EXPONENTIAL GROWTH OF BRANCHING PROCESSES IN A GENERAL CONTEXT OF LIFETIMES AND BIRTHTIMES DEPENDENCE

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Abstract. We study the exponential growth of branching processes with ancestral dependence. We suppose here that the lifetimes of the cells are dependent random variables, that the numbers of new cells are random and dependent. Lifetimes and new cells's numbers are also assumed to be dependent. Applying the spectral study of Laplace-type operators recently made in [15], we illustrate our results in the Markov context, for which the exponential growth property is linked to the Laplace transform of the lifetimes of the cells.

Contents

1. Introduction

Mathematical models for the growth of populations have been widely studied and applied in many fields especially animal, cell biology (mitosis), plant and forestry sciences. Branching process is a mathematical model often used to model reproduction (see for instance [4], [12] and [18] or more recently, [2], [3], [6] and [16] and references therein). It is described as follows. A single ancestor object (that may be a particle, a neutron, a cosmic ray, a cell and

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so on) is born at time 0. It lives for a random time. At the moment of death, the object produces a random number of progenies. In the classical case (i.e. independence context), each of the first generation progeny behaves, independently of each other and of the ancestor: the objects do not interfere with one another. Many authors were interested by generalizing this classical model by trying to introduce dependence in the above model (see, for instance, [6], [16], [21] and references therein).

In the spirit of the branching processes and for the sake of generalization, we consider in this paper a model of reproduction with a random number of children and with random life duration. As done in [2], [6] and [16], our approach to develop this mathematical model, is to associate to each object v a parameter x_v, called its characteristics. This parameter may depend on its energy, its growth, its position or on other non-observed factors and determines its number of children and its life duration. We consider a stationary context: the sequence of characteristics of two lines are assumed to have the same finite dimensional distributions (a line is an infinite succession of objects starting from the initial one and linked from one to the next one by a parent-child relation). The classical independent identically distributed (i.i.d. for short) case above described is a particular case of stationarity. Next we will use, as was done by many authors (see for instance [2], [6] and [16]), Markov processes as mathematical models for this characteristics process. We discuss, in particular, the linear autoregressive process.

This paper is specifically concerned with the mitosis model and from now on, an object will be a cell. This mitosis model starts with one single initial cell. After a random time, this initial cell is divided into a random number of cells and the process continues. For technical reasons we will suppose that the random number of children is always larger than 2. We summarize our model, used throughout the paper, as follows,

(A) to each cell v, is associated a parameter $x_v \in \mathbb{X}$, called its characteristics, (with $(\mathbb{X}, \mathcal{X})$ a measurable space) which determines its lifetime $\xi(x_v)$ and the number of new cells $\kappa(x_v)$ in which the cell splits at the end of its lifetime (where ξ and κ are two measurable functions with values in $[0, +\infty)$ and in $\mathbb{Z}_+$ respectively);
(B) there exists a process $(X_n)_n$ with values in $\mathbb{X}$ such that, for each line $(v_n)_{n\geq 0}$ of cells, the characteristics along this line is given by a copy of $(X_n)_{n\geq 0}$ (these copies are not assumed to be mutually independent);
(C) $\kappa(x) \geq 2$ for any x, i.e. each cell gives birth to more than two children.

The present work deals with the so-called Belleman-Harris age-dependent branching process $(N_t)_{t\geq 0}$ where, for every $t \geq 0$, N_t denotes the number of cells alive at time t (see [4] and [12] for more about). More precisely we study the first and second moment of $(N_t)_t$, as well as the exponential growth behavior of N_t (as t tends to infinity) which will be linked to an extended Malthusian parameter ν defined below.

Roughky speaking, our above assumptions mean that the characteristics of the successive cells of a same line are modeled by a reference process $(X_n)_n$, and that, the characteristics of each cell determine both its number of children and its life duration. In particular, in our more general setting, the number of children and the life duration of a cell are not assumed to be independent and are neither assumed to be independent for two cells of a same line, nor of two different lines. For the study of the first moment of $(N_t)_t$, the characteristics of two different lines will be assumed to have the same distribution, but not to be independent (conditionally to its last common ancestor). For the study of the auto-covariances of $(N_t)_t$, we require a bit more in terms of stationary: we will moreover assume that

(D) for every non negative integer k, there exists a process $(X_n^{(k)})_n$ such that any couple of characteristics of two different lines with last common ancestor of generation k is a copy of $((X_n)_n, (X_n^{(k)})_n)$.

This includes the case where the children are driven by conditionally independent processes (as in [16]), but also the case in which the children have the same characteristic, and a lot of other situations. In our applications $(X_n)_n$ will be a Markov process.

The exponential growth behavior of N_t as t tends to infinity was studied in the book of Harris [12] in the case where $(X_n)_n$ is a sequence of i.i.d. rv's, that is when the lifetimes are modeled by a sequence of i.i.d. random variables independent of the random numbers of the news cells which are also assumed to be i.i.d. The growth rate ν_0 (also called the Malthusian parameter) was defined, in this context, as the positive root of the equation,

$$\mathbb{E}[\kappa(X_1)]\,\mathbb{E}\left[e^{-\nu_0\xi(X_1)}\right] = 1, \tag{1}$$

as soon as the distribution of $\xi(X_1)$ is not lattice (cf. [12, Theorem 17.1]). Louhichi and Ycart [21] extend some results of Harris to the case where the lifetimes are a sequence of dependent random variables and when each cell is divided, after a random lifetime, into two cells: $(X_n)_n$ is a stationary process and $\kappa(x) = 2$ for any x. Under those assumptions the Malthusian parameter ν_1 is expressed in terms of the Laplace transform of the random variable S_n

$$S_n := \sum_{k=0}^{n} \xi(X_k) \tag{2}$$

which models the birth date of the $(n+1)$-th individual of a same line. More precisely,

$$\nu_1 = \inf\left\{\gamma > 0,\ \sum_{n\geq 0} 2^n \mathbb{E}\left[e^{-\gamma S_n}\right] < \infty\right\}. \tag{3}$$

In this paper, since the lifetimes and the numbers of new cells are dependent random variables, the growth rate of N_t is given by

$$\nu := \inf\left\{\gamma > 0,\ \sum_{n\geq 0} g_n(\gamma) < \infty\right\}, \tag{4}$$

where $g_n(\gamma)$ is expressed in terms of a Laplace-type transform of S_n, that is

$$g_n(\gamma) := \mathbb{E}\left[\left(\prod_{j=0}^{n-1}\kappa(X_j)\right)(\kappa(X_n)-1)e^{-\gamma S_n}\right].$$

One task of the paper, is to give exact evaluations of $\mathbb{E}[N_t]$ and $\mathbb{E}[N_t N_{t+\tau}]$, for any $t, \tau \geq 0$, under general and minimal conditions on the characteristics process (X_n) as described above. Those calculations are the main ingredients to get the convergence almost surely of $e^{-\nu t}N_t$ to a non-negative random variable W. A second task of the paper is to discuss the conditions yielding the previous results and to give some Markovian models for which the growth parameter ν is finite.

This paper is organized as follows. In Subsection 2.1 we give with proofs an exact evaluation of $\mathbb{E}[N_t]$ (see Proposition 2.2). An immediate consequence of this calculation, when ν is supposed finite, is the convergence in mean of $e^{-\nu t}\mathbb{E}[N_t]$ to some constant C_ν given by

$$C_\nu := \lim_{\gamma\to 0} \frac{\gamma}{\gamma+\nu} \sum_{n\geq 0} g_n(\gamma+\nu),$$

see Corollary 2.3 for a precise statement. As noticed in Subsection 2.2, from a multiplicative ergodicity behavior of $g_n(\gamma)$, see Definition 2.4, one can deduce that ν and C_ν are both finite.

The multiplicative ergodicity property is particularly useful in the context of additive functional of Markov chains with Markov kernel P and initial law μ. In Subsection 2.2 we present two instances of Markov models satisfying this multiplicative ergodicity property (and then for which ν and C_ν are both finite): first a toy model involving some Knudsen gas in Theorem 2.6 (see [5] for other results on Knudsen gases); second the model of linear autoregressive processes in Theorem 2.7. Both Theorems 2.6 and 2.7 are obtained under weak integrability assumptions on the observable ξ (the lifetime). The proof of these two theorems is based on the fact that

$$\forall n\geq 1, \quad g_n(\gamma) = \mu\left(\kappa\, e^{-\gamma\xi}\, P_\gamma^{n-1}\left(Ph_{\kappa,\gamma}\right)\right),$$

where P_γ is a Laplace-type kernel associated with P, ξ and κ (see Formula (13) for more details about the notations), and where $h_{\kappa,\gamma}$ is given by the formula $h_{\kappa,\gamma} := \big(\kappa-1\big)\, e^{-\gamma\xi}$. Then the multiplicative ergodicity property can be proved in the case when the Laplace kernels P_γ satisfy some nice spectral properties on a suitable Banach space $\mathcal{B}$ (see (14)). Then ν is proved to be finite and given by

$$\nu = \inf\{\gamma>0,\ r(\gamma)<1\},$$

where $r(\gamma)$ denotes the spectral radius of P_γ on $\mathcal{B}$. The main lines of our spectral approach, based on the quasi-compactness property (see [13]) and the Keller and Liverani perturbation theorem (see [17, 1]), are summarized at the end of Subsection 2.2. The complete procedure, together with further references related to the spectral method, are presented in [15].

In Subsection 3.1 we study the behavior of $\mathbb{E}[N_tN_{t+\tau}]$, for any $t>0$, $\tau\geq 0$, in the very general setting of dependence (cf. Proposition 3.2). Under assumptions (A)-(D), Proposition 3.2 states a formula for $\mathbb{E}[N_tN_{t+\tau}]$ in terms of expectations of functionals of $((X_n)_n, (X_n^{(k)})_n)$ The behavior of $\mathbb{E}[N_tN_{t+\tau}]$ is the main ingredient for the study of the quadratic mean and of the almost sure convergence of $e^{-\nu t}N_t$ as t tends to infinity (see Corollary 3.3). The purpose of Subsection 3.2 is to discuss Corollary's 3.3 assumptions, yielding the almost sure convergence of $e^{-\nu t}N_t$ as t tends to infinity, in the particular case where lifetimes and new cells numbers are independent and new cells numbers are modelled by a sequence of iid random variables. A main step is then to establish that, for some $\delta>0$ (see Lemma 3.9),

$$\left| e^{-\nu t}\sum_{n\geq 0}\kappa_1^n(\kappa_1-1)\mathbb{P}(\sum_{i=1}^{n+1}\xi(X_i)\leq t|X_0=x) - \tilde{C}_0(x)\right| \leq \frac{\Psi_0(x)}{2\pi}e^{-\delta t}, \tag{5}$$

where $\kappa_1 = \mathbb{E}[\kappa(X_1)]$ (see Lemma 3.9 for the definitions of the functions $\tilde{C}_0$ and Ψ_0). The bound (5) is the main tool to obtain (see Proposition 3.10 for more details),

$$\mathbb{E}[N_t] = e^{\nu t}\mathbb{E}[\kappa(X_0)]\mathbb{E}[e^{-\nu\xi(X_0)}\tilde{C}_0(X_0)][1+O(e^{-\epsilon_1 t})],\ as\ t\to\infty,\ \text{for some } \epsilon_1>0.$$

Under additional assumptions, the bound (5) also allows us to obtain (see Proposition 3.11),

$$\mathbb{E}[N_t N_{t+\tau}] = e^{\nu(2t+\tau)}\kappa_2 \sum_{k=0}^{\infty} \kappa_1^k \mathbb{E}[\tilde{C}_0^2(X_k)e^{-2\nu S_k}](1+ae^{-\epsilon_1 t}),\ as\ t\to\infty,$$

where $\kappa_2 = \mathbb{E}[\kappa(X_1)(\kappa(X_1)-1)]$ and where a, ϵ_1 are positive constants independent of t and τ. Lemma 3.9 gives sufficient conditions ensuring (5). Theorem 3.8 studies the convergence in mean quadratic and almost surely of $e^{-\nu t}N_t$ to a random variable W and gives the expressions of the first and of the second moment of this limit W. In Subsection 3.3 we discuss mainly the conditions of Lemma 3.9 (and then sufficient conditions for the bound (5)).

2. Behavior of the first moment, multiplicative ergodicity, examples

2.1. First moment of N_t.

The following proposition evaluates, for any $t>0$, the expectation of N_t, when it exists, in terms of the lifetimes $(\xi(X_i))_{i\geq 0}$ and of the numbers of new cells $(\kappa(X_i))_{i\geq 0}$. For this first result, we only assume our general stationary assumptions (A)-(C) that can be reformulated as follows.

Hypothesis 2.1. *There exist a family $\{X_{0,k_1,\dots,k_n},\ n\geq 0,\ k_1,..,k_n\geq 1\}$ of $\mathbb{X}$-valued random variables defined on the same probability space and two functions $\kappa:\mathbb{X}\to\mathbb{Z}_+$ and $\xi:\mathbb{X}\to[0,+\infty)$ such that, for every sequence $(k_i)_{i\geq 1}$ of positive integers,*

- *$(X_{0,k_1,\dots,k_n})_{n\geq 0}$ has the same distribution as $(X_n := X_{0,1^n})_{n\geq 0}$, where we use the notation $0,1^n$ to denote $0,k_1,\dots,k_n$ when $k_1=\dots=k_n=1$,*

and such that, setting $D_{0,k_1,\dots,k_n} := \kappa(X_{0,k_1,\dots,k_n})$ and $T_{0,k_1,\dots,k_n} := \xi(X_{0,k_1,\dots,k_n})$, the following model hypothesis holds

- *for every $x\in\mathbb{X}$, if a cell has characteristics x, then it has $\kappa(x)\geq 2$ children and a life duration of $\xi(x)$,*
- *X_0 is the characteristics of the initial cell (and so D_0 and T_0 are respectively its number of children and its life duration),*
- *for every $k\in\{1,\cdots,D_0\}$, $X_{0,k}$ is the characteristics of the k-th child of the initial cell (and so $D_{0,k}$ and $T_{0,k}$ are respectively its number of children and its life duration),*
- *more generally, for every $n\geq 0$, for every positive integers $k_1,\dots k_n$ such that $k_i\leq D_{0,k_1,\dots,k_{i-1}}$ (for every $i=1,\dots,n$), $X_{0,k_1,\dots,k_n}$ is the characteristics of the k_n-th child of the k_{n-1}-th child of the $\cdots$ of the k_1-th child of the initial cell (and so $D_{0,k_1,\cdots,k_n}$ and $T_{0,k_1,\cdots,k_n}$ are respectively its number of children and its lifetime).*

Proposition 2.2. *Assume Hypothesis 2.1. Let $t>0$ be fixed. If $\sum_{n\geq 0}\mathbb{E}\left[\left(\prod_{j=0}^n \kappa(X_j)\right)\mathbf{1}_{\{S_n\leq t\}}\right]<\infty$, then $\mathbb{E}[N_t]<\infty$ and*

$$\mathbb{E}[N_t] = 1+\sum_{n\geq 0}\mathbb{E}\left[\left(\prod_{j=0}^{n-1}\kappa(X_j)\right)(\kappa(X_n)-1)\,\mathbf{1}_{\{S_n\leq t\}}\right] \tag{6}$$

(with the usual convention $\prod_{j=0}^{-1}\kappa(X_j)=1$).

Proof. For every $n\geq 0$, we write $\Sigma_n(t)$ for the number of cells of generation n alive at time t. Observe that $\mathbb{E}[\Sigma_0(t)] = \mathbb{P}(\xi(X_0)>t)$ and that, for every $n\geq 1$ (with the convention

$k_0=0$),

$$\mathbb{E}[\Sigma_n(t)]=\mathbb{E}\left[\sum_{k_1=1}^{D_0}\sum_{k_2=1}^{D_{0,k_1}}\cdots\sum_{k_n=1}^{D_{0,k_1,\cdots,k_{n-1}}}\mathbf{1}_{\{T_0+T_{0,k_1}+\cdots+T_{0,k_1,\cdots,k_{n-1}}\le t<T_0+T_{0,k_1}+\cdots+T_{0,k_1,\cdots,k_n}\}}\right].$$

We write, since $D_0=\kappa(X_0)$, i.e, D_0 is a measurable function of X_0,

$$\begin{aligned}&\mathbb{E}[\Sigma_n(t)]\\&=\mathbb{E}\left[\sum_{k_1=1}^{D_0}\mathbb{E}\left[\sum_{k_2=1}^{D_{0,k_1}}\cdots\sum_{k_n=1}^{D_{0,k_1,k_2,\cdots,k_{n-1}}}\mathbf{1}_{\{T_0+T_{0,k_1}+\cdots+T_{0,k_1,k_2,\cdots,k_{n-1}}\le t<T_0+T_{0,k_1}+\cdots+T_{0,k_1,k_2,\cdots,k_n}\}}\middle| X_0\right]\right].\end{aligned}$$

Since the vector $(X_0,X_{0,k_1},\cdots,X_{0,k_1,k_2,\cdots,k_n})$ is distributed as $(X_0,X_{0,1},\cdots,X_{0,1,k_2,\cdots,k_n})$, we obtain a. s.

$$\begin{aligned}&\mathbb{E}\left[\sum_{k_2=1}^{D_{0,k_1}}\cdots\sum_{k_n=1}^{D_{0,k_1,k_2,\cdots,k_{n-1}}}\mathbf{1}_{\{T_0+T_{0,k_1}+\cdots+T_{0,k_1,k_2,\cdots,k_{n-1}}\le t<T_0+T_{0,k_1}+\cdots+T_{0,k_1,k_2,\cdots,k_n}\}}\middle| X_0\right]\\&=\mathbb{E}\left[\sum_{k_2=1}^{D_{0,1}}\cdots\sum_{k_n=1}^{D_{0,1,k_2,\cdots,k_{n-1}}}\mathbf{1}_{\{T_0+T_{0,1}+\cdots+T_{0,1,k_2,\cdots,k_{n-1}}\le t<T_0+T_{0,1}+\cdots+T_{0,1,k_2,\cdots,k_n}\}}\middle| X_0\right].\end{aligned}$$

We obtain, collecting the two last equalities and using some properties of the conditional expectation,

$$\begin{aligned}&\mathbb{E}[\Sigma_n(t)]\\&=\mathbb{E}\left[D_0\mathbb{E}\left[\sum_{k_2=1}^{D_{0,1}}\cdots\sum_{k_n=1}^{D_{0,1,k_2,\cdots,k_{n-1}}}\mathbf{1}_{\{T_0+T_{0,1}+\cdots+T_{0,1,k_2,\cdots,k_{n-1}}\le t<T_0+T_{0,1}+\cdots+T_{0,1,k_2,\cdots,k_n}\}}\middle| X_0\right]\right]\\&=\mathbb{E}\left[\mathbb{E}\left[D_0\sum_{k_2=1}^{D_{0,1}}\cdots\sum_{k_n=1}^{D_{0,1,k_2,\cdots,k_{n-1}}}\mathbf{1}_{\{T_0+T_{0,1}+\cdots+T_{0,1,k_2,\cdots,k_{n-1}}\le t<T_0+T_{0,1}+\cdots+T_{0,1,k_2,\cdots,k_n}\}}\middle| X_0\right]\right]\\&=\mathbb{E}\left[D_0\sum_{k_2=1}^{D_{0,1}}\cdots\sum_{k_n=1}^{D_{0,1,k_2,\cdots,k_{n-1}}}\mathbf{1}_{\{T_0+T_{0,1}+\cdots+T_{0,1,k_2,\cdots,k_{n-1}}\le t<T_0+T_{0,1}+\cdots+T_{0,1,k_2,\cdots,k_n}\}}\right].\end{aligned}$$

Now, starting with the last term above and using, analogously, the fact that $D_0,D_{0,1}$ are $(X_0,X_{0,1})$-measurable and that the distribution of $(X_0,X_{0,1},X_{0,1,k_2},\cdots,X_{0,1,k_2,\cdots,k_n})$ does not depend on k_2, we obtain

$$\begin{aligned}&\mathbb{E}[\Sigma_n(t)]=\\&\mathbb{E}\left[D_0D_{0,1}\mathbb{E}\left[\sum_{k_3=1}^{D_{0,1,1}}\cdots\sum_{k_n=1}^{D_{0,1,1,k_3,\cdots,k_{n-1}}}\mathbf{1}_{\{T_0+T_{0,1}+\cdots+T_{0,1,1,k_3,\cdots,k_{n-1}}\le t<T_0+T_{0,1}+\cdots+T_{0,1,1,k_3,\cdots,k_n}\}}\middle| X_0,X_{0,1}\right]\right]\\&=\mathbb{E}\left[D_0D_{0,1}\sum_{k_3=1}^{D_{0,1,1}}\cdots\sum_{k_n=1}^{D_{0,1,1,k_3,\cdots,k_{n-1}}}\mathbf{1}_{\{T_0+T_{0,1}+\cdots+T_{0,1,1,k_3,\cdots,k_{n-1}}\le t<T_0+T_{0,1}+\cdots+T_{0,1,1,k_3,\cdots,k_n}\}}\right].\end{aligned}$$

We deduce, iterating those arguments (recall that $1^m = 1, \cdots, 1$ (m times)), that for every $n \geq 1$,

$$\mathbb{E}[\Sigma_n(t)] = \mathbb{E}\left[D_0 D_{0,1} \cdots D_{0,1^{n-1}} \mathbf{1}_{\{T_0+T_{0,1}+\cdots+T_{0,1^{n-1}} \leq t < T_0+T_{0,1}+\cdots+T_{0,1^n}\}}\right].$$

Now we get, since the vector $(D_0, \cdots, D_{0,1^{n-1}}, T_0, \cdots, T_{0,1^n})$ is distributed as the vector $(\kappa(X_0), \cdots, \kappa(X_{n-1}), \xi(X_0), \cdots, \xi(X_n))$,

$$\begin{aligned} &\mathbb{E}\left[D_0 D_{0,1} \cdots D_{0,1^{n-1}} \mathbf{1}_{\{T_0+T_{0,1}+\cdots+T_{0,1^{n-1}} \leq t < T_0+T_{0,1}+\cdots+T_{0,1^n}\}}\right] \\ &= \mathbb{E}\left[\left(\prod_{j=0}^{n-1} \kappa(X_j)\right)\left(\mathbf{1}_{\{S_{n-1}\leq t\}} - \mathbf{1}_{\{S_n \leq t\}}\right)\right]. \end{aligned}$$

Hence, we obtain by collecting the two last equalities,

$$\mathbb{E}[\Sigma_n(t)] = \mathbb{E}\left[\left(\prod_{j=0}^{n-1} \kappa(X_j)\right)\left(\mathbf{1}_{\{S_{n-1}\leq t\}} - \mathbf{1}_{\{S_n \leq t\}}\right)\right].$$

Now we get, since $\sum_{n\geq 0} \mathbb{E}\left[\left(\prod_{j=0}^{n} \kappa(X_j)\right) \mathbf{1}_{\{S_n \leq t\}}\right] < \infty$ and $N_t = \mathbf{1}_{T_0 > t} + \sum_{n=1}^{\infty} \Sigma_n(t)$ a. s.,

$$\begin{aligned} \mathbb{E}[N_t] &= \mathbb{P}(\xi(X_0) > t) + \sum_{n=1}^{\infty} \mathbb{E}\left[\left(\prod_{j=0}^{n-1} \kappa(X_j)\right)\left(\mathbf{1}_{\{S_{n-1}\leq t\}} - \mathbf{1}_{\{S_n \leq t\}}\right)\right] \\ &= \mathbb{P}(\xi(X_0) > t) + \sum_{n=0}^{\infty} \mathbb{E}\left[\left(\prod_{j=0}^{n} \kappa(X_j)\right) \mathbf{1}_{\{S_n \leq t\}}\right] - \sum_{n=1}^{\infty} \mathbb{E}\left[\left(\prod_{j=0}^{n-1} \kappa(X_j)\right) \mathbf{1}_{\{S_n \leq t\}}\right] \\ &= \mathbb{P}(\xi(X_0) > t) + \mathbb{E}\left[\kappa(X_0)\mathbf{1}_{\xi(X_0)\leq t}\right] + \sum_{n\geq 1} \mathbb{E}\left[\left(\prod_{j=0}^{n-1} \kappa(X_j)\right)(\kappa(X_n) - 1)\mathbf{1}_{\{S_n \leq t\}}\right] \\ &= 1 + \mathbb{E}\left[(\kappa(X_0) - 1)\mathbf{1}_{\xi(X_0)\leq t}\right] + \sum_{n\geq 1} \mathbb{E}\left[\left(\prod_{j=0}^{n-1} \kappa(X_j)\right)(\kappa(X_n) - 1)\mathbf{1}_{\{S_n \leq t\}}\right]. \end{aligned}$$

□

It follows from Proposition 2.2 that $\mathbb{E}[N_t] < \infty$ if ν defined by (4) is finite (using the fact that $\mathbf{1}_{\{S_n \leq t\}} \leq e^{\gamma t} e^{-\gamma S_n}$ for any positive γ). Now using Proposition 2.2 and arguing exactly as for the proof of Theorem 2.1 in [21], we obtain the following exponential behavior in mean of $\mathbb{E}[N_t]$ in a very general setting of dependence with the use of the function G given by

$$G(\gamma) := \sum_{n\geq 0} g_n(\gamma), \quad \text{recall that} \quad g_n(\gamma) = \mathbb{E}\left[\left(\prod_{j=0}^{n-1} \kappa(X_j)\right)(\kappa(X_n) - 1)e^{-\gamma S_n}\right]. \qquad (7)$$

Corollary 2.3. *Assume Hypothesis 2.1 and that $\nu < \infty$ and that the following limit exists*

$$C_\nu := \lim_{\gamma \to 0} \frac{\gamma}{\gamma + \nu} G(\nu + \gamma). \qquad (8)$$

Then,
$$\lim_{t\to\infty}\frac{1}{t}\int_0^t e^{-\nu s}\mathbb{E}[N_s]ds = C_\nu. \tag{9}$$

2.2. **Multiplicative ergodicity, application to Markov chains.** In order to study the above function $G(\cdot)$, and so ν and C_ν, we adapt the notion of "multiplicative ergodicity", as introduced in [19] and [20], to our context.

Definition 2.4. *Let $\gamma_1 > 0$. We say that $(S_n, \kappa(X_n))_n$ is* **multiplicatively ergodic** *on $J = [0, \gamma_1)$ if there exist two continuous maps A and ρ from J to $(0, +\infty)$ such that, for every compact subset K of $(0, \gamma_1)$, there exist $M_K > 0$ and $\theta_K \in (0, 1)$ such that, for every $n \geq 1$,*
$$\forall \gamma \in K, \quad |g_n(\gamma) - A(\gamma)(\rho(\gamma))^n| \leq M_K(\rho(\gamma)\theta_K)^n. \tag{10}$$
When $\kappa(\cdot)$ is constant, we will simply say that $(S_n)_n$ is multiplicatively ergodic on J.

Remark 2.5. *Assume that $(S_n, \kappa(X_n))_n$ is multiplicatively ergodic on $J = [0, \gamma_1)$. Then*

- *For every $\gamma \in J$ we have: $G(\gamma) = \sum_{n\geq 0} g_n(\gamma) < \infty \iff \rho(\gamma) < 1$.*
- *For every compact subset K of J, we obtain from the definition of ν in (4) that*
$$\forall \gamma \in K \cap (\nu, +\infty), \quad \left|G(\gamma) - \frac{A(\gamma)}{1-\rho(\gamma)}\right| \leq \frac{M_K}{1-\rho(\gamma)\theta_K}.$$
- *$\nu < \gamma_1$ means that*
$$\nu = \inf\{\gamma \in J \ : \ \rho(\gamma) < 1\} < \gamma_1. \tag{11}$$
- *If moreover ρ is differentiable at ν with $\rho(\nu) = 1$ and $\rho'(\nu) \neq 0$, then (8) follows with $C_\nu = -\frac{A(\nu)}{\nu\rho'(\nu)}$. Actually, to obtain (11), we can relax the continuity assumptions on A and ρ on $J = [0, \gamma_1)$. For (8), we just need the continuity of A and the differentiability of ρ at ν (with $\rho'(\nu) \neq 0$).*

The multiplicative ergodicity property is specially adapted for additive functional of Markov chains, that is: $X = (X_n)_n$ is a Markov chain on $(\mathbb{X}, \mathcal{X})$ with Markov kernel $P(x, dy)$, invariant probability π, and initial distribution μ (i.e. μ is the distribution of X_0). Below we present two Markov models satisfying the multiplicative ergodicity property. The first one is a toy model, namely: at each step, either we follow a Markov chain $Z = (Z_n)_n$ (with probability $(1-\alpha)$) or we generate an independent random variable with distribution the invariant probability measure of Z (with probability α). See [5] for more about this model.

Theorem 2.6 (Knudsen gas)**.** *Let $\mathbb{X} := \mathbb{R}^d$, let π be some Borel probability measure on $\mathbb{X}$, and let U a Markov operator with stationary probability π. We fix $\alpha \in (1/2, 1)$. Let $X = (X_n)_n$ be a Markov chain with transition kernel $P := \alpha\pi + (1-\alpha)U$. Assume that the initial distribution μ admits a density (with respect to π) having a moment of order p for some $p > 1$. Moreover assume that $\kappa \equiv 2$ and that $\pi(\xi > 0) = 1$. Then $(S_n)_n$ is multiplicatively ergodic on some interval $[0, \gamma_1]$ with $\rho(0) = 2$ and $\gamma_1 > 0$ such that $\rho(\gamma_1) < 1$. Thus ν defined by (4) is finite.*
If, moreover, $\pi(\xi^\tau) < \infty$ for some $\tau \in (1, p/(p-1))$, then (8) is well defined and Property (9) holds with $C_\nu \in (0, +\infty)$.

Note that, for this example, as for the next one, our moment assumptions are very weak (no exponential moment is needed, our assumptions only involve finite order moment assumptions).

Theorem 2.7 (Linear autoregressive model). *Let $\mathbb{X} := \mathbb{R}$ and $X_n = \alpha X_{n-1} + \vartheta_n$ for $n \geq 1$, where X_0 is a real-valued random variable, $\alpha \in (-1, 1)$, and $(\vartheta_n)_{n\geq 1}$ is a sequence of i.i.d. real-valued random variables independent of X_0. Let $r_0 > 0$. We assume that ϑ_1 has a continuous Lebesgue probability density function $p > 0$ on $\mathbb{X}$ satisfying the following condition: for all $x_0 \in \mathbb{R}$, there exist a neighbourhood V_{x_0} of x_0 and a non-negative function $q_{x_0}(\cdot)$ such that $y \mapsto (1 + |y|)^{r_0}\, q_{x_0}(y)$ is Lebesgue-integrable and such that* [1]

$$\forall y \in \mathbb{R},\ \forall v \in V_{x_0},\ p(y + v) \leq q_{x_0}(y). \tag{12}$$

Assume that the initial distribution μ is either the stationary probability measure π or δ_x for some $x \in \mathbb{R}$. Let N_0 be a positive integer. Assume that κ is bounded, that $\lim_{|x|\to+\infty} \xi(x) = +\infty$, that the Lebesgue measure of the set $[\xi = 0]$ is zero, and that $\sup_{x\in\mathbb{R}} \frac{\xi(x)}{(1+|x|)^{r_0}} < \infty$.

Then $(S_n, \kappa(X_n))_n$ is multiplicatively ergodic on $J = [0, +\infty)$ with $\lim_{\gamma\to 0_+} \rho(\gamma) \geq 2$ and $\lim_{\gamma\to+\infty} \rho(\gamma) = 0$. Thus ν given by (4) is well defined (and is independent of the choice of the initial distribution μ).
If moreover there exists $\tau > 0$ such that $\sup_{x\in\mathbb{R}} \frac{\xi(x)^{1+\tau}}{(1+|x|)^{r_0}} < \infty$ [2]*, then the constant C_ν given by (8) is well defined in $(0, +\infty)$ and Property (9) holds.*

The proof of Theorems 2.6 and 2.7 is based on the spectral study of Laplace operators associated with (P, ξ, κ). A general setting of this method is provided in [15], together with the complete proof of Theorems 2.6 and 2.7 (see [15, Theorems 5.1 and 6.1]). Let us give the main lines of this spectral method.

• *The Laplace kernel associated with* (P, ξ, κ). We assume that, for every $n \geq 1$, the random variable $\prod_{j=0}^{n} \kappa(X_j)$ is integrable. We set $h_{\kappa,\gamma} := \big(\kappa - 1\big)\, e^{-\gamma\xi}$. Let $\gamma \in (0, +\infty)$. For $n \geq 1$,

$$\begin{aligned} g_n(\gamma) &= \mathbb{E}\left[\left(\prod_{j=0}^{n-1} \kappa(X_j) e^{-\gamma\xi(X_j)}\right) h_{\kappa,\gamma}(X_n)\right] \\ &= \mathbb{E}\left[\left(\prod_{j=0}^{n-1} \kappa(X_j) e^{-\gamma\xi(X_j)}\right) (P h_{\kappa,\gamma})(X_{n-1})\right], \end{aligned}$$

with $(Ph)(x) := \int_{\mathbb{X}} h(y)\, P(x, dy)$. If $n \geq 2$, we continue and obtain

$$g_n(\gamma) = \mathbb{E}\left[\left(\prod_{j=0}^{n-2} \kappa(X_j) e^{-\gamma\xi(X_j)}\right) (P_\gamma(P h_{\kappa,\gamma}))(X_{n-2})\right],$$

with $P_\gamma h := P(h\kappa e^{-\gamma\xi})$. An easy induction gives

$$\forall n \geq 1, \quad g_n(\gamma) = \mu\Big(\kappa\, e^{-\gamma\xi}\, P_\gamma^{n-1}\, (P h_{\kappa,\gamma})\Big). \tag{13}$$

• *A spectral multiplicative ergodicity property.* The first step of the spectral procedure is to find an interval $J_0 \subset [0, \infty)$ and a suitable Banach space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ on which, for every $\gamma \in J_0$, the Laplace kernel P_γ continuously acts on $\mathcal{B}$ and has the following spectral properties: there

---

[1] Note that ϑ_1 admits a moment of order r_0.

[2] Recall that $\int_{\mathbb{R}} |x|^{r_0}\, d\pi(x) < \infty$ under the assumptions of Theorem 2.7 (see [7, 8]). Hence $\sup_{x\in\mathbb{R}} \frac{\xi(x)^{1+\tau}}{(1+|x|)^{r_0}} < \infty$ implies that $\int_{\mathbb{R}} |\xi|^{1+\tau}\, d\pi < \infty$.

exists a map $\gamma \mapsto \Pi_\gamma$ from J_0 into the space $\mathcal{L}(\mathcal{B})$ of bounded linear operators on $\mathcal{B}$ such that, for every compact subset K of J_0, there exist $\theta_K \in (0,1)$ and $M_K \in (0,+\infty)$ such that

$$\forall \gamma \in K,\ \forall f \in \mathcal{B}, \quad \left\| P_\gamma^n f - r(\gamma)^n \Pi_\gamma f \right\|_{\mathcal{B}} \le M_K \left(\theta_K\, r(\gamma)\right)^n \|f\|_{\mathcal{B}} \tag{14}$$

where $r(\gamma)$ is the spectral radius of P_γ. Then, using Definition 2.4, Remark 2.5 and Formula (13), the following assertions hold:

(i) If the functions $\gamma \mapsto r(\gamma)$ and $\gamma \mapsto B(\gamma) := \mu\left(\kappa\, e^{-\gamma\xi}\Pi_\gamma(Ph_{\kappa,\gamma})\right)$ are continuous from J_0 to $(0,+\infty)$, then $(S_n, \kappa(X_n))_n$ is multiplicatively ergodic on J_0 with $A(\gamma) := \frac{B(\gamma)}{r(\gamma)}$ and $\rho(\gamma) = r(\gamma)$.

(ii) If moreover $\inf\limits_{\gamma \in J_0} r(\gamma) < 1 < \sup\limits_{\gamma \in J_0} r(\gamma)$, then ν is finite and

$$\nu = \inf\{\gamma > 0 \,:\, r(\gamma) < 1\}. \tag{15}$$

(iii) If furthermore the functions $r(\cdot)$ and $B(\cdot)$ are C^1-smooth on J_0, and if $r'(\nu) \neq 0$, then the constant C_ν of (8) is well defined and finite, and Property (9) holds true.

• *On Property (14).* A natural way to obtain (14) is to use quasi-compactness. More precisely assume that, for some fixed $\gamma \ge 0$, P_γ continuously acts on $\mathcal{B}$ and that

(a) $r(\gamma) > 0$ and P_γ is quasi-compact on $\mathcal{B}$,

(b) $r(\gamma)$ is the only eigenvalue of modulus $r(\gamma)$ for P_γ, and $r(\gamma)$ is a first order pole of P_γ with moreover $\dim \operatorname{Ker}(P_\gamma - r(\gamma)I) = \dim \operatorname{Ker}(P_\gamma - r(\gamma)I)^2 = 1$.

Then P_γ satisfies (14) with $K = \{\gamma\}$. Condition (a) may be investigated by applying the quasi-compactness criteria of [13]. Condition (b) may be studied by using standard arguments of positive operators. The case when $\mathcal{B}$ is a Banach lattice is specially adapted to the study of (b), and further useful properties on $r(\cdot)$ may be obtained in this case, as for instance the non-increasingness of $r(\cdot)$. For the Knudsen gas (Theorem 2.6), the conditions (a) and (b) are fulfilled on every $\mathbb{L}^a(\pi)$ $(a > 1)$, where $\mathbb{L}^a(\pi)$ denotes the usual Lebesgue space associated with the P-invariant probability distribution π. For linear autoregressive models (Theorem 2.7), setting $V(x) := (1+|x|)^{r_0}$, the conditions (a) and (b) are proved to hold on the Banach space $\mathcal{B}_a = \mathcal{C}_{V^a}$ for each $a \in (0,1]$, where $(\mathcal{C}_{V^a}, \|\cdot\|_{V^a})$ denotes the space of continuous fonctions $f : \mathbb{R} \to \mathbb{C}$ such that the limits $\lim_{x\to-\infty} f(x)/V(x)^a$ and $\lim_{x\to+\infty} f(x)/V(x)^a$ exist in $\mathbb{C}$ and are equal, and where $\|f\|_{V^a} = \sup_{x\in\mathbb{R}} |f(x)|/V(x)^a$. The Banach spaces involved in these two instances are Banach lattices.

• *The use of weak perturbation theory.* The above conditions (a) and (b) only give Property (14) for $K = \{\gamma\}$. To obtain (14) with the desired uniformity property with respect to any compact subset K of J_0, and moreover to obtain the continuity of the functions $r(\cdot)$ and $B(\cdot)$, a natural way is to apply the perturbation theory of bounded linear operators. Unfortunately, the classical operator perturbation method [22, 23, 9, 10] does not apply to our context. Indeed, because we do not assume any exponential moment condition on ξ, the map $\gamma \mapsto P_\gamma$ is (in general) **not continuous** from $(0,+\infty)$ to $\mathcal{L}(\mathcal{B})$. For instance, for linear autoregressive models (respectively for the Knudsen gas), the map $\gamma \mapsto P_\gamma$ is not continuous in general from $(0,+\infty)$ to $\mathcal{L}(\mathcal{B}_a)$ with $\mathcal{B}_a = \mathcal{C}_{V^a}$ (respectively with $\mathcal{B}_a = \mathbb{L}^a(\pi)$), but only from $(0,+\infty)$ to $\mathcal{L}(\mathcal{B}_a, \mathcal{B}_b)$ for $0 \le a < b \le 1$ (respectively for $1 \le b < a$). The same problem occurs in the study of the C^1-regularity of $r(\cdot)$ and $B(\cdot)$. This is the reason why we use the

Keller-Liverani perturbation theorem [17] in [15]. The price to pay is to consider "a chain of Banach spaces" instead of a single one, according to the approach used in [14].

3. Behavior of the second moment and almost sure convergence

3.1. **Second moment and applications.** To study the behaviour of $\mathbb{E}[N_t N_{t+\tau}]$ (Proposition 3.2), as well as the convergence of $(e^{-\nu t}N_t)_{t\geq 0}$ when $t \to +\infty$ (Corollary 3.3), let us introduce an additional assumption involving the characteristics for lines of cells coinciding up to the k-th generation (this is a reformulation of our Assumptions (A)-(D)).

Hypothesis 3.1. *Hypothesis 2.1 holds true. Moreover, for each $k \in \mathbb{N}$, there exists a process $X^{(k)} = (X_n^{(k)})_{n\geq 0}$ such that*

$$\begin{cases} (X_n^{(k)})_{0\leq n\leq k} = (X_n)_{0\leq n\leq k} & a.s. \\ (X_n^{(k)})_{n\geq 0} = (X_n)_{n\geq 0} & \text{in law}, \end{cases} \tag{16}$$

and such that, for every couple of sequences of positive integers $(m_i)_{i\geq 1}$ and $(\ell_i)_{i\geq 1}$ such that $m_1 = \ell_1, ..., m_k = \ell_k$ and $\ell_{k+1} \neq m_{k+1}$, $((X_{0,m_1,...,m_n})_n, (X_{0,\ell_1,...,\ell_n})_n)$ has the same distribution as $(X, X^{(k)})$.

Now define, for any integers $n \geq 1, m \geq 1$ and $\min(n,m)-1 \geq k \geq 0$ the random variables $A_{n,m,k}$ as follows:

$$\begin{aligned} A_{n,m,k} \quad = \quad & \left(\prod_{i=0}^{n-2} \kappa(X_i)\right) \left(\prod_{j=\min(k+1,n-1)}^{m-2} \kappa(X_j^{(k)})\right) \\ & \left(\prod_{j\in\{k\}\setminus\{n-1,m-1\}} (\kappa(X_j)-1)\right) (\kappa(X_{n-1})-1) \left(\kappa(X_{m-1}^{(k)})-1\right), \end{aligned}$$

with the usual convention $\prod_{i=k+1}^{\ell} \cdots = 1$ if $\ell \leq k$. Define also $S_n^{(k)} := \sum_{j=0}^{n} \xi(X_j^{(k)})$. The main result of this section is the following proposition.

Proposition 3.2. *Assume that Hypothesis 3.1 holds. Let $t > 0$ and $\tau \geq 0$ be fixed. If $\sum_{n\geq 0} \mathbb{E}\left[\left(\prod_{j=0}^{n} \kappa(X_j)\right) \mathbf{1}_{\{S_n \leq t+\tau\}}\right] < \infty$, then*

$$\mathbb{E}[N_t N_{t+\tau}] = \mathbb{E}[N_t] + \mathbb{E}[N_{t+\tau}] - 1 + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{\min(n,m)-1} \mathbb{E}\left[A_{n,m,k}\mathbf{1}_{\{S_{n-1}\leq t, S_{m-1}^{(k)} \leq t+\tau\}}\right]. \tag{17}$$

Proof. We have, using the notations of the proof of Proposition 2.2, $N_t = \mathbf{1}_{\{T_0>t\}} + \sum_{n=1}^{\infty} \Sigma_n(t) \quad a.s.$, with

$$\Sigma_n(t) = \sum_{k_1=1}^{D_0} \sum_{k_2=1}^{D_{0,k_1}} \cdots \sum_{k_n=1}^{D_{0,k_1,\cdots,k_{n-1}}} \left(\mathbf{1}_{\{T_0+T_{0,k_1}+\cdots+T_{0,k_1,\cdots,k_{n-1}}\leq t\}} - \mathbf{1}_{\{T_0+T_{0,k_1}+\cdots+T_{0,k_1,\cdots,k_n}\leq t\}}\right).$$

We deduce from $\sum_{n\geq 0} \mathbb{E}\left[\left(\prod_{j=0}^{n} \kappa(X_j)\right) \mathbf{1}_{\{S_n \leq t+\tau\}}\right] < \infty$ that $\sum_{n\geq 0} \mathbb{E}\left[\left(\prod_{j=0}^{n} \kappa(X_j)\right) \mathbf{1}_{\{S_n \leq t\}}\right] < \infty$, and then, arguing exactly as in the proof of Proposition 2.2,

$$\sum_{n=1}^{\infty}\mathbb{E}\left[\sum_{k_1=1}^{D_0}\sum_{k_2=1}^{D_{0,k_1}}\cdots\sum_{k_n=1}^{D_{0,k_1,\cdots,k_{n-1}}}\mathbf{1}_{\{T_0+T_{0,k_1}+\cdots+T_{0,k_1,\cdots,k_{n-1}}\le t\}}\right]=\sum_{n=1}^{\infty}\mathbb{E}\left[\left(\prod_{j=0}^{n-1}\kappa(X_j)\right)\mathbf{1}_{\{S_{n-1}\le t\}}\right]<\infty.$$

Hence,

$$\sum_{n=1}^{\infty}\sum_{k_1=1}^{D_0}\sum_{k_2=1}^{D_{0,k_1}}\cdots\sum_{k_n=1}^{D_{0,k_1,\cdots,k_{n-1}}}\mathbf{1}_{\{T_0+T_{0,k_1}+\cdots+T_{0,k_1,\cdots,k_{n-1}}\le t\}}<\infty\quad a.s..$$

Therefore $N_t=1+\sum_{n=1}^{\infty}\tilde\Sigma_n(t)\quad a.s.$, with

$$\tilde\Sigma_n(t)=\sum_{k_1=1}^{D_0}\sum_{k_2=1}^{D_{0,k_1}}\cdots\sum_{k_{n-1}=1}^{D_{0,k_1,\cdots,k_{n-2}}}(D_{0,k_1,\cdots,k_{n-1}}-1)\mathbf{1}_{\{T_0+T_{0,k_1}+\cdots+T_{0,k_1,\cdots,k_{n-1}}\le t\}}.$$

Consequently,

$$\begin{aligned}N_tN_{t+\tau}&=1+\sum_{n=1}^{\infty}\tilde\Sigma_n(t)+\sum_{n=1}^{\infty}\tilde\Sigma_n(t+\tau)+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\tilde\Sigma_n(t)\tilde\Sigma_m(t+\tau)\\&=N_t+N_{t+\tau}-1+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\tilde\Sigma_n(t)\tilde\Sigma_m(t+\tau)\,.\end{aligned}$$

For any positive integers n,m and $t>0,\tau\ge 0$, we have,

$$\tilde\Sigma_n(t)\tilde\Sigma_m(t+\tau)=\sum_{k=0}^{\min(n-1,m-1)}\sum_{(\boldsymbol{\ell},\tilde{\boldsymbol{\ell}})\in E_{n,m,k}}(D_{\boldsymbol{\ell}}-1)(D_{\tilde{\boldsymbol{\ell}}}-1)\mathbf{1}_{\{\mathcal{S}_{n-1}(\boldsymbol{\ell})\le t,\mathcal{S}_{m-1}(\tilde{\boldsymbol{\ell}})\le t+\tau\}}$$

where $E_{n,m,k}$ is the set of $(\boldsymbol{\ell},\tilde{\boldsymbol{\ell}})$, with $\boldsymbol{\ell}=(0,\ell_1,...,\ell_{n-1})\in\{0\}\times(\mathbf{N}\setminus\{0\})^{n-1}$ and $\tilde{\boldsymbol{\ell}}=(0,\tilde\ell_1,...,\tilde\ell_{m-1})\in\{0\}\times(\mathbf{N}\setminus\{0\})^{n-1}$ having the same coordinates up to time k, i.e. such that $\min\{j=0,...,\min(n,m)\ :\ \ell_j\neq\tilde\ell_j\}=k+1$, with the notation $\mathcal{S}_{n-1}(\boldsymbol{\ell}):=T_0+T_{0,\ell_1}+\cdots+T_{0,\ell_1,\cdots,\ell_{n-1}}$. We conclude by proceeding exactly as in the proof of Proposition 2.2. □

As it was done in [12] in the case of independence (cf. Lemma 19.1 and Theorem 21.1 there), Proposition 3.2 is the main ingredient for the proofs of the quadratic mean and of the almost sure convergence of $e^{-\nu t}N_t$, where the growth rate ν given in (4) is assumed to be finite. This is the purpose of the next corollary.

Corollary 3.3. *Assume that the assumptions of Proposition 3.2 are satisfied, that $\nu<\infty$, that $\limsup_{t\to\infty}e^{-\nu t}\mathbb{E}[N_t]<\infty$ and that there exists $K>0$ such that*

$$\limsup_{t\to\infty}\sup_{\tau\ge0}\left|e^{-\nu(2t+\tau)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{\min(n,m)-1}\mathbb{E}\left[A_{n,m,k}\mathbf{1}_{\{S_{n-1}\le t,S_{m-1}^{(k)}\le t+\tau\}}\right]-K\right|=0.\tag{18}$$

Then there exists a square integrable random variable W such that $e^{-\nu t}N_t$ converges in quadratic mean to W as t tends to infinity.

If moreover the convergence in (18) is exponentially fast and if $W>0$ then $e^{-\nu t}N_t$ converges almost surely to W as t tends to infinity.

Proof of Corollary 3.3. Clearly,

$$\begin{aligned}&\mathbb{E}\left[\left(e^{-\nu t}N_t - e^{-\nu(t+\tau)}N_{t+\tau}\right)^2\right]\\ &= e^{-2\nu t}\mathbb{E}\left[N_t^2\right] + e^{-2\nu(t+\tau)}\mathbb{E}\left[N_{t+\tau}^2\right] - 2e^{-2\nu t-\nu\tau}\mathbb{E}\left[N_tN_{t+\tau}\right].\end{aligned}$$

Now Proposition 3.2 gives,

$$\begin{aligned}&e^{-2\nu t-\nu\tau}\mathbb{E}[N_tN_{t+\tau}]\\ &= e^{-2\nu t-\nu\tau}\mathbb{E}[N_t] + e^{-2\nu t-\nu\tau}\left(\mathbb{E}[N_{t+\tau}]-1\right)\\ &+ e^{-2\nu t-\nu\tau}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{\min(n,m)-1}\mathbb{E}\left[A_{n,m,k}\mathbf{1}_{\{S_{n-1}\leq t, S_{m-1}^{(k)}\leq t+\tau\}}\right].\end{aligned}$$

Thanks to the assumptions of Corollary 3.3, the two first terms of the right hand side of the last equality tends to 0 as t tends to infinity. While the third term tends to K. Those three limits hold for any $\tau \geq 0$ and uniformly in τ. Hence,

$$\lim_{t\to\infty}\mathbb{E}\left[\left(e^{-\nu t}N_t - e^{-\nu(t+\tau)}N_{t+\tau}\right)^2\right] = K + K - 2K = 0,$$

for any $\tau \geq 0$, uniformly in τ. The Cauchy criterion ensures then the convergence in quadratic mean of $e^{-\nu t}N_t$ as t tends to infinity to a random variable W with finite second moment.

For the last point, we deduce from Proposition 3.2 that $\int_0^\infty \mathbb{E}\left[\left(e^{-\nu t}N_t - W\right)^2\right]dt < \infty$. This yields (arguing as for the proof of Theorem 21.1 in [12]) the almost sure convergence, as t tends to infinity, of $e^{-\nu t}N_t$ to W. □

We will apply these results in the two following sections under some additional independence assumptions.

3.2. **Some extensions of Harris' results.** For further results, we will make the following stronger assumption reinforcing (A)-(D) and involving some independence assumptions.

Hypothesis 3.4. *Hypothesis 3.1 holds.* $(X_n)_n$ *is a stationary sequence of random variables,* $(\kappa(X_n))_n$ *is a sequence of i.i.d. square integrable random variables of expectation* κ_1*, which is independent of* $(\xi(X_n))_n$*. Moreover, for all* $k \in \mathbb{N}$*,* $(X_n^{(k)})_{n\geq k+1}$ *and* $(X_n)_{n\geq k+1}$ *are independent given* X_k*. Finally the number* ν *(as defined in (4)) satisfies*

$$\forall x \in \mathbb{X}, \quad \nu = \inf\left\{\gamma > 0,\ \sum_{n\geq 0}\kappa_1^n\mathbb{E}\left[e^{-\gamma S_{n+1}}|X_0 = x\right] < \infty\right\} < \infty. \tag{19}$$

We set $\kappa_2 := \mathbb{E}[\kappa(X_1)(\kappa(X_1)-1)]$.

Remark 3.5. *Observe that under Hypothesis 3.4,*

$$\mathbb{E}\left[\left(\prod_{j=0}^{n}\kappa(X_j)\right)\mathbf{1}_{\{S_n\leq t\}}|X_0\right] = \kappa_1^n\kappa(X_0)\mathbb{E}\left[\mathbf{1}_{\{S_n\leq t\}}|X_0\right] \leq \kappa_1^{n+1}\kappa(X_0)\mathbb{E}\left[e^{-\gamma(S_n-t))}|X_0\right].$$

Hence, Proposition 2.2 applies and (6) *can be rewritten*

$$\mathbb{E}[N_t] = 1 + \sum_{n\geq 0}\kappa_1^n(\kappa_1 - 1)\mathbb{P}\left(S_n \leq t\right).$$

Remark 3.6 (Bifurcating Markov chains). *Hypothesis 3.4 can be satisfied by bifurcating Markov chains as defined, for instance, in Section 3 of* [21] *or by Definition 1.1 in* [6]*, where an explicit model is constructed there. This model is similar to that introduced in* [11]*. It supposes that the characteristics X_{v0} and X_{v1} of two daughters are linked to the mother's one X_v through the following auto-regressive equations: for any $v \in \mathbb{T}$,*

$$\begin{cases} X_{v0} = & \rho_m X_v + \sqrt{1-\rho_m^2}\epsilon_{v0} \\ X_{v1} = & \rho_m X_v + \sqrt{1-\rho_m^2}(\rho_s\epsilon_{v0} + \sqrt{1-\rho_s^2}\epsilon_{v1}) \end{cases}$$

where $\rho_m, \rho_s \in]-1,1[$ and they play the role of the mother's and sister's correlations, $(\epsilon_v)_{v\in\mathbb{T}}$ is a sequence of independent standard normal law and $X_0 = \epsilon_0$. Contrarily to [6]*, the purpose is to evaluate the Laplace transform of $S_n = \sum_{k=0}^n \xi(X_k)$. In* [21]*, an explicit computation of this Laplace transform is done in the case where*

$$\xi(x) = a + b(x+c)^2,$$

see Proposition 4.1 there.

Define $f_{x,0}(t) = (\kappa_1 - 1)e^{-\nu t}\sum_{n\geq 0}\kappa_1^n \mathbb{P}\left(S_{n+1} - S_0 \leq t | X_0 = x\right)$. We will make the following assumption involving the Laplace transform $\tilde{f}_{x,0}$ of $f_{x,0}$:

$$\forall\gamma > 0, \quad \tilde{f}_{x,0}(\gamma) = \int_0^\infty e^{-\gamma t} f_{x,0}(t)dt = \frac{\kappa_1 - 1}{\gamma+\nu}\sum_{n\geq 0}\kappa_1^n \mathbb{E}\left[e^{-(\gamma+\nu)(S_{n+1}-S_0)}|X_0 = x\right]. \quad (20)$$

Hypothesis 3.7. *Suppose that there exist two positive reals $\delta < \nu$ and ϵ such that, for any x, the Laplace transform $\tilde{f}_{x,0}$, extended on the complex plane, satisfies the following conditions:*

(1) $\tilde{f}_{x,0}$ is analytic in $\{z = u + iy, |u| < \delta + \epsilon, y \in \mathbb{R}\} \setminus \{0\}$,
(2) $\tilde{f}_{x,0}$ has a simple pole at 0, with residue $\tilde{C}_0(x)$,
(3) $\int_{-\infty}^{+\infty} |\tilde{f}_{x,0}(\delta + iy)|dy < \infty$,
(4) $\lim_{y\to\pm\infty} \tilde{f}_{x,0}(u + iy) = 0$, uniformly in $u \in [-\delta, \delta]$,
(5) $\Psi_0(x) := \int_{-\infty}^{+\infty} |\tilde{f}_{x,0}(-\delta + iy)|dy < \infty$.

We generalize the approach of [21] to obtain the following result extending [12, Theorem 19.1] to the case where the lifetimes are dependent.

Theorem 3.8. *Assume Hypotheses 3.4 and 3.7 with $(X_n)_n$ a Markov process and that*

$$\mathbb{E}\left[e^{-\nu\xi(X_0)}\tilde{C}_0(X_0)\right] + \mathbb{E}\left[e^{-(\nu-\delta)\xi(X_0)}\Psi_0(X_0)\right] + \sum_{k=0}^\infty \kappa_1^k \mathbb{E}\left[\tilde{C}_0^2(X_k)e^{-2\nu S_k}\right] < \infty,$$

$$\text{and} \qquad \sum_{k=0}^\infty \kappa_1^k \mathbb{E}\left[\tilde{C}_0(X_k)\Psi_0(X_k)e^{-(2\nu-\delta)S_k} + \Psi_0^2(X_k)e^{-2(\nu-\delta)S_k}\right] < \infty,$$

$$\sum_{k=0}^\infty \kappa_1^k \mathbb{E}\left[(\tilde{C}_0(X_k) + \Psi_0(X_k) + k)e^{-(\nu+\delta)S_k}\right] < \infty, \quad (21)$$

then there exists a square integrable random variable W such that $e^{-\nu t}N_t$ converges in quadratic mean to W as t tends to infinity, with

$$\mathbb{E}[W] = \kappa_1 \mathbb{E}\left[e^{-\nu\xi(X_0)}\tilde{C}_0(X_0)\right]$$

$$Var(W) = \kappa_2 \sum_{k=0}^{\infty} \kappa_1^k \mathbb{E}\left[\tilde{C}_0^2(X_k)e^{-2\nu S_k}\right] - \kappa_1^2\left(\mathbb{E}\left[e^{-\nu\xi(X_0)}\tilde{C}_0(X_0)\right]\right)^2.$$

If, moreover, $W > 0$ almost surely then $e^{-\nu t}N_t$ converges almost surely to W.

The rest of this subsection is devoted to the proof of Theorem 3.8.

Define, for $r \le m$, the partial sums $S_{r,m} := \sum_{i=r}^{m} \xi(X_i)$ $S_{r,m}^{(k)} := \sum_{i=r}^{m} \xi(X_i^{(k)})$ (so that $S_n = S_{0,n}$ and $S_n^{(k)} = S_{0,n}^{(k)}$, with the usual convention, for $r > m$, $S_{r,m}^{(k)} = S_{r,m} = 0$.

The proof of the next lemma is an immediate consequence of Lemma 2.2 in [21] and we omit it.

Lemma 3.9. *Assume Hypothesis 3.7. Then, for any $t > 0$,*

$$\left| e^{-\nu t}\sum_{n\ge 0}\kappa_1^n(\kappa_1 - 1)\mathbb{P}(S_{1,n+1} \le t | X_0 = x) - \tilde{C}_0(x)\right| \le \frac{\Psi_0(x)}{2\pi}e^{-\delta t}. \tag{22}$$

Lemma 3.9 together with Proposition 2.2 allows us to give an exact asymptotic behavior of $\mathbb{E}[N_t]$ as t tends to infinity, as shows the following proposition.

Proposition 3.10. *Assume Hypotheses 3.4 and 3.7 are satisfied for some positive δ strictly less than ν. If $\mathbb{E}[e^{-\nu\xi(X_0)}\tilde{C}_0(X_0)] < \infty$ and if $\mathbb{E}(e^{-(\nu-\delta)\xi(X_0)}\Psi_0(X_0)) < \infty$, then $\mathbb{E}[N_t] < \infty$ and there exists $\epsilon_1 > 0$ such that*

$$\mathbb{E}[N_t] = e^{\nu t}\kappa_1\mathbb{E}[e^{-\nu\xi(X_0)}\tilde{C}_0(X_0)](1 + O(e^{-\epsilon_1 t})),\ \textit{as } t \to \infty\,.$$

Proof. Recall that $f_{x,0}(u) = e^{-\nu u}\sum_{n\ge1}\kappa_1^{n-1}(\kappa_1 - 1)\mathbb{P}(S_{1,n} \le u | X_0 = x)$. Moreover, due to Remark 3.5,

$$\begin{aligned}
\mathbb{E}[N_t] &= \sum_{n\ge1}\kappa_1^n(\kappa_1-1)\mathbb{P}(S_n \le t)\\
&= \mathbb{E}\left[\sum_{n\ge1}\kappa_1^n(\kappa_1-1)\mathbb{P}(S_{1,n} \le t - \xi(X_0)|X_0)\mathbf{1}_{\{\xi(X_0)\le t\}}\right]\\
&= e^{\nu t}\kappa_1\mathbb{E}\left[e^{-\nu\xi(X_0)}f_{X_0,0}(t-\xi(X_0))\mathbf{1}_{\{\xi(X_0)\le t\}}\right]\\
&= e^{\nu t}\kappa_1\mathbb{E}\left[e^{-\nu\xi(X_0)}\left(f_{X_0,0}(t-\xi(X_0)) - \tilde{C}_0(X_0)\right)\mathbf{1}_{\{\xi(X_0)\le t\}}\right]\\
&\quad + e^{\nu t}\kappa_1\mathbb{E}\left[e^{-\nu\xi(X_0)}\tilde{C}_0(X_0)\mathbf{1}_{\{\xi(X_0)\le t\}}\right].
\end{aligned}$$

Now Lemma 3.9 gives,

$$\mathbb{E}\left[e^{-\nu\xi(X_0)}\left(f_{X_0,0}(t-\xi(X_0)) - \tilde{C}_0(X_0)\right)\mathbf{1}_{\{\xi(X_0)\le t\}}\right] \le \frac{e^{-\delta t}}{2\pi}\mathbb{E}\left[\Psi_0(X_0)e^{-(\nu-\delta)\xi(X_0)}\right].$$

and so

$$\mathbb{E}[N_t] \;=\; e^{\nu t}\kappa_1\left[\mathbb{E}\left(e^{-\nu\xi(X_0)}\tilde{C}_0(X_0)\mathbf{1}_{\{\xi(X_0)\le t\}}\right)+e^{-\delta t}D(t)\right] \tag{23}$$

with $0<D(t)\le(2\pi)^{-1}\mathbb{E}\left(\Psi_0(X_0)e^{-(\nu-\delta)\xi(X_0)}\right)$. This allows to deduce thanks to Proposition 2.2 that $\mathbb{E}[N_t]<\infty$ and

$$\mathbb{E}[N_t]=e^{\nu t}\kappa_1\mathbb{E}[e^{-\nu\xi(X_0)}\tilde{C}_0(X_0)\mathbf{1}_{\{\xi(X_0)\le t\}}]\left[1+e^{-\nu t}A(t)+e^{-\delta t}D(t)\right],$$

with $0<A(t)<\sup_{t>0}A(t)<\infty, 0<D(t)<\sup_{t>0}D(t)<\infty$. □

The following proposition gives an exact asymptotic behavior of $\mathbb{E}[N_tN_{t+\tau}]$ under further assumptions.

Proposition 3.11. *Assume Hypotheses 3.4 and 3.7 are satisfied with $(X_n)_n$ a Markov process. Suppose also that $\sum_{k=0}^{\infty}\kappa_1^k\mathbb{E}[\tilde{C}_0^2(X_k)e^{-2\nu S_k}]<\infty$. If Condition (21) is satisfied, then, for any $t>0$, $\tau\ge0$, $\mathbb{E}[N_tN_{t+\tau}]<\infty$ and*

$$\mathbb{E}[N_tN_{t+\tau}]=e^{\nu(2t+\tau)}\kappa_2\sum_{k=0}^{\infty}\kappa_1^k\mathbb{E}\left[\tilde{C}_0^2(X_k)e^{-2\nu S_k}\right](1+ae^{-\epsilon_1 t}),\ \textit{as } t\to\infty,$$

where a and ϵ_1 are positive constants independent of t and τ.

Proof. The task is to apply Proposition 3.2. For $0\le k\le\min(n,m)-1$, we have (recall that $S_{m-1}^{(k)}=\sum_{i=0}^{m-1}\xi(X_i^{(k)})$)

$$\mathbb{E}\left[A_{n,m,k}\mathbf{1}_{\{S_{n-1}\le t,\ S_{m-1}^{(k)}\le t+\tau\}}\right]=\mathbb{E}\left[A_{n,m,k}\right]\mathbb{P}\left(S_{n-1}\le t,\ S_{m-1}^{(k)}\le t+\tau\right).$$

Observe that

$$\mathbb{E}\left[A_{n,m,k}\right]=\begin{cases}\mathbb{E}[(\kappa(X_0)-1)^2]\kappa_1^{n-1} & \text{if } k=n-1=m-1\\ \kappa_2(\kappa_1-1)\kappa_1^{\max(n-2,m-2)} & \text{if } (k=m-1 \textit{ ou } k=n-1),\ n\ne m\\ \kappa_2(\kappa_1-1)^2\kappa_1^{n-2+m-2-k} & \text{if } (k\ne m-1 \textit{ et } k\ne n-1).\end{cases}$$

Now we have for $k\ne m-1$,

$$\begin{aligned}&\mathbb{P}\left(S_{n-1}\le t,\ S_{m-1}^{(k)}\le t+\tau\right)=\\&\mathbb{E}\left[\mathbb{P}\left(S_{k+1,n-1}\le t-S_k,\ \ S_{k+1,m-1}^{(k)}\le t+\tau-S_k|S_k,X_k\right)\mathbf{1}_{\{S_k\le t\}}\right]\\&=\mathbb{E}\left[\mathbb{P}\left(S_{k+1,n-1}\le t-S_k|S_k,X_k\right)\mathbb{P}\left(S_{k+1,m-1}^{(k)}\le t+\tau-S_k|S_k,X_k\right)\mathbf{1}_{\{S_k\le t\}}\right].\end{aligned}$$

When $k=m-1$ (and then necessarily $m\le n$ and $S_{m-1}\le S_{n-1}$), $S_{m-1}^{(m-1)}=\sum_{i=0}^{m-1}\xi(X_i^{(m-1)})$ and by Hypothesis 3.1, $(X_i^{(m-1)})_{1\le i\le m-1}=(X_i)_{1\le i\le m-1}$ a.s. Hence in this case $S_{m-1}^{(m-1)}=S_{m-1}$ a.s. and then

$$\mathbb{P}\left(S_{n-1}\le t,\ S_{m-1}^{(m-1)}\le t+\tau\right)=\mathbb{P}\left(S_{n-1}\le t,\ S_{m-1}\le t+\tau\right)=\mathbb{P}\left(S_{n-1}\le t\right).$$

Consequently (recall that $S^{(m-1)}_{m,m-1}=0$),

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{m=1}^{n}\mathbb{E}\left[A_{n,m,m-1}\mathbf{1}_{\{S_{n-1}\le t,\ S_{m-1}+S^{(m-1)}_{m,m-1}\le t+\tau\}}\right]\\
&=\sum_{n=1}^{\infty}\sum_{m=1}^{n-1}\kappa_1^{n-2}(\kappa_1-1)\kappa_2\mathbb{P}\left(S_{n-1}\le t\right)+\sum_{n=1}^{\infty}\kappa_1^{n-1}\mathbb{E}[(\kappa(X_0)-1)^2]\mathbb{P}\left(S_{n-1}\le t\right)\\
&=(\kappa_1-1)\kappa_2\sum_{n=1}^{\infty}(n-1)\kappa_1^{n-2}\mathbb{P}\left(S_{n-1}\le t\right)+\mathbb{E}[(\kappa(X_0)-1)^2]\sum_{n=1}^{\infty}\kappa_1^{n-1}\mathbb{P}\left(S_{n-1}\le t\right).
\end{aligned}$$

The last equality, together with the fact that

$$\mathbb{P}\left(S_{n-1}\le t\right)=\mathbb{P}\left(e^{-(\nu+\delta)S_{n-1}}\ge e^{-(\nu+\delta)t}\right)\le e^{(\nu+\delta)t}\mathbb{E}[e^{-(\nu+\delta)S_{n-1}}],$$

proves that, for any $0<\delta<\nu$

$$\begin{aligned}
&e^{-\nu(2t+\tau)}\sum_{n=1}^{\infty}\sum_{m=1}^{n}\mathbb{E}\left[A_{n,m,m-1}\mathbf{1}_{\{S_{n-1}\le t,\ S_{m-1}+S^{(m-1)}_{m,m-1}\le t+\tau\}}\right]\\
&\le e^{-t(\nu-\delta)}(\kappa_1-1)\kappa_2\sum_{n=1}^{\infty}(n-1)\kappa_1^{n-2}\mathbb{E}[e^{-(\nu+\delta)S_{n-1}}]\\
&+e^{-t(\nu-\delta)}\mathbb{E}[(\kappa(X_0)-1)^2]\sum_{n=1}^{\infty}\kappa_1^{n-1}\mathbb{E}[e^{-(\nu+\delta)S_{n-1}}]. \qquad (24)
\end{aligned}$$

We also have,

$$\begin{aligned}
&\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\sum_{k=0}^{\min(n,m)-2}\mathbb{E}\left[A_{n,m,k}\mathbf{1}_{\{S_{n-1}\le t,\,S_k+S^{(k)}_{k+1,m-1}\le t+\tau\}}\right] \qquad (25)\\
&=\kappa_2\sum_{k=0}^{\infty}\kappa_1^k\mathbb{E}\left[\sum_{n\ge k+2}\kappa_1^{n-2-k}(\kappa_1-1)\mathbb{P}\left(S_{k+1,n-1}\le t-S_k|S_k,X_k\right)\right.\\
&\left.\times\sum_{m\ge k+2}\kappa_1^{m-2-k}(\kappa_1-1)\mathbb{P}\left(S^{(k)}_{k+1,m-1}\le t+\tau-S_k|S_k,X_k\right)\mathbf{1}_{\{S_k\le t\}}\right].
\end{aligned}$$

Now the bound (22) gives, letting $a_n(k)=\kappa_1^{n-2-k}(\kappa_1-1)$,

$$\begin{aligned}
&|e^{-\nu(t-\sum_{i=0}^k\xi(x_i))}\sum_{n\ge k+2}a_n(k)\mathbb{P}(S_{k+1,n-1}\le t-\sum_{i=0}^k\xi(x_i)|X_k=x_k)-\tilde{C}_0(x_k)|\\
&\le\Psi_0(x_k)e^{-\delta(t-\sum_{i=0}^k\xi(x_i))},
\end{aligned}$$

and

$$\begin{aligned}
&|e^{-\nu(t+\tau-\sum_{i=0}^k\xi(x_i))}\sum_{m\ge k+2}a_m(k)\mathbb{P}(S^{(k)}_{k+1,m-1}\le t+\tau-\sum_{i=0}^k\xi(x_i)|X_k=x_k)-\tilde{C}_0(x_k)|\\
&\le\Psi_0(x_k)e^{-\delta(t+\tau-\sum_{i=0}^k\xi(x_i))}.
\end{aligned}$$

The two last bounds together with (25) give then

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\sum_{k=0}^{\min(n,m)-2}\mathbb{E}\left[A_{n,m,k}\mathbf{1}_{\{S_{n-1}\le t,\ S_k+S^{(k)}_{k+1,m-1}\le t+\tau\}}\right]$$
$$= e^{\nu(2t+\tau)}\kappa_2\left[\sum_{k=0}^{\infty}\kappa_1^k\mathbb{E}(\tilde{C}_0^2(X_k)e^{-2\nu S_k}\mathbf{1}_{S_k\le t})+e^{-\delta t}a\right], \tag{26}$$

where $a\le 2\sum_{k=0}^{\infty}\kappa_1^k\mathbb{E}[\tilde{C}_0(X_k)\Psi_0(X_k)e^{-(2\nu-\delta)S_k}]+e^{-\delta t}\sum_{k=0}^{\infty}\kappa_1^k\mathbb{E}[\Psi_0^2(X_k)e^{-2(\nu-\delta)S_k}]$.
Finally,

$$\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}\mathbb{E}\left[A_{n,m,n-1}\mathbf{1}_{\{S_{n-1}\le t,\ S_{n-1}+S^{(k)}_{n,m-1}\le t+\tau\}}\right] \tag{27}$$
$$=\kappa_2(\kappa_1-1)\sum_{n=1}^{\infty}\mathbb{E}\left[\sum_{m=n+1}^{\infty}\kappa_1^{m-2}\mathbb{P}\left(S^{(n-1)}_{n,m-1}\le t+\tau-S_{n-1}|S_{n-1},X_{n-1}\right)\mathbf{1}_{\{S_{n-1}\le t\}}\right]$$
$$=\kappa_2(\kappa_1-1)\mathbb{E}\left[\sum_{n=1}^{\infty}\kappa_1^{n-1}\mathbf{1}_{\{S_{n-1}\le t\}}\sum_{m=n+1}^{\infty}\kappa_1^{m-n-1}\mathbb{P}\left(S^{(n-1)}_{n,m-1}\le t+\tau-S_{n-1}|S_{n-1},X_{n-1}\right)\right].$$

Now the bound (22) gives, letting $b_n(m)=(\kappa_1-1)\kappa_1^{m-n-1}$,

$$|e^{-\nu(t+\tau-\sum_{i=0}^{n-1}\xi(x_i))}\sum_{m\ge n+1}b_n(m)\mathbb{P}(S^{(n-1)}_{n,m-1}\le t+\tau-\sum_{i=0}^{n-1}\xi(x_i)|X_{n-1}=x_{n-1})-\tilde{C}_0(x_{n-1})|$$
$$\le \Psi_0(x_{n-1})e^{-\delta(t+\tau-\sum_{i=0}^{n-1}\xi(x_i))}. \tag{28}$$

We have also,

$$e^{-\nu t}\sum_{n\ge1}\kappa_1^{n-1}\mathbb{E}\left(\tilde{C}_0(X_{n-1})e^{-\nu S_{n-1}}\mathbf{1}_{\{S_{n-1}\le t\}}\right)\le e^{-(\nu-\delta)t}\sum_{n\ge1}\kappa_1^{n-1}\mathbb{E}\left[\tilde{C}_0(X_{n-1})e^{-(\nu+\delta)S_{n-1}}\right], \tag{29}$$

and

$$e^{-\nu t}\mathbb{E}\left[\sum_{n\ge1}\kappa_1^{n-1}\mathbf{1}_{\{S_{n-1}\le t\}}\Psi_0(X_{n-1})e^{-\nu S_{n-1}}e^{-\delta(t+\tau-S_{n-1})}\right]$$
$$\le e^{-(\nu+\delta)t}\sum_{n\ge1}\kappa_1^{n-1}\mathbb{E}\left(\mathbf{1}_{\{S_{n-1}\le t\}}\Psi_0(X_{n-1})e^{-(\nu-\delta)S_{n-1}}\right)$$
$$\le e^{-(\nu-\delta)t}\sum_{n\ge1}\kappa_1^{n-1}\mathbb{E}\left[\Psi_0(X_{n-1})e^{-(\nu+\delta)S_{n-1}}\right] \tag{30}$$

We get collecting Inequalities (27), (28), (29) and (30)

$$e^{-\nu(2t+\tau)}\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}\mathbb{E}\left[A_{n,m,n-1}\mathbf{1}_{\{S_{n-1}\le t,\ S_{n-1}+S^{(k)}_{n,m-1}\le t+\tau\}}\right]$$
$$\le e^{-(\nu-\delta)t}\kappa_2\sum_{n\ge1}\kappa_1^{n-1}\mathbb{E}\left((\tilde{C}_0(X_{n-1})+\Psi_0(X_{n-1}))e^{-(\nu+\delta)S_{n-1}}\right). \tag{31}$$

We then obtain combining (24), (26) and (31),

$$e^{-\nu(2t+\tau)} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{k=0}^{\min(n,m)-1} \mathbb{E}\left[A_{n,m,k} \mathbf{1}_{\{S_{n-1} \le t,\ S_k + S_{k+1,m-1}^{(k)} \le t+\tau\}}\right]$$
$$= \kappa_2 \sum_{k=0}^{\infty} \kappa_1^k \mathbb{E}[\tilde{C}_0^2(X_k) e^{-2\nu S_k} \mathbf{1}_{\{S_k \le t\}}] + e^{-(\nu-\delta)t} A + e^{-\delta t} \kappa_2 a, \tag{32}$$

where a is as defined in (26) and

$$A \le \kappa_2 \sum_{n \ge 1} \kappa_1^{n-1} \mathbb{E}\left[(\tilde{C}_0(X_{n-1}) + \Psi_0(X_{n-1})) e^{-(\nu+\delta)S_{n-1}}\right]$$
$$+(\kappa_1 - 1)\kappa_2 \sum_{n=1}^{\infty} (n-1)\kappa_1^{n-2} \mathbb{E}[e^{-(\nu+\delta)S_{n-1}}] + \mathbb{E}[(\kappa(X_0) - 1)^2] \sum_{n=1}^{\infty} \kappa_1^{n-1} \mathbb{E}[e^{-(\nu+\delta)S_{n-1}}].$$

Consequently, we obtain collecting (32) together with (17) and Proposition 3.10, that there exists $\epsilon_1 > 0$, such that, for any $t, \tau \ge 0$,

$$\mathbb{E}[N_t N_{t+\tau}] = e^{\nu(2t+\tau)} \kappa_2 \sum_{k=0}^{\infty} \kappa_1^k \mathbb{E}[\tilde{C}_0^2(X_k) e^{-2\nu S_k}](1 + c_{t,\tau} e^{-\epsilon_1 t}), \text{ as } t \to \infty,$$

where $\sup_{t,\tau} c_{t,\tau} < \infty$. □

Proof of Theorem 3.8. The bound (32) proves that the convergence in (18) is exponentially fast and satisfied with

$$K = \kappa_2 \sum_{k=0}^{\infty} \kappa_1^k \mathbb{E}[\tilde{C}_0^2(X_k) e^{-2\nu S_k}].$$

Proposition 3.10 ensures that $\limsup_{t\to\infty} e^{-\nu t} \mathbb{E}[N_t] < \infty$. Hence, due to Corollary 3.3, as $t \to +\infty$, $(e^{-\nu t} N_t)_t$ converges in quadratic mean to some random variable W. Therefore $\mathbb{E}[W] = \lim_{t\to\infty} \mathbb{E}[e^{-\nu t} N_t)]$ and $\mathbb{E}[W^2] = \lim_{t\to\infty} \mathbb{E}[e^{-2\nu t} N_t^2]$ and so, due to respectively Propositions 3.10 and 3.11 (with $\tau = t$),

$$\mathbb{E}[W] = \kappa_1 \mathbb{E}[e^{-\nu \xi(X_0)} \tilde{C}_0(X_0)] \quad \text{and} \quad \mathbb{E}[W^2] = \kappa_2 \sum_{k=0}^{\infty} \kappa_1^k \mathbb{E}[\tilde{C}_0^2(X_k) e^{-2\nu S_k}].$$

□

3.3. **About Hypothesis 3.7.** The purpose of this section is to discuss Hypothesis 3.7 yielding to the key bound (22). We assume, along this subsection, Hypothesis 3.4 with $(X_n)_n$ a Markov process.

3.3.1. *The i.i.d. case.* If moreover the lifetimes are i.i.d., then the growth rate ν as defined in (19) is also that defined in (1), that is $\kappa_1 \mathbb{E}\left[e^{-\nu \xi(X_0)}\right] = 1$. The following lemma gives additional assumptions ensuring (22) in this i.i.d. case. Although this case is classical, its study is important to make comparison with previous results (obtained with other methods of proofs). Also proofs for more general results will be obtained in the spirit of this classical case, see the paragraph below.

Lemma 3.12 (i.i.d. case). *Assume Hypothesis 3.4, that $(\xi(X_n))_n$ is a sequence of i.i.d. random variables admitting a density g with moment of order p for some $p>1$. If, for some $M>0$, $\nu>\delta>0$,*

$$\inf_{y\in\mathbb{R}}\left|1-\kappa_1\mathbb{E}\left[e^{-(-\delta+\nu+iy)\xi(X_0)}\right]\right|>0 \quad \textit{and} \quad \inf_{|y|\geq M}\inf_{u\in[-\delta,\delta]}|1-\kappa_1\mathbb{E}\left[e^{-(u+\nu+iy)\xi(X_0)}\right]|>0$$

then Hypothesis 3.7 is satisfied for any $x\in\mathbb{X}$, with

$$\tilde{C}_0(x)=\frac{\kappa_1-1}{\kappa_1^2\nu}\left(\mathbb{E}\left[\xi(X_0)e^{-\nu\xi(X_0)}\right]\right)^{-1},$$

$$\Psi_0(x)\leq Const.\left(\inf_{y\in\mathbb{R}}\left|1-\kappa_1\mathbb{E}\left[e^{-(-\delta+\nu+iy)\xi(X_0)}\right]\right|\right)^{-1}\left[\int_{-\infty}^{+\infty}g^p(u)du\right]^{1/(p-1)}.$$

Remark 3.13. *Note that if $\xi(X_0)$ is exponentially distributed with parameter λ, then for any $\delta>0$ sufficiently small,*

$$\nu=\lambda(\kappa_1-1)$$
$$\inf_{y\in\mathbb{R}}\left|1-\kappa_1\mathbb{E}\left[e^{-(-\delta+\nu+iy)\xi(X_0)}\right]\right|>0$$
$$\inf_{|y|\geq M}\inf_{u\in[-\delta,\delta]}\left|1-\kappa_1\mathbb{E}\left[e^{-(u+\nu+iy)\xi(X_0)}\right]\right|\geq M\lambda\kappa_1(\lambda+\delta)^{-1}((\delta+\lambda+\nu)^2+M^2)^{-1/2}>0$$

Remark 3.14. *Due to Lemma 3.12, Propositions 3.10 and 3.11 are respectively Theorem 17.2 and Lemma 18.1 in* [12] *(m, $h''(1)$ and n_1 there being respectively κ_1, κ_2 and the constant function $\tilde{C}_0$).*

Proof of Lemma 3.12. The task is to check the conditions on the Laplace transform $\tilde{f}_{x,0}$ (as calculated in (20)). In this i.i.d. case, we have, for $\gamma=s+iy$ and $s>0$,

$$\tilde{f}_{x,0}(\gamma)=\frac{\kappa_1-1}{\gamma+\nu}\frac{\mathbb{E}[e^{-(\gamma+\nu)\xi(X_0)}]}{1-\kappa_1\mathbb{E}[e^{-(\gamma+\nu)\xi(X_0)}]}.$$

The right hand side of the last equality is analytic in a sufficiently narrow strip $\{z=u+iy,|u|\leq\delta+\epsilon,y\in\mathbb{R}\}\setminus\{0\}$ for $\delta+\epsilon<\nu$. It has a simple pole at 0 with residue $\tilde{C}_0(x)$, since

$$\lim_{z\to 0}z\tilde{f}_{x,0}(z)=\frac{\kappa_1-1}{\kappa_1^2\nu}\left(\mathbb{E}\left[\xi(X_0)e^{-\nu\xi(X_0)}\right]\right)^{-1}=\tilde{C}_0(x).$$

We have, since $|1-\kappa_1\mathbb{E}[e^{-(\delta+\nu+iy)\xi(X_0)}]|$ is bounded below by a strictly positive constant (this follows from $\sup_{y\in\mathbb{R}}|\mathbb{E}[e^{-(\delta+\nu+iy)\xi(X_0)}]|<\mathbb{E}[e^{-\nu\xi(X_0)}]=1/\kappa_1$), and letting $p'=p/(p-1)$,

$$\int_{-\infty}^{+\infty}|\tilde{f}_{x,0}(\delta+iy)|dy\leq\mathrm{Cst}\int_{-\infty}^{+\infty}\frac{|\mathbb{E}[e^{-(\delta+iy+\nu)\xi(X_0)}]|}{\sqrt{(\nu+\delta)^2+y^2}}dy$$
$$\leq\mathrm{Cst}\left(\int_{-\infty}^{+\infty}|\mathbb{E}[e^{-(\delta+iy+\nu)\xi(X_0)}]|^{p'}dy\right)^{1/p'}\left(\int_{-\infty}^{+\infty}((\nu+\delta)^2+y^2)^{-p/2}dy\right)^{1/p}.$$

Now we have, arguing as for the proof of Lemma 3 in Harris (1963) page 163,

$$\int_{-\infty}^{+\infty}|\mathbb{E}[e^{-(\delta+iy+\nu)\xi(X_0)}]|^{p'}dy\leq\tilde{C}_p\left[\int_{-\infty}^{+\infty}e^{-p(\delta+\nu)t}g^p(t)dt\right]^{1/(p-1)}.$$

Consequently $\int_{-\infty}^{+\infty} |\tilde{f}_{x,0}(\delta+iy)|dy < \infty$ since the density g is supposed to be in $\mathcal{L}^p$. Using the same arguments, we prove that $\int_{-\infty}^{+\infty} |\tilde{f}_{x,0}(-\delta+iy)|dy < \infty$, in fact,

$$\int_{-\infty}^{+\infty} |\tilde{f}_{x,0}(-\delta+iy)|dy$$
$$\leq \kappa_1 \tilde{C}_p \left(\inf_{y\in\mathbb{R}} \left|1-\kappa_1\mathbb{E}[e^{-(-\delta+\nu+iy)\xi(X_0)}]\right|\right)^{-1} \left[\int_{-\infty}^{+\infty} g^p(t)dt\right]^{1/p} \left(\int_{-\infty}^{+\infty} ((\nu-\delta)^2+y^2)^{-p/2}dy\right)^{1/p},$$

which is finite by the requirements of the lemma. Now, we have for any $u \in [-\delta,\delta]$ and for any $|y| > M > 0$,

$$|\tilde{f}_{x,0}(u+iy)| \leq (\kappa_1-1)((\nu+u)^2+y^2)^{-1/2} \left|1-\kappa_1\mathbb{E}[e^{-(u+\nu+iy)\xi(X_0)}]\right|^{-1}$$
$$\leq \frac{\kappa_1}{|y|}\left(\inf_{|y|\geq M}\inf_{u\in[-\delta,\delta]}\left|1-\kappa_1\mathbb{E}[e^{-(u+\nu+iy)\xi(X_0)}]\right|\right)^{-1}$$

this proves that, for all $M > 0$, $\sup_{|y|>M}\sup_{u\in[-\delta,\delta]}\left|\tilde{f}_{x,0}(u+iy)\right| < \infty$ and then $\lim_{y\to\pm\infty}\tilde{f}_{x,0}(u+iy) = 0$, uniformly in $u \in [-\delta,\delta]$. Hypothesis (3.7) is then satisfied. □

3.3.2. *Multiplicative ergodic case.* The purpose of this paragraph is to prove that Hypothesis 3.7 can also be satisfied by Markov chains having multiplicative ergodic sums (cf. Definition 3.3 and Section 3 in [21]), the multiplicative ergodic property implying that, for any $\gamma > 0$, any $x \in \mathbb{X}$ and any $n \in \mathbb{N}$ (recall that, $S_{1,n+1} = \sum_{i=1}^{n+1}\xi(X_i)$)

$$\mathbb{E}\left[e^{-\gamma S_{1,n+1}}|X_0 = x\right] = \alpha(\gamma,x)L^{n+1}(\gamma) + r_{n+1}(\gamma,x) \quad (33)$$

for suitable non-negative functions α, L and $(r_n)_n$. We suppose here that,

(a) For all $x \in \mathbb{X}$, the functions $\alpha(\cdot,x)$, L and $r_n(\cdot,x)$ can be extended to analytic functions in $\{z = u+iy, |u| \leq \delta+\epsilon < \nu, y \in \mathbb{R}\}$
(b) L is positive and non-increasing on $\mathbb{R}_+^*$. The equation $\kappa_1 L(z) = 1$, has a unique positive solution in $\mathbb{C}$, denoted by ν.
(c) The mapping L is holomorphic at ν and $L'(\nu) < 0$.
(d) The series $\sum_{n>0}\kappa_1^n r_n(\gamma,x)$ converges uniformly in γ in a neighborhood of ν uniformly in x.
(e) There exists $p' > 1$ such that $\int_{-\infty}^{\infty}|\alpha(\pm\delta+\nu+iy,x)L(\pm\delta+\nu+iy)|^{p'}dy < \infty$ and that $\int_{-\infty}^{\infty}|\sum_{n\geq 0}\kappa_1^n r_n(\pm\delta+\nu+iy,x)|^{p'}dy < \infty$.
(f) $\inf_{y\in\mathbb{R}}|1-\kappa_1 L(\pm\delta+iy+\nu)| > 0$, $\infty > \sup_{|y|>M}\sup_{u\in[-\delta,\delta]}|\alpha(u+iy+\nu,x)L(u+iy+\nu)| > 0$, $\infty > \sup_{|y|>M}\sup_{u\in[-\delta,\delta]}\sum_{n\geq 0}\kappa_1^n|r_n(u+iy+\nu,x)| > 0$ and $\inf_{|y|>M}\inf_{u\in[-\delta,\delta]}|1-\kappa_1 L(u+iy+\nu)| > 0$, for some $M > 0$.

Remark 3.15. *Assume 3.4 and 3.7 with $(X_n)_n$ a Markov process, the multiplicative ergodic property stated in (33) ensures (10), in fact, in this case*

$$g_n(\gamma) = (\kappa_1-1)\kappa_1^n L^n(\gamma)\mathbb{E}[\alpha(\gamma,X_0)] + (\kappa_1-1)\kappa_1^n\mathbb{E}[r_n(\gamma,X_0)],\ \textit{for any } \gamma > 0.$$

Lemma 3.16. *Hypothesis 3.7 is satisfied under Conditions (a)-$\cdots$-(f) with*

$$\tilde C_0(x) = -\frac{(\kappa_1-1)}{\kappa_1^2\nu L'(\nu)}\alpha(\nu,x)$$

$$\Psi_0(x) \le \kappa_1\left(\inf_{y\in\mathbb{R}}|1-\kappa_1 L(-\delta+iy+\nu,x)|\right)^{-1}\times$$

$$\left(\int_{-\infty}^{\infty}|\alpha(-\delta+\nu+iy,x)L(-\delta+\nu+iy)|^{p'}dy\right)^{1/p'}\left(\int_{-\infty}^{+\infty}((\nu-\delta)^2+y^2)^{-p/2}dy\right)^{1/p}$$

$$+\kappa_1\left(\int_{-\infty}^{\infty}|\sum_{n\ge0}\kappa_1^n r_{n+1}(-\delta+\nu+iy,x)|^{p'}dy\right)^{1/p'}\left(\int_{-\infty}^{+\infty}((\nu-\delta)^2+y^2)^{-p/2}dy\right)^{1/p}$$

for any $x\in\mathbb{X}$.

Proof of Lemma 3.16. We have, in this case,

$$\tilde f_{x,0}(\gamma) = \frac{\kappa_1-1}{\gamma+\nu}\sum_{n\ge0}\kappa_1^n\mathbb{E}[e^{-(\gamma+\nu)S_{1,n+1}}|X_0=x]$$
$$= \frac{\kappa_1-1}{\gamma+\nu}\frac{\alpha(\gamma+\nu,x)L(\gamma+\nu)}{1-\kappa_1L(\gamma+\nu)}+\frac{\kappa_1-1}{\gamma+\nu}\sum_{n\ge0}\kappa_1^n r_{n+1}(\gamma+\nu,x),$$

which can be extended thanks to Conditions (a) and (d) to an analytic function in $\{z = u+iy, |u|\le\delta+\epsilon<\nu, y\in\mathbb{R}\}\setminus\{0\}$. Conditions (a), (b), (c) and (d) allow to deduce that

$$\lim_{z\to0} z\tilde f_{x,0}(z) = -\frac{(\kappa_1-1)}{\kappa_1^2\nu L'(\nu)}\alpha(\nu,x) =: \tilde C_0(x).$$

We have arguing as for the proof of Lemma 3.12, for any $|y|>M$ and any $u\in[-\delta,\delta]$,

$$|\tilde f_{x,0}(u+iy)| \le \frac{\kappa_1}{|y|}\frac{\sup_{|y|>M}\sup_{u\in[-\delta,\delta]}|\alpha(u+iy+\nu,x)L(u+iy+\nu)|}{\inf_{|y|>M}\inf_{u\in[-\delta,\delta]}|1-\kappa_1L(u+iy+\nu)|}$$
$$+\frac{\kappa_1}{|y|}\sup_{|y|>M}\sup_{u\in[-\delta,\delta]}\sum_{n\ge0}\kappa_1^n|r_{n+1}(u+iy+\nu,x)|,$$

which proves, thanks to (f), that $\lim_{y\to\pm\infty}\sup_{u\in[-\delta,\delta]}|\tilde f_{x,0}(u+iy)|=0$. Now,

$$\int|\tilde f_{x,0}(\pm\delta+iy)|dy \le \kappa_1\left(\inf_{y\in\mathbb{R}}|1-\kappa_1L(\pm\delta+iy+\nu)|\right)^{-1}\times$$

$$\left(\int_{-\infty}^{\infty}|\alpha(\pm\delta+\nu+iy,x)L(\pm\delta+\nu+iy)|^{p'}dy\right)^{1/p'}\left(\int_{-\infty}^{+\infty}((\nu\pm\delta)^2+y^2)^{-p/2}dy\right)^{1/p}$$

$$+\kappa_1\left(\int_{-\infty}^{\infty}|\sum_{n\ge0}\kappa_1^n r_{n+1}(\pm\delta+\nu+iy,x)|^{p'}dy\right)^{1/p'}\left(\int_{-\infty}^{+\infty}((\nu\pm\delta)^2+y^2)^{-p/2}dy\right)^{1/p},$$

which is finite thanks to Conditions (e) and (f). Hypothesis (3.7) is then satisfied. □

Remark 3.17. *In* [3]*, the authors consider the classical continuous time Galton-Watson tree where each branch lives during an independent exponential time and splits into a random number of new branches given by independent random variables. More precisely, let N_t^* be* *the size of the living population V_t at time t and let $(X_t)_{t\geq 0}$ be a Markov chain indexed by this continuous time Galton-Watson tree. The authors focus on the following probability measure,*

$$\frac{I_{N_t^*>0}}{N_t^*}\sum_{u\in V_t}\delta_{X_t^u}(dx).$$

Their study uses the a.s. limiting behavior of N_t^*. Since the purpose of our paper is to study this a.s. limiting behavior for a general Galton-Watson tree, one may wonder if it is possible to generalize the result of [3] *to a general Galton-Watson tree (i.e. without assuming independence between the branch lifetimes and the number of the new branches).*

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