu_n(f)\right| + |\mu_n(f)|\,\left|\pi(1_S) - \frac{1}{\mu_n(1_{\mathbb{X}})}\right| \\
&\le C_\gamma\, \gamma^n\, \pi(V)\, \|f\|_V + \|f\|_V \frac{L}{1-\delta}\, \frac{C_\gamma\, \gamma^n\, \pi(V)}{1 - C_\gamma\gamma^n},
\end{aligned}$$

from (19) and from the previous lemma. This provides the inequality (18). □

Remark 3.1 *Theorem 3.1 asserts the existence of a unique invariant probability when $\mathbb{X}$ is a general state space and P satisfies the conditions (**D**)-(**M**). Under topological assumptions on $\mathbb{X}$ such a statement can be simply obtained by using Prohorov's theorem. This is the case when P satisfies the drift condition (**D**) provided that $\mathbb{X}$ is a separable complete metric space and that V has compact level sets (for completenes a proof is postponed to Proposition B.1).*

Remark 3.2 *It follows from (9) and (7) that $\beta_k = \nu(P-T)^{k-1}$ for every $k \ge 1$, so that the series representation (14) of π reduces to $\pi = \pi(1_S)\,\nu \sum_{k=0}^{+\infty}(P-T)^k$. Such a representation is well known when P satisfies the Doeblin condition (i.e. $\mathbb{X}$ is a small set, e.g. see [LC14]) and when P is irreducible and recurrent positive by using the renewal theory, see [Num84, p. 74]. Note that Theorem 3.1 gives this formula with the additional geometric rate (15) which is central for analysing the power series introduced in the next section.*

4 Some relevant power series

In this short section some power series related to the $\beta_k(\cdot)$'s are introduced and we prove a result that highlights the interest of Property (15) and the role of Assumption (**SA**). For every $\tau > 0$ we set $D(0,\tau) := \{z \in \mathbb{C} : |z| < \tau\}$ and $\overline{D}(0,\tau) := \{z \in \mathbb{C} : |z| \le \tau\}$.

Proposition 4.1 *Assume that P satisfies (**D**)-(**M**). Then, for every $f \in \mathcal{B}_V$, the radius of convergence of the power series*

$$B_f(z) := \sum_{k=1}^{+\infty} \beta_k(f)\, z^k$$

is larger than $1/r$. The functions $B_{1_{\mathbb{X}}}$ and B_{1_S} (i.e. B_f for $f := 1_{\mathbb{X}}$ and $f := 1_S$) satisfy

$$\forall z \in D(0,1/r), \quad (1-z)B_{1_{\mathbb{X}}}(z) = \nu(1_{\mathbb{X}})\, z\, \big(1 - B_{1_S}(z)\big). \tag{21}$$

*Under the additional assumption (**SA**), $z = 0$ is the unique zero of $B_{1_{\mathbb{X}}}(\cdot)$ in $\overline{D}(0,1)$.*

Proof. The assertion on the radius of convergence follows from (15). Next, set $a_{-1} := 1$ and $\forall j \geq 0$, $a_j := \nu(P^j 1_S)$. Let $f \in \mathcal{B}_V$. Then (6) rewrites as

$$\forall n \geq 1, \quad \nu(P^{n-1}f) = \sum_{k=1}^{n} \beta_k(f) a_{n-k-1}. \tag{22}$$

Note that the radius of convergence of the power series $N_f(z) := \sum_{n=0}^{+\infty} \nu(P^n f) z^n$ is larger than 1 since $\sup_n \nu(|P^n f|) \leq \|f\|_V \sup_n \nu(P^n V) < \infty$ from (**D**)-(**M**). It follows from (22) that for every $z \in D(0,1)$

$$\sum_{n=1}^{+\infty} \nu(P^{n-1}f) z^n = \sum_{n=1}^{+\infty}\sum_{k=1}^{n} \beta_k(f) a_{n-k-1} z^n = \sum_{k=1}^{+\infty} \beta_k(f) z^k \sum_{n=k}^{+\infty} a_{n-k-1} z^{n-k},$$

so that: $\forall z \in D(0,1)$, $zN_f(z) = B_f(z)\big(1 + zN_{1_S}(z)\big)$. We obtain with $f := 1_{\mathbb{X}}$ and $f := 1_S$

$$\forall z \in D(0,1), \quad \nu(1_{\mathbb{X}}) \frac{z}{1-z} = B_{1_{\mathbb{X}}}(z)\big(1 + zN_{1_S}(z)\big) \quad zN_{1_S}(z)\big(1 - B_{1_S}(z)\big) = B_{1_S}(z). \tag{23}$$

The second equality gives $(1 + zN_{1_S}(z))(1 - B_{1_S}(z)) = 1$, and multiplying the first one by $(1 - B_{1_S}(z))$ provides (21) on $D(0,1)$. The extension of (21) to the open disk $D(0,1/r)$ follows from the principle of analytic continuation.

Now we prove the last assertion of Proposition 4.1. Note that $B_{1_{\mathbb{X}}}(0) = 0$. The first equality in (23) shows that, for every $z \in D(0,1)$, $z \neq 0$, we have $B_{1_{\mathbb{X}}}(z) \neq 0$ since $z/(1-z) \neq 0$. Now assume that there exists $z_0 \in \mathbb{C}$ such that $|z_0| = 1$, $z_0 \neq 1$, and $B_{1_{\mathbb{X}}}(z_0) = 0$. Then $B_{1_S}(z_0) = 1$ from (21), which is impossible since

$$\sum_{k=1}^{+\infty} \beta_k(1_S)\, z^k = 1,\ z \in \mathbb{C},\ |z| = 1 \implies z = 1. \tag{24}$$

Indeed set $z := e^{i\vartheta}$ with $\vartheta \in [0, 2\pi[$. Then the equality $\sum_{k=1}^{+\infty} \beta_k(1_S)\, z^k = 1$ provides $\sum_{k=1}^{+\infty} \beta_k(1_S)\big(1 - \cos(k\vartheta)\big) = 1$ since $\sum_{k=1}^{+\infty} \beta_k(1_S) = 1$. We deduce from $\beta_1(1_S) = \nu(1_S) > 0$ that $\cos(\vartheta) = 1$, that is $z = 1$. We have proved by a reductio ad absurdum that $B_{1_{\mathbb{X}}}(z_0) \neq 0$ for every $z_0 \in \mathbb{C}$ such that $|z_0| = 1$, $z_0 \neq 1$. Finally note that $B_{1_{\mathbb{X}}}(1) = 1/\pi(1_S) \neq 0$. □

5 V-geometrical ergodicity and bound of the spectral gap

In this section an original proof of the V-geometrical ergodicity of P under the three assumptions (**D**)-(**M**)-(**SA**) is derived from the previous statements. We also provide a new bound of the spectral gap $\rho_V(P)$ of P on $\mathcal{B}_V$ defined in (5). This bound is related to the real number $r \in [0,1)$ of Theorem 2.1 and to the following real number ϱ_S only depending on the action of the iterates of P on the small set S in (**M**):

$$\varrho_S := \limsup_{n \to +\infty} \left(\big\|(P^n - \Pi)1_S\big\|_V \right)^{\frac{1}{n}}. \tag{25}$$

Under Assumptions (**D**)-(**M**), Proposition 4.1 is used in order to define

$$\theta := \min\big\{|z| : 1 < |z| < 1/r,\ B(z) = 0\big\} \quad \text{where } B(z) \equiv B_{1_{\mathbb{X}}}(z) = \sum_{k=1}^{+\infty} \beta_k(1_{\mathbb{X}})\, z^k, \tag{26}$$

with the convention $\theta := 1/r$ when the previous set is empty.

Proposition 5.1 *Assume that P satisfies **(D)**-**(M)**-**(SA)**. Then we have: $\varrho_S \le \theta^{-1} < 1$.*

Proof. For every $0 < \tau < 1/r$, the function $B(\cdot)$ is analytic on $\overline{D}(0,\tau)$ from Proposition 4.1 so has a finite number of zeros in $\overline{D}(0,\tau)$. From this fact and from the definition of θ, it follows that $\theta > 1$. Next, the following result is used to derive the inequality $\varrho_S \le \theta^{-1}$.

Lemma 5.2 *Let $\phi \in \mathcal{B}'_V$, and for every $j \ge 0$ set $\sigma_j := \phi\big((P^j - \Pi)\, 1_S\big)$. Then the power series $\sigma(z) := \sum_{k=0}^{+\infty} \sigma_k\, z^k$ has a radius of convergence larger that θ.*

Proof. The radius of convergence of $\sigma(z)$ is larger than 1 since $(\sigma_k)_{k\ge0}$ is clearly bounded from above by $2\|\phi\|'_V$. Next, we deduce from the definitions of T_n and π in (7) and (14) that

$$\begin{aligned}(T_n - P^n)1_{\mathbb{X}} = T_n 1_{\mathbb{X}} - 1_{\mathbb{X}} = T_n 1_{\mathbb{X}} - \Pi\, 1_{\mathbb{X}} &= \sum_{k=1}^{n} \beta_k(1_{\mathbb{X}}) P^{n-k} 1_S - \Big(\sum_{k=1}^{+\infty} \beta_k(1_{\mathbb{X}})\Big)\pi(1_S)1_{\mathbb{X}} \\ &= \sum_{k=1}^{n} \beta_k(1_{\mathbb{X}})\big(P^{n-k} - \Pi\big)1_S \ - \Big(\sum_{k=n+1}^{+\infty} \beta_k(1_{\mathbb{X}})\Big)\Pi\, 1_S.\end{aligned}$$

Composing on the left by ϕ this equality we obtain that

$$\forall n \ge 1, \quad \sum_{k=1}^{n} \beta_k(1_{\mathbb{X}})\sigma_{n-k} = h_n$$

where $(h_n)_{n\ge1}$ is a sequence of complex numbers (depending on ϕ) such that, for every $\gamma \in (r,1)$, $|h_n| = \mathrm{O}(\gamma^n)$ from Corollary 2.1 and (15). Then

$$\forall z \in D(0,1), \quad \sum_{n=1}^{+\infty}\sum_{k=1}^{n} \beta_k(1_{\mathbb{X}})\sigma_{n-k}\, z^n = B(z)\sigma(z) = h(z) \quad \text{where } \ h(z) := \sum_{n=1}^{+\infty} h_n z^n. \tag{27}$$

Note that $h(z)$ (as $B(z)$) has a radius of convergence larger than $1/r$ since we have, for every $\gamma \in (r,1)$, $|h_n| = \mathrm{O}(\gamma^n)$. Moreover, first $z = 0$ is the only zero of $B(\cdot)$ on $D(0,\theta)$ from Proposition 4.1 and from the definition of θ, second $z = 0$ is a simple zero of $B(\cdot)$ since $\beta_1(1_{\mathbb{X}}) = \nu(1_{\mathbb{X}}) > 0$. Thus, for every $z \in D(0,\theta)$, $B(z) = z\xi(z)$ with $\xi(z) = \sum_{k=0}^{+\infty} \beta_{k+1}(1_{\mathbb{X}})z^k$ having a radius of convergence larger than $1/r$ and having no zero in $D(0,\theta)$. It follows from (27) that $z \mapsto z\,\sigma(z)$ coincides on $D(0,1)$ with the function h/ξ which is analytic on $D(0,\theta)$ since $1/r \ge \theta$ and ξ does not vanish on $D(0,\theta)$. Therefore the power series $\sum_{k=0}^{+\infty} \sigma_k\, z^{k+1}$ has a radius of convergence larger that θ. $\square$

Now the proof of Proposition 5.1 can be completed. Let $\phi \in \mathcal{B}'_V$. Then Lemma 5.2 and the Cauchy-Hadamard formula give

$$\limsup_{k\to+\infty} \big|\phi\big((P^k - \Pi)1_S\big)\big|^{1/k} \le \theta^{-1}.$$

Let $\varepsilon > 0$. We have proved that: $\forall \phi \in \mathcal{B}'_V,\ \sup_{n\geq 0} \big(\theta^{-1}+\varepsilon\big)^{-n} \big|\phi\big((P^n - \Pi)1_S\big)\big| < \infty$. It follows from a classical corollary of the Banach-Steinhaus theorem that

$$\sup_{n\geq 0} \big(\theta^{-1}+\varepsilon\big)^{-n} \big\|(P^n - \Pi)1_S\big\|_V < \infty. \tag{28}$$

This give $\varrho_S \leq \theta^{-1}+\varepsilon$, thus $\varrho_S \leq \theta^{-1}$ since ε is arbitrary. □

We are now in position to state the main result of this section.

Theorem 5.1 *Assume that P satisfies **(D)**-**(M)**-**(SA)**. Then P is V-geometrically ergodic. Moreover*

$$\rho_V(P) \leq \max\big(r, \varrho_S\big) \leq \theta^{-1} < 1$$

where r, ϱ_S and θ are defined in (10), (25) and (26) respectively. More precisely

(i) $\rho_V(P) = \varrho_S \leq \theta^{-1}$ *when* $r \leq \varrho_S$*;* *(ii)* $\rho_V(P) \leq r$ *when* $r > \varrho_S$.

Proof. Let $n \geq 1$. We have

$$T_n - \mu_n(\cdot)\Pi\, 1_S = \sum_{k=1}^{n} \beta_k(\cdot)\Big(P^{n-k}1_S - \Pi\, 1_S\Big)$$

from (7) and (13). From Proposition 5.1 we know that $\varrho_S < 1$. Let $\gamma \in (r, 1)$, $\varrho \in (\varrho_S, 1)$. Set $\alpha := \max(\gamma, \varrho)$, and define $D_\varrho := \sup_{n\geq 0} \varrho^{-n} \big\|P^n 1_S - \Pi\, 1_S\big\|_V < \infty$. Then

$$\begin{aligned} \|T_n - \mu_n(\cdot)\Pi\, 1_S\|_V \leq \sum_{k=1}^{n} \|\beta_k\|'_V \big\|P^{n-k}1_S - \Pi\, 1_S\big\|_V &\leq \nu(V) C_\gamma\, D_\varrho \sum_{k=1}^{n} \gamma^{k-1} \varrho^{n-k} \\ &\leq \frac{\nu(V) C_\gamma\, D_\varrho}{\gamma}\, n\, \alpha^n \end{aligned}$$

from (15) and from the definitions of D_ϱ and α. Moreover note that

$$\|\mu_n(\cdot)\Pi\, 1_S - \Pi\|_V = \|\pi(1_S)\mu_n(\cdot) - \pi(\cdot)\|'_V \leq \frac{LC_\gamma}{1-\delta}\, \gamma^n$$

from (19) and Lemma 3.1. Then

$$\begin{aligned} \|P^n - \Pi\|_V &\leq \|P^n - T_n\|_V + \|T_n - \mu_n(\cdot)\Pi\, 1_S\|_V + \|\mu_n(\cdot)\Pi\, 1_S - \Pi\|_V \\ &\leq C_\gamma\, \gamma^n + \frac{\nu(V) C_\gamma\, D_\varrho}{\gamma}\, n\, \alpha^n + \frac{LC_\gamma}{1-\delta}\, \gamma^n \end{aligned}$$

from Corollary 2.1 and from the previous inequalities. It follows from the definition of $\rho_V(P)$ in (5) that $\rho_V(P) \leq \alpha$, thus P is V-geometrically ergodic. Next, since γ and ϱ are arbitrarily close to r and ϱ_S respectively, we obtain that $\rho_V(P) \leq \max\big(r, \varrho_S\big)$. Inequality $\max\big(r, \varrho_S\big) \leq \theta^{-1}$ holds since $\varrho_S \leq \theta^{-1}$ from Proposition 5.1 and since $r \leq \theta^{-1}$ from the definition of θ. Next, if $r \leq \varrho_S$, then $\rho_V(P) \leq \varrho_S$, thus $\rho_V(P) = \varrho_S$ since $\varrho_S \leq \rho_V(P)$ from the definitions of $\rho_V(P)$ and ϱ_S. This gives (i)-(ii). □

Remark 5.1 *As already mentioned, the geometric approximation (12) as well as the geometrical rate for $\|\beta_k\|'_V$ in (15) are central in the proof of Proposition 5.1. Indeed this ensures that the radius of convergence of both power series $B(\cdot)$ and $h(\cdot)$ in (27) are larger than $1/r$. In this regard note that, if the function $B(\cdot)$ in (26) has no zero in the annulus $\{1 < |z| < 1/r\}$, then $\rho_V(P) \le r$ from Theorem 5.1 since $\theta = 1/r$ in this case. By contrast, if $B(\cdot)$ has a zero in the annulus $\{1 < |z| < 1/r\}$, then Inequality $\rho_V(P) > r$ may occur: in this case the convergence rate $O\big((r+\varepsilon)^n\big)$ in both inequalities (12) and (18) is better than $O\big((\rho_V(P)+\varepsilon)^n\big)$ in (3).*

Remark 5.2 *The main results of this paper extend when conditions **(D)**-**(M)**-**(SA)** hold for some iterate P^N with $N > 1$ (in place of P). Indeed Theorem 2.1 and Theorem 3.1 then apply to P^N. In particular P^N has a unique invariant probability π which is also P-invariant. In the same way Theorem 5.1 asserts that P^N is V-geometrically ergodic, provided that the small set associated with P^N satisfies Assumption **(SA)**. Then it easily follows that P is V-geometrically ergodic with spectral gap $\rho_V(P) = (\rho_V(P^N))^{1/N}$.*

A Complements thanks to quasi-compactness

In this appendix we give another proof of the V-geometric ergodicity of P under the assumptions (**D**)-(**M**)-(**SA**) by using quasi-compactness arguments. Moreover the link between $\rho_V(P)$ and the real number θ of (26) is made clearer. The proofs below are independent from that of Theorem 5.1, in particular ϱ_S in (25) is not used.

Various equivalent definitions of the essential spectral radius of a bounded linear operator on a Banach space can be found in the literature in link with, either the essential spectrum, or the quasi-compactness property, e.g. see [Hen93, Hen07] for a general context and [Hen06, HL14b, HL14a] in the framework of V-geometrically ergodic Markov kernels. What is needed to know here on quasi-compactness is summarized in the assertions **(qc1)-(qc2)** below.

Let $r_{ess}(P)$ denote the essential spectral radius of P on $\mathcal{B}_V$. In [HL14a, Section 5] it is proved that, under Assumptions (**D**)-(**M**), P is a power-bounded (i.e. $\sup_n \|P^n\|_V < \infty$) and quasi-compact operator on $\mathcal{B}_V$ with

$$r_{ess}(P) \le r \tag{29}$$

where $r \in [0,1)$ is defined in (10). The elements β_k of $\mathcal{B}'_V$ are defined in (6). Recall that $B_{1_{\mathbb{X}}}(z) = \sum_{k=1}^{+\infty} \beta_k(1_{\mathbb{X}}) z^k$ is well-defined and analytic on $D(0,1/r)$ from Proposition 4.1, and that the real number θ in (26) is defined as follows:

(a) If $B_{1_{\mathbb{X}}}$ has at least one zero in the annulus $\{1 < |z| < 1/r\}$, then θ is the smallest one in modulus, in particular $1 < \theta < 1/r$ in this case.

(b) If $B_{1_{\mathbb{X}}}$ has no zero in the annulus $\{1 < |z| < 1/r\}$, then $\theta = 1/r$ by convention.

Theorem A.1 *Assume that P satisfies **(D)**-**(M)**-**(SA)**. Then P is V-geometrically ergodic, and $\rho_V(P) \le \theta^{-1}$. More precisely*

(a') $\rho_V(P) = \theta^{-1} < 1$ when $\theta < 1/r$; *(b') $\rho_V(P) \le r$ when $\theta = 1/r$.*

The proof of Theorem A.1 involves the adjoint operator of P acting on $\mathcal{B}'_V$, which is denoted by P^*. We know that P and P^* have the same spectral values, but quasi-compactness gives much more spectral informations. Indeed, since under Conditions (**D**)-(**M**) P is quasi-compact on $\mathcal{B}_V$ with spectral radius $r(P)=1$, then so is P^* on $\mathcal{B}'_V$ with the same spectral radius and the same essential spectral radius. Using (29) the important things to remember from quasi-compactness in the next proofs are the following facts, see [HL14b] for details.

- For every $a \in (r,1)$ there are finitely number of spectral values λ of P (or of P^*) such that $a \leq |\lambda| \leq 1$ and they are in fact eigenvalues of both P and P^*.
- For every $a \in (r,1)$, denoting by $\mathcal{V}_a$ the set of eigenvalues λ of P (or of P^*) such that $a \leq |\lambda| < 1$, the following assertions hold:

  (qc1) If $\mathcal{V}_a \neq \emptyset$, then $\rho_V(P) = \max\{|\lambda|,\, \lambda \in \mathcal{V}_a\}$.

  (qc2) If $\mathcal{V}_a = \emptyset$, then $\rho_V(P) \leq a$.

Proof of Theorem A.1. From Proposition 4.1, $B_{1_S}(z) := \sum_{k=1}^{+\infty} \beta_k(1_S) z^k$ is well-defined on $D(0, 1/r)$. First we prove the folowing lemma.

Lemma A.1 *Assume that P satisfies Conditions (**D**)-(**M**). Let λ be an eigenvalue of P^* such that $r < |\lambda| \leq 1$. Then the subspace $E_\lambda := \{\psi \in \mathcal{B}'_V : \psi \circ P = \lambda\, \psi\}$ is spanned by the following absolutely convergent series $\psi_\lambda := \sum_{k=1}^{+\infty} \lambda^{-k}\, \beta_k$ in $\mathcal{B}'_V$. Moreover we have $B_{1_S}(\lambda^{-1}) = 1$.*

Proof. The absolute convergence of the series $\sum_{k=1}^{+\infty} \lambda^{-k}\, \beta_k$ in $\mathcal{B}'_V$ follows from (15) applied with $\gamma \in (r, |\lambda|)$. Let $\psi \in E_\lambda$, $\psi \neq 0$. Then composing on the left by ψ in (12) gives

$$\lambda^n \psi = \psi(1_S) \sum_{k=1}^{n} \lambda^{n-k}\, \beta_k + O(\gamma^n),$$

so that $\psi = \psi(1_S) \sum_{k=1}^{n} \lambda^{-k}\, \beta_k + O((\gamma/\lambda)^n)$. Hence $\psi = \psi(1_S)\, \psi_\lambda$. Since $\psi \neq 0$, we have $\psi(1_S) > 0$, thus $1 = \sum_{k=1}^{+\infty} \lambda^{-k}\, \beta_k(1_S)$. □

Let us establish that P is V-geometrically ergodic. From $P1_{\mathbb{X}} = 1_{\mathbb{X}}$, we know that $\lambda = 1$ is an eigenvalue of P, thus of P^*. Next, from Lemma A.1 we know that $\lambda = 1$ is a simple eigenvalue of P^*, more precisely the subspace $E_1 := \{\psi \in \mathcal{B}'_V : \psi P = \psi\}$ is spanned by $\psi_1 := \sum_{k=1}^{+\infty} \lambda^{-k}\, \beta_k$ which is a bounded positive measure from Proposition 2.1 and (15). In particular the probability measure π on $(\mathbb{X}, \mathcal{X})$ defined by $\pi = \psi_1/\psi_1(1_{\mathbb{X}})$ is P-invariant (as already stated in Theorem 3.1). Furthermore Lemma A.1 and Property (24) due to (**SA**) ensure that $\lambda = 1$ is the single eigenvalue with modulus one of P^*, thus of P. Then we deduce from the quasi-compactness of P^* on $\mathcal{B}'_V$ and from the previous facts that there exists on $\mathcal{B}'_V$ a rank-one eigen-projector Π'_1 belonging to $\lambda = 1$ such that the sequence $(P^{*n} - \Pi'_1)_n$ converges to 0 with geometric rate in operator norm on $\mathcal{B}'_V$. Similarly the quasi-compactness of P on $\mathcal{B}_V$ ensures that there exists on $\mathcal{B}_V$ a finite-rank eigen-projector Π_1 belonging to $\lambda = 1$ such that the sequence $(P^n - \Pi_1)_n$ converges to 0 with geometric rate in operator norm on $\mathcal{B}_V$. We have $\Pi_1^* = \Pi'_1$ since $P^n - \Pi_1$ and $P^{*n} - \Pi_1^*$ have the same norm. It follows that Π_1 is rank-one, more precisely Π_1 writes as $\Pi_1 = \pi(\cdot)h$ for some $h \in \mathcal{B}_V$. From $P^n 1_{\mathbb{X}} = 1_{\mathbb{X}}$ we obtain that $1_{\mathbb{X}} = \pi(1_{\mathbb{X}})h = h$, thus $\Pi_1 = \pi(\cdot)1_{\mathbb{X}}$. This proves that P is V-geometrically ergodic.

Now applying the above statements **(qc1)**-**(qc2)**, Properties (a') and (b') of Theorem A.1 follow from the next proposition and from the definition of θ in (a)-(b). More precisely, in case (a), apply Proposition A.2 and **(qc1)** with some (any) a such that $r < a < \theta^{-1}$; in case (b), apply Proposition A.2 and **(qc2)** with any $a \in (r,1)$ (arbitrarily close to r). □

Proposition A.2 *Assume that P satisfies Conditions **(D)**-**(M)**. Let λ be a complex number such that $r < |\lambda| < 1$. Then the three following assertions are equivalent.*

(i) λ is an eigenvalue of P (or of P^*) on $\mathcal{B}_V$.

(ii) $B_{1_{\mathbb{X}}}(\lambda^{-1}) = 0$.

(iii) $B_{1_S}(\lambda^{-1}) = 1$.

In others words, if the set of eigenvalues λ of P (or of P^*) on $\mathcal{B}_V$ such that $r < |\lambda| < 1$ is non-empty, then it coincides with the set $\{1/z : B_{1_{\mathbb{X}}}(z) = 0,\ 1 < |z| < 1/r\}$.

Proof. The equivalence $(ii) \Leftrightarrow (iii)$ follows from Proposition 4.1. Recall that $T\cdot = \nu(\cdot)1_S$.

Lemma A.3 *Let $z \in \mathbb{C}$ such that $|z| < 1/r$. Then the series $\mathfrak{B}(z) := \sum_{k=1}^{+\infty} z^k \beta_k$ absolutely converges in $\mathcal{B}'_V$, and satifies*

$$\forall z \in D(0, 1/r), \quad z\,\mathfrak{B}(z) \circ P = \mathfrak{B}(z) - z\,\nu + z\mathfrak{B}(z) \circ T. \tag{30}$$

Proof. The above stated absolute convergence follows from (15). Recall that T_k for $k \geq 1$ is defined in Proposition 2.1. Setting $T_0 := 0$ we obtain that for every $k \geq 1$

$$P^k - T_k := (P - T)^k = (P^{k-1} - T_{k-1})(P - T) = P^k - P^{k-1}T - T_{k-1}P + T_{k-1}T$$

so that

$$\forall k \geq 1, \quad T_k - T_{k-1}P = P^{k-1}T - T_{k-1}T = (P^{k-1} - T_{k-1})T. \tag{31}$$

Let $z \in D(0, 1/r)$. Then

$$\begin{aligned}
\mathfrak{B}(z) \circ P &= \sum_{k=1}^{+\infty} z^k\,\nu \circ \big(P^k - T_{k-1} \circ P\big) && \text{(from (9))} \\
&= \sum_{k=1}^{+\infty} z^k\,\nu \circ (P^k - T_k) + \sum_{k=1}^{+\infty} z^k\,\nu \circ \big(T_k - T_{k-1}P\big) \\
&= \sum_{k=1}^{+\infty} z^k\,\beta_{k+1} + \sum_{k=1}^{+\infty} z^k\,\nu \circ (P^{k-1} - T_{k-1})T && \text{(from (9) and (31))} \\
&= \frac{1}{z}\big(\mathfrak{B}(z) - z\nu\big) + \mathfrak{B}(z) \circ T && \text{(from (9))}
\end{aligned}$$

from which we deduce (30). □

Now let $\lambda \in \mathbb{C}$ be such that $r < |\lambda| < 1$. If λ is an eigenvalue of P (or of P^*), then (iii) holds from Lemma A.1. Conversely, assume that $B_{1_S}(\lambda^{-1}) = 1$ and set $\psi := \sum_{k=1}^{+\infty} \lambda^{-k}\,\beta_k$ in $\mathcal{B}'_V$. Note that $\psi = \mathfrak{B}(\lambda^{-1})$ and that $\psi \neq 0$ since $\psi(1_S) = B_{1_S}(\lambda^{-1}) = 1$. Lemma A.3 gives

$$\psi \circ P = \lambda\,\psi - \nu + \psi \circ T.$$

Moreover, using $T\cdot = \nu(\cdot)1_S$ we obtain that

$$\psi \circ T = \sum_{k=1}^{+\infty} \lambda^{-k}\,\beta_k \circ T = \bigg(\sum_{k=1}^{+\infty} \lambda^{-k}\,\beta_k(1_S)\bigg)\nu(\cdot) = B_{1_S}(\lambda^{-1})\,\nu(\cdot) = \nu(\cdot)$$

from which it follows that $\psi \circ P = \lambda\,\psi$. Hence λ is an eigenvalue of P^*. □

Remark A.1 *Note that Equality (21) of Proposition 4.1 can be deduced from Lemma A.3. Indeed we have $\mathfrak{B}(z)(1_{\mathbb{X}}) = B_{1_{\mathbb{X}}}(z)$ and $\mathfrak{B}(z)(1_S) = B_{1_S}(z)$. Then (30) applied to $1_{\mathbb{X}}$ and $T1_{\mathbb{X}} = \nu(1_{\mathbb{X}})1_S$ easily provide (21).*

B Existence of π in a separable complete metric state space

The following result gives the existence of a P-invariant probability under the drift condition (**D**), even under the weaker condition **(WD)** : $\exists\delta \in (0,1),\ \exists L > 0,\ PV \le \delta V + L1_{\mathbb{X}}$. A proof is provided since we do not succeed in finding simple arguments for this statement in the literature.

Proposition B.1 *Let $(\mathbb{X}, d)$ be a separable complete metric space and $V : \mathbb{X} \to [1, +\infty)$ be a continuous function such that the set $\{V \le \alpha\}$ is compact for every $\alpha \in (0, +\infty)$. If P satisfies Condition* **(WD)***, then there exists a P-invariant probability measure π such that $\pi(V) < \infty$.*

Proof. We know from the proof of Theorem 2.1 that P is power-bounded on $\mathcal{B}_V$. Let $x_0 \in \mathbb{X}$. Then $K := \sup_n (P^n V)(x_0) < \infty$. Let π_n, $n \ge 1$, be the probability measure on $(\mathbb{X}, \mathcal{X})$ defined by: $\forall B \in \mathcal{X},\ \pi_n(1_B) = (1/n)\sum_{k=0}^{n-1}(P^k 1_B)(x_0)$. Then Markov's inequality gives: $\forall n \ge 1,\ \forall \alpha \in (0, +\infty),\quad \pi_n\big(1_{\{V>\alpha\}}\big) \le \pi_n(V)/\alpha \le K/\alpha$. Thus the sequence $(\pi_n)_{n\ge1}$ is tight, and we can select a subsequence $(\pi_{n_k})_{k\in\mathbb{N}}$ weakly converging to a probability measure π, which is clearly P-invariant. For $p \in \mathbb{N}^*$, set $V_p(\cdot) = \min(V(\cdot), p)$. Then $\forall k \ge 0,\ \forall p \ge 0,\ \pi_{n_k}(V_p) \le \pi_{n_k}(V) \le K$. Since V_p is continuous and bounded on $\mathbb{X}$, we obtain: $\forall p \ge 0,\ \lim_k \pi_{n_k}(V_p) = \pi(V_p) \le K$. The monotone convergence theorem then gives $\pi(V) < \infty$. $\square$

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