A picture of the ODE's flow in the torus: from everywhere or almost-everywhere asymptotics to homogenization of transport equations

Marc Briane & Loïc Hervé

Univ Rennes, INSA Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France

mbriane@insa-rennes.fr & loic.herve@insa-rennes.fr

Contents

Abstract

In this paper, we study various aspects of the ODE's flow induced by a periodic vector field b in the torus. We present a picture of all logical connections between: the everywhere or the a.e. asymptotics of the flow, the rectification of b, the ergodicity of the flow, the unit set condition for Herman's rotation set, the unit set condition for the set D_b composed of the means of b related to the invariant measures being absolutely continuous with respect to Lebesgue's measure, the homogenization of the linear transport equation with oscillating velocity $b(x/\varepsilon)$. The main result of the paper is that the a.e. asymptotics of the flow, the unit set condition for D_b and the homogenization of the transport equation with divergence free b, are equivalent. Extending the two-dimensional results on Stepanoff flow to any dimension, we show that the flow may be ergodic without satisfying the everywhere asymptotics.

Keywords: ODE's flow, transport equation, asymptotics of the flow, homogenization, rectification of fields, invariant measure, rotation set, ergodic

Mathematics Subject Classification: 34E10, 35B27, 37C10, 37C40

1 Introduction

In this paper we study various aspects of the ODE's flow X in the torus Y_d

$$\begin{cases} \dfrac{\partial X}{\partial t}(t,x) = b(X(t,x)), & t \in \mathbb{R} \\ X(0,x) = x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where b is a $\mathbb{Z}^d$-periodic vector field in $C^1(\mathbb{R}^d)^d$ (denoted by $b \in C^1_\sharp(Y_d)^d$), which completely determines the flow X.

First, we are interested in the asymptotics of the flow X depending on whether it can hold almost-everywhere (a.e.), or everywhere (e.) in Y_d, namely

$$\exists \lim_{t\to\infty} \frac{X(t,x)}{t} \quad \mu\text{-a.e. } x \in Y_d \quad \text{or} \quad \exists \lim_{t\to\infty} \frac{X(t,x)}{t} \quad \forall\, x \in Y_d,$$

for some probability measure μ on Y_d. If the flow X is ergodic with respect to some invariant probability measure μ, *i.e.* that μ agrees with its image measure $\mu_{X(t,\cdot)}$ for any $t \in \mathbb{R}$ (see Section 1.2 below), then Birkhoff's theorem (see, *e.g.*, [9, Theorem 1, Section 2, Chapter 1]) ensures that

$$\lim_{t\to\infty} \frac{X(t,x)}{t} = \lim_{t\to\infty} \left(\frac{1}{t} \int_0^t b(X(s,x))\,ds \right) = \int_{Y_d} b(y)\,d\mu(y) \quad \mu\text{-a.e. } x \in Y_d.$$

The non empty set $\mathscr{I}_b$ composed of the invariant probability measures for the flow X plays a fondamental role in ergodic theory. Associated with the set $\mathscr{I}_b$, the rotations sets of [18] are strongly connected to the asymptotic behavior of the flow. In particular, the compact convex Herman rotation set [13] defined by

$$\mathsf{C}_b := \left\{ \int_{Y_d} b(y)\,d\mu(y) : \mu \in \mathscr{I}_b \right\} \tag{1.2}$$

characterizes the everywhere asymptotics of the flow, since by [8, Proposition 2.1] we have for any $\zeta \in \mathbb{R}^d$,

$$\mathsf{C}_b = \{\zeta\} \quad \Leftrightarrow \quad \forall\, x \in Y_d, \ \lim_{t\to\infty} \frac{X(t,x)}{t} = \zeta. \tag{1.3}$$

We also consider the subset D_b of C_b defined by

$$\mathsf{D}_b := \left\{ \int_{Y_d} b(y)\,\sigma(y)\,dy : \sigma \in L^1_\sharp(Y_d) \text{ and } \sigma(y)\,dy \in \mathscr{I}_b \right\}, \tag{1.4}$$

which is *a priori* less interesting than Herman's rotation set, since it may be empty and it is not compact in general. Actually, the set D_b characterizes the almost-everywhere asymptotics of the flow. More precisely, assuming the existence of an a.e. positive invariant density function with respect to Lebesgue's measure, we prove that for any $\zeta \in \mathbb{R}^d$ (see Proposition 2.1),

$$\mathsf{D}_b = \{\zeta\} \quad \Leftrightarrow \quad \lim_{t\to\infty} \frac{X(t,x)}{t} = \zeta \ \text{ a.e. } x \in Y_d. \tag{1.5}$$

On the other hand, as a natural association with flow (1.1), we consider the linear transport equation with oscillating data

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t}(t,x) - b(x/\varepsilon) \cdot \nabla_x u_\varepsilon(t,x) = f(t,x,x/\varepsilon) & \text{in } (0,T) \times \mathbb{R}^d \\ u_\varepsilon(0,x) = u_0(x,x/\varepsilon) & \text{for } x \in \mathbb{R}^d, \end{cases} \tag{1.6}$$

2

where $f(t, x, y)$ and $u_0(x, y)$ are suitably regular and $\mathbb{Z}^d$-periodic functions with respect to variable y. In their famous paper [10] DiPerna and Lions showed the strong proximity between ODE's flows (1.1) and transport equations, in particular when the velocity has a good divergence. In the context of homogenization, the linear transport equation with oscillating data (1.6) as $\varepsilon \to 0$ was widely studied in the literature. Tartar [25] proved that in general homogenization of first-order equations leads to nonlocal effects. These effects were studied carefully in [2] for equation (1.6). In order to avoid any anomalous effective effect, namely to get a homogenized transport equation of same nature, it is thus necessary to assume some additional condition. Assuming that the vector field b is divergence free and the associated flow X is ergodic, Brenier [4] first obtained the weak convergence of the solution u_ε to the transport equation. Following this seminal work, the homogenization of the transport equation was obtained for instance in [11, 12, 15, 24] with various conditions, but which are all based on the ergodicity of the flow. Extending the result of [4] with ergodicity arguments, Peirone [21] proved the convergence of the solution to the two-dimensional transport equation (1.6) with $f(t, x, y) = 0$ and $u_0(x, y)$ independent of y, under the sole assumption that b is a non vanishing field in $C^1_\sharp(Y_2)^2$. More recently, the homogenization of the transport equation with $f(t, x, y) = 0$ and $u_0(x, y)$ independent of y, was derived in [6] (see [7] for a non periodic framework) under the global rectification of the vector field b, which is not an ergodic condition, *i.e.* the existence of a C^2-diffeomorphism Ψ on Y_d and of a vector $\xi \in \mathbb{R}^d$ such that

$$\forall\, y \in Y_d, \quad \nabla\Psi(y)\, b(y) = \xi. \tag{1.7}$$

This result was extended in [8] replacing the classical ergodic condition by the unit rotation set condition $\#\mathsf{C}_b = 1$, or, equivalently, the everywhere asymptotics (1.3) of the flow.

In the present paper, we prove (see Theorem 3.2) that the homogenization of the transport equation (1.6) with a divergence free velocity field, holds if, and only if, one of the equivalent conditions of (1.5) is satisfied. It is a quite new result beyond all the former results based on the sufficient conditions induced either by the ergodic condition, or by the unit Herman's rotation set condition. The proof of this result which is partly based on two-scale convergence [19, 1], clearly shows (see Remark 3.1) the difference between the ergodic approach of [15], and the present approach through the unit set condition (1.5) which turns out to be optimal.

Therefore, we establish strong connections between the three following *a priori* foreign notions: the oscillations in the transport equation (1.6), the means of b only related to the invariant measures for the flow X which are absolutely continuous with respect to Lebesgue's measure, and finally the almost-everywhere asymptotics of X. More generally, owing to this new material we do build the complete array of all logical connections between the following seven conditions (see Theorem 4.1 and Figure 1 below):

- the global rectification (1.7) of the vector field b,
- the ergodicity of the flow X (1.1) related to an invariant probability measure which is absolutely continuous with respect to Lebesgue's measure,
- the everywhere asymptotics of the flow X in (1.3),
- the almost-everywhere asymptotics of the flow X in (1.5),
- the unit set condition for Herman's rotation set C_b (1.2),
- the unit set condition for D_b (1.4),
- the homogenization of the transport equation (1.6) when b is divergence free in $\mathbb{R}^d$.

In addition, the following pairs of conditions cannot be compared in general:

- the global rectification of b and the ergodicity of X,

- the ergodicity of X and the everywhere asymptotics of X,
- the ergodicity of X and the unit set condition for C_b.

The proof of the three last items involves the Stepanoff flow [23] (see Example 4.1) in which the vector field b has a non empty finite zero set, and is parallel to a fixed direction $\xi \in \mathbb{R}^d$ with incommensurable coordinates. Using a purely ergodic approach, Oxtoby [20] and later Marchetto [17] proved that any two-dimensional flow homeomorphic to a Stepanoff flow admits a unique invariant probability measure μ for the flow which does not load the zero set of b, that μ is absolutely continuous with respect to Lebesgue's measure on Y_2, and finally that the flow is ergodic with respect to μ. Moreover, the set D_b is a unit set, but the rotation set C_b is a closed line segment of $\mathbb{R}^2$, possibly not reduced to a unit set. We extend (see Proposition 5.1) the two-dimensional results of [20, 17] on the Stepanoff flow to any dimension $d \geq 2$, thanks to a new and elementary approach based on classical tools of PDE's analysis. Finally, owing to another two-dimensional flow (see Example 5.1 and Proposition 5.4) we obtain that the set D_b may be either empty or a singleton, while the rotation set C_b is a closed line segment of $\mathbb{R}^2$ possibly not reduced to a singleton.

1.1   Notation

- $(e_1, \ldots, e_d)$ denotes the canonical basis of $\mathbb{R}^d$.

- "$\cdot$" denotes the scalar product and $|\cdot|$ the euclidian norm in $\mathbb{R}^d$.

- Y_d, $d \geq 1$, denotes the d-dimensional torus $\mathbb{R}^d/\mathbb{Z}^d$, which is identified to the unit cube $[0, 1)^d$ in $\mathbb{R}^d$.

- $C_c^k(\mathbb{R}^d)$, $k \in \mathbb{N} \cup \{\infty\}$, denotes the space of the real-valued functions in $C^k(\mathbb{R}^d)$ with compact support in $\mathbb{R}^d$.

- $C_\sharp^k(Y_d)$, $k \in \mathbb{N} \cup \{\infty\}$, denotes the space of the real-valued functions $f \in C^k(\mathbb{R}^d)$ which are $\mathbb{Z}^d$-periodic, *i.e.*
$$\forall k \in \mathbb{Z}^d, \ \forall x \in \mathbb{R}^d, \quad f(x + k) = f(x). \tag{1.8}$$

- The abbreviations "a.e." for almost everywhere, and "e." for everywhere will be used throughout the paper. The simple mention "a.e." refers to the Lebesgue measure on $\mathbb{R}^d$.

- dx or dy denotes the Lebesgue measure on $\mathbb{R}^d$.

- For a Borel measure μ on Y_d, extended by $\mathbb{Z}^d$-periodicity to a Borel measure $\tilde{\mu}$ on $\mathbb{R}^d$ (see definition (1.21) below), a $\tilde{\mu}$-measurable function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be $\mathbb{Z}^d$-periodic $\tilde{\mu}$-a.e. in $\mathbb{R}^d$, if
$$\forall k \in \mathbb{Z}^d, \quad f(\cdot + k) = f(\cdot) \ \ \tilde{\mu}\text{-a.e. on } \mathbb{R}^d. \tag{1.9}$$

- For a Borel measure μ on Y_d, $L_\sharp^p(Y_d, \mu)$, $p \geq 1$, denotes the space of the μ-measurable functions $f : Y_d \to \mathbb{R}$ such that $\int_{Y_d} |f(x)|^p \, d\mu(x) < \infty$.

- $L_\sharp^p(Y_d)$, $p \geq 1$, simply denotes the space of the Lebesgue measurable functions f in $L_{\text{loc}}^p(\mathbb{R}^d)$, which are $\mathbb{Z}^d$-periodic dx-a.e. in $\mathbb{R}^d$.

- $\mathscr{M}_{\text{loc}}(\mathbb{R}^d)$ denotes the space of the non negative Borel measures on $\mathbb{R}^d$, which are finite on any compact set of $\mathbb{R}^d$.

4

- $\mathscr{M}_\sharp(Y_d)$ denotes the space of the non negative Radon measures on Y_d, and $\mathscr{M}_p(Y_d)$ denotes the space of the probability measures on Y_d.

- $\mathscr{D}'(\mathbb{R}^d)$ denotes the space of the distributions on $\mathbb{R}^d$.

- For a Borel measure μ on Y_d and for $f \in L^1_\sharp(Y_d, \mu)$, $\mu(f)$ denotes the μ-mean of f on Y_d

$$\mu(f) := \int_{Y_d} f(y) \, d\mu(y), \tag{1.10}$$

which is simply denoted by $\overline{f}$ when μ is Lebesgue's measure. The same notation is used for a vector-valued function in $L^1_\sharp(Y_d, \mu)^d$.

- The notation $\mathscr{I}_b$ in (1.18) will be used throughout the paper.

1.2 Definitions and recalls

Let $b : \mathbb{R}^d \to \mathbb{R}^d$ be a vector-valued function in $C^1_\sharp(Y_d)^d$. Consider the dynamical system

$$\begin{cases} \dfrac{\partial X}{\partial t}(t, x) = b(X(t, x)), & t \in \mathbb{R} \\ X(0, x) = x \in \mathbb{R}^d. \end{cases} \tag{1.11}$$

The solution $X(\cdot, x)$ to (1.11) which is known to be unique (see, $e.g.$, [14, Section 17.4]) induces the dynamic flow X associated with the vector field b, defined by

$$\begin{aligned} X : \ \mathbb{R} \times \mathbb{R}^d \ &\to \ \mathbb{R}^d \\ (t, x) \ &\mapsto \ X(t, x), \end{aligned} \tag{1.12}$$

which satisfies the semi-group property

$$\forall \, s, t \in \mathbb{R}, \ \forall \, x \in \mathbb{R}^d, \quad X(s + t, x) = X(s, X(t, x)). \tag{1.13}$$

The flow X is actually well defined in the torus Y_d, since

$$\forall \, t \in \mathbb{R}, \ \forall \, x \in \mathbb{R}^d, \ \forall \, k \in \mathbb{Z}^d, \quad X(t, x + k) = X(t, x) + k. \tag{1.14}$$

Property (1.14) follows immediately from the uniqueness of the solution $X(\cdot, x)$ to (1.11) combined with the $\mathbb{Z}^d$-periodicity of b.

A possibly signed Borel measure μ on Y_d is said to be *invariant for the flow X* if

$$\forall \, t \in \mathbb{R}, \ \forall \, \psi \in C^0_\sharp(Y_d), \quad \int_{Y_d} \psi\big(X(t, y)\big) \, d\mu(y) = \int_{Y_d} \psi(y) \, d\mu(y). \tag{1.15}$$

For a non negative Borel measure μ on Y_d, a function $f \in L^1_\sharp(Y_d, \mu)$ is said to be *invariant for the flow X* with respect to μ, if

$$\forall \, t \in \mathbb{R}, \quad f \circ X(t, \cdot) = f(\cdot) \quad \mu\text{-a.e. in } Y_d. \tag{1.16}$$

The flow X is said to be *ergodic* with respect to some invariant probability measure μ, if

$$\forall \, f \in L^1_\sharp(Y_d, \mu), \text{ invariant for } X \text{ w.r.t. } \mu, \quad f = \mu(f) \ \mu\text{-a.e. in } Y_d. \tag{1.17}$$

Then, define the set

$$\mathscr{I}_b := \big\{ \mu \in \mathscr{M}_p(Y_d) : \mu \text{ invariant for the flow } X \big\}, \tag{1.18}$$

where $\mathscr{M}_p(Y_d)$ is the set of probability measures on Y_d. From the set of invariant probability measures we define the so-called Herman [13] rotation set

$$\mathsf{C}_b := \left\{ \mu(b) = \int_{Y_2} b(y)\, d\mu(y) : \mu \in \mathscr{I}_b \right\}, \tag{1.19}$$

and its subset

$$\mathsf{D}_b := \left\{ \overline{\sigma\, b} = \int_{Y_2} b(y)\, \sigma(y)\, dy : \sigma \in L^1_\sharp(Y_d) \text{ and } \sigma(y)\, dy \in \mathscr{I}_b \right\} \tag{1.20}$$

which is restricted to the invariant probability measures which are absolutely continuous with respect to Lebesgue's measure. If there is no such invariant measure, then the set D_b is empty (see Remark 4.1).

We have the following characterization of an invariant measure known as Liouville's theorem, which can also be regarded as a divergence-curl result with measures (see [8, Proposition 2.2] and [8, Remark 2.2] for further details).

Proposition 1.1 (Liouville's theorem) *Let $b \in C^1_\sharp(Y_d)^d$, and let $\mu \in \mathscr{M}_\sharp(Y_d)$. We define the Borel measure $\tilde{\mu} \in \mathscr{M}_{\mathrm{loc}}(\mathbb{R}^d)$ on $\mathbb{R}^d$ by*

$$\int_{\mathbb{R}^d} \varphi(x)\, d\tilde{\mu}(x) = \int_{Y_d} \varphi_\sharp(y)\, d\mu(y), \quad \text{where} \quad \varphi_\sharp(\cdot) := \sum_{k \in \mathbb{Z}^d} \varphi(\cdot + k) \quad \text{for } \varphi \in C^0_c(\mathbb{R}^d). \tag{1.21}$$

Then, the three following assertions are equivalent:

(i) μ *is invariant for the flow X, i.e. (1.15) holds,*

(ii) $\tilde{\mu}\, b$ *is divergence free in the space $\mathbb{R}^d$, i.e.*

$$\mathrm{div}\, (\tilde{\mu}\, b) = 0 \quad \text{in } \mathscr{D}'(\mathbb{R}^d), \tag{1.22}$$

(iii) $\mu\, b$ *is divergence free in the torus Y_d, i.e.*

$$\forall \psi \in C^1_\sharp(Y_d), \quad \int_{Y_d} b(y) \cdot \nabla \psi(y)\, d\mu(y) = 0. \tag{1.23}$$

Remark 1.1 *Since any function $\psi \in C^\infty_\sharp(Y_d)$ can be represented as a function $\varphi_\sharp$ for a suitable function $\varphi \in C^\infty_c(\mathbb{R}^d)$ (see [5, Lemma 3.5]), we deduce that the mapping*

$$\mathscr{M}_\sharp(Y_d) \;\to\; \left\{ \nu \in \mathscr{M}_{\mathrm{loc}}(\mathbb{R}^d) : \forall \varphi \in C^0_c(\mathbb{R}^d), \; \varphi_\sharp = 0 \Rightarrow \int_{\mathbb{R}^d} \varphi(x)\, d\nu(x) = 0 \right\}$$

$$\mu \;\mapsto\; \tilde{\mu}$$

is bijective. Therefore, the measure $\tilde{\mu}$ of (1.21) completely characterizes the measure μ.

By virtue of [8, Proposition 2.1] (see also [18]) Herman's set C_b satisfies the following result.

Proposition 1.2 ([8, 18]) *Let $b \in C^1_\sharp(Y_d)^d$. Then, we have for any $\zeta \in \mathbb{R}^d$,*

$$\mathsf{C}_b = \{\zeta\} \quad \Leftrightarrow \quad \forall\, x \in Y_d, \;\; \lim_{t \to \infty} \frac{X(t, x)}{t} = \zeta. \tag{1.24}$$

2 Preliminary results

We have the following characterization of the singleton condition satisfied by D_b, which has to be compared to the one satisfied by C_b in Proposition 1.2 above.

Proposition 2.1 *Let $b \in C^1_\sharp(Y_b)^d$ be such that there exists an a.e. positive function $\sigma_0 \in L^1_\sharp(Y_d)$ with $\overline{\sigma_0} = 1$, satisfying $\mathrm{div}(\sigma_0\, b) = 0$ in $\mathbb{R}^d$. Then, the flow X associated with b satisfies for any $\zeta \in \mathbb{R}^d$,*

$$\mathsf{D}_b = \{\zeta\} \quad \Leftrightarrow \quad \lim_{t\to\infty} \frac{X(t,x)}{t} = \zeta, \quad a.e.\ x \in Y_d. \tag{2.1}$$

Proof. First of all, by virtue of the Birkhoff theorem applied with the invariant measure $\sigma_0(x)\, dx$ with the a.e. positive function $\sigma_0 \in L^1_\sharp(Y_d)$, combined with the uniform boundedness of $X(t,x)/t$ for $t \in \mathbb{R}$ and $x \in Y_d$, there exists a function $\xi \in L^\infty_\sharp(Y_d)$ which is invariant for the flow X with respect to Lebesgue's measure, such that

$$\lim_{t\to\infty} \frac{X(t,x)}{t} = \xi(x) \quad \text{a.e. } x \in Y_d.$$

Hence, by Lebesgue's theorem we get that for any invariant measure $\sigma(x)\, dx$ with $\sigma \in L^1_\sharp(Y_d)$,

$$
\begin{aligned}
\int_{Y_d} b(x)\, \sigma(x)\, dx &= \lim_{t\to\infty} \frac{1}{t} \int_0^t \left(\int_{Y_d} b(X(s,x))\, \sigma(x)\, dx \right) ds \\
&= \int_{Y_d} \lim_{t\to\infty} \left(\frac{X(t,x) - x}{t} \right) \sigma(x)\, dx \\
&= \int_{Y_d} \xi(x)\, \sigma(x)\, dx.
\end{aligned}
\tag{2.2}
$$

($\Rightarrow$) Assume that $\mathsf{D}_b = \{\zeta\}$ for some $\zeta \in \mathbb{R}^d$. Then, we have for any invariant measure $\sigma(x)\, dx$ with $\sigma \in L^1_\sharp(Y_d)$,

$$\int_{Y_d} b(x)\, \sigma(x)\, dx = \zeta \int_{Y_d} \sigma(x)\, dx,$$

which by (2.2) implies that

$$\int_{Y_d} (\xi(x) - \zeta)\, \sigma(x)\, dx = 0. \tag{2.3}$$

On the other hand, since the non negative and the non positive parts $(\xi - \zeta)^\pm$ of $\xi - \zeta$ are also invariant functions for the flow X with respect to Lebesgue's measure, by Lemma 2.2 below the measures $(\xi(x) - \zeta)^\pm \sigma_0(x)\, dx$ are invariant for X. Therefore, putting the measures $\sigma(x)\, dx = (\xi(x) - \zeta)^\pm \sigma_0(x)\, dx$ in equality (2.3) we get that

$$\int_{Y_d} (\xi(x) - \zeta)\, (\xi(x) - \zeta)^\pm \sigma_0(x)\, dx = \pm \int_{Y_d} \left[(\xi(x) - \zeta)^\pm\right]^2 \sigma_0(x)\, dx = 0,$$

which due to the *a.e.* positivity of σ_0, implies the right hand-side of (2.1).

($\Leftarrow$) Conversely, we deduce immediately from (2.2) that for any invariant measure $\sigma(x)\, dx$ with $\sigma \in L^1_\sharp(Y_d)$,

$$\int_{Y_d} b(x)\, \sigma(x)\, dx = \zeta \int_{Y_d} \sigma(x)\, dx,$$

which yields $\mathsf{D}_b = \{\zeta\}$. $\qquad\square$

Lemma 2.2 *Let $b \in C^1_\sharp(Y_d)^d$ be a vector field in $\mathbb{R}^d$ such that there exists an a.e. positive function $\sigma_0 \in L^1_\sharp(Y_d)$ with $\overline{\sigma_0} = 1$, satisfying $\mathrm{div}(\sigma_0 \, b) = 0$ in $\mathbb{R}^d$. Then, a function f in $L^\infty_\sharp(Y_d)$ is invariant for the flow X with respect to Lebesgue's measure if, and only if, the signed measure $f(x) \, \sigma_0(x) \, dx$ is invariant for X.*

Proof. First of all, for any $t \in \mathbb{R}$, $X(t, \cdot)$ is a C^1-diffeomorphism on $\mathbb{R}^d$ with reciprocal $X(-t, \cdot)$, as a consequence of the semi-group property (1.13) satisfied by the flow X. Moreover, by virtue of Liouville's theorem the jacobian determinant of $X(t, \cdot)$ is given by

$$\forall t \in \mathbb{R}, \ \forall x \in Y_d, \quad J(t, x) := \det\big(\nabla_x X(t, x)\big) = \exp\left(\int_0^t (\mathrm{div}\, b)(X(s, x)) \, ds\right). \quad (2.4)$$

Since by Proposition 1.1 the measure $\widetilde{\sigma_0(x) \, dx} = \sigma_0(x) \, dx$ (due to the $\mathbb{Z}^d$-periodicity of σ_0) is invariant for the flow X, we have for any function $\varphi \in C^0_c(\mathbb{R}^d)$ and any $t \in \mathbb{R}$,

$$\varphi_\sharp(X(-t, \cdot)) = \big(\varphi(X(-t, \cdot))\big)_\sharp \quad \text{by (1.14),}$$

and

$$\int_{\mathbb{R}^d} \varphi(x) \, \sigma_0(x) \, dx = \int_{Y_d} \varphi_\sharp(x) \, \sigma_0(x) \, dx$$

$$= \int_{Y_d} \varphi_\sharp(X(-t, x)) \, \sigma_0(x) \, dx = \int_{Y_d} \big(\varphi(X(-t, x))\big)_\sharp \, \sigma_0(x) \, dx$$

$$= \int_{\mathbb{R}^d} \varphi(X(-t, x)) \, \sigma_0(x) \, dx \underbrace{=}_{x = X(t, y)} \int_{\mathbb{R}^d} \varphi(y) \, \sigma_0(X(t, y)) \, J(t, y) \, dy.$$

This implies that the jacobian determinant $J(t, \cdot)$ satisfies the relation

$$\forall t \in \mathbb{R}, \quad \sigma_0(X(t, y)) \, J(t, y) = \sigma_0(y) \ \text{ a.e. } \ y \in \mathbb{R}^d. \quad (2.5)$$

Now, let $f \in L^\infty_\sharp(Y_d)$. From (2.5) we deduce that for any function $\varphi \in C^0_c(\mathbb{R}^d)$ and any $t \in \mathbb{R}$,

$$\int_{\mathbb{R}^d} \varphi(X(-t, x)) \, f(x) \, \sigma_0(x) \, dx \underbrace{=}_{x = X(t, y)} \int_{\mathbb{R}^d} \varphi(y) \, f(X(t, y)) \, \sigma_0(X(t, y)) \, J(t, y) \, dy$$

$$= \int_{\mathbb{R}^d} \varphi(y) \, f(X(t, y)) \, \sigma_0(y) \, dy.$$

By virtue of Remark 1.1 combined with the $\mathbb{Z}^d$-periodicity of the function f, the former equality also reads as

$$\forall \psi \in C^0_\sharp(Y_d), \ \forall t \in \mathbb{R}, \quad \int_{Y_d} \psi(X(-t, x)) \, f(x) \, \sigma_0(x) \, dx = \int_{Y_d} \psi(x) \, f(X(t, x)) \, \sigma_0(x) \, dx. \quad (2.6)$$

Therefore, due to the a.e. positivity of σ_0, the function $f \in L^\infty_\sharp(Y_d)$ is invariant for the flow X with respect to Lebesgue's measure, *i.e.* $f(X(\cdot, x)) = f(x)$ a.e. $x \in Y_d$, if, and only if, the signed measure $f(x) \, \sigma_0(x) \, dx$ is invariant for the flow X. $\square$

3 A NSC for homogenization of the transport equation

First of all, recall the definition of the two-scale convergence introduced by Nguetseng [19] and Allaire [1], which is easily extended to the time dependent case.

8

Definition 3.1 *Let $T \in (0, \infty)$.*

a) *A sequence $u_\varepsilon(t, x)$ in $L^2((0, T) \times \mathbb{R}^d)$ is said to* two-scale converge *to a function $U(t, x, y)$ in $L^2([0, T] \times \mathbb{R}^d; L^2_\sharp(Y_d))$, if we have for any function $\varphi \in C^0_c([0, T] \times \mathbb{R}^d; C^0_\sharp(Y_d))$ with compact support in $[0, T] \times \mathbb{R}^d \times Y_d$,*

$$\lim_{\varepsilon \to 0} \int_{(0,T) \times \mathbb{R}^d} u_\varepsilon(t, x)\, \varphi(t, x, x/\varepsilon)\, dtdx = \int_{(0,T) \times \mathbb{R}^d \times Y_d} U(t, x, y)\, \varphi(t, x, y)\, dtdxdy, \qquad (3.1)$$

b) *According to [1, Definition 1.4] any function $\psi(t, x, y) \in C^0_c([0, T] \times \mathbb{R}^d; L^2_\sharp(Y_d))$ with compact support in $[0, T] \times \mathbb{R}^d \times Y_d$, is said to be an* admissible function for two-scale convergence, *if $(t, x) \mapsto \psi(t, x, x/\varepsilon)$ is Lebesgue measurable and*

$$\lim_{\varepsilon \to 0} \int_{(0,T) \times \mathbb{R}^d} \psi^2(t, x, x/\varepsilon)\, dtdx = \int_{(0,T) \times \mathbb{R}^d \times Y_d} \psi^2(t, x, y)\, dtdxdy. \qquad (3.2)$$

Then, we have the following two-scale convergence compactness result.

Theorem 3.1 ([1], Theorem 1.2, Remark 1.5) *Any sequence $u_\varepsilon(t, x)$ which is bounded in $L^2((0, T) \times \mathbb{R}^d)$ two-scale converges, up to extract a subsequence, to some function $U(t, x, y)$ in $L^2((0, T) \times \mathbb{R}^d; L^2_\sharp(Y_d))$. Moreover, equality (3.1) holds true for any admissible function (3.2).*

Let $b(y) \in C^1_\sharp(Y_d)^d$ be a vector field, let $u_0(x, y) \in C^0_c(\mathbb{R}^d; L^2_\sharp(Y_d))$ be an admissible function with compact support in $\mathbb{R}^d \times Y_d$, and let $f(t, x, y) \in C^0_c([0, T] \times \mathbb{R}^d; L^\infty_\sharp(Y_d))$ be an admissible function with compact support in $[0, T] \times \mathbb{R}^d \times Y_d$. Consider the linear transport equation with oscillating data

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t}(t, x) - b(x/\varepsilon) \cdot \nabla_x u_\varepsilon(t, x) = f(t, x, x/\varepsilon) & \text{in } (0, T) \times \mathbb{R}^d \\ u_\varepsilon(0, x) = u_0(x, x/\varepsilon) & \text{for } x \in \mathbb{R}^d, \end{cases} \qquad (3.3)$$

which by [10, Proposition II.1, Theorem II.2] has a unique solution in $L^\infty((0, T); L^2(\mathbb{R}^d))$. We have the following criterion for the homogenization of equation (3.3).

Theorem 3.2 *Let b be a divergence free vector field in $C^1_\sharp(Y_d)^d$, and let X be the flow (1.11) associated with b. Then, we have the equivalence of the two following assertions:*

(i) *There exists $\zeta \in \mathbb{R}^d$ such that the flow X satisfies the asymptotics*

$$\lim_{t \to \infty} \frac{X(t, x)}{t} = \zeta, \quad a.e. \ x \in Y_d, \qquad (3.4)$$

or, equivalently, $\mathsf{D}_b = \{\zeta\}$.

(ii) *There exists $\zeta \in \mathbb{R}^d$ such that for any admissible functions $u_0(x, y) \in C^0_c(\mathbb{R}^d; L^2_\sharp(Y_d))$ with compact support in $[0, T] \times \mathbb{R}^d$, and $f(t, x, y) \in C^0_c([0, T] \times \mathbb{R}^d; L^\infty_\sharp(Y_d))$ with compact support in $[0, T] \times \mathbb{R}^d \times Y_d$, the solution u_ε to (3.3) converges weakly in $L^2((0, T) \times \mathbb{R}^d)$ to the solution $u(t, x)$ to the transport equation*

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) - \zeta \cdot \nabla_x u(t, x) = \overline{f(t, x, \cdot)} & \text{in } (0, T) \times \mathbb{R}^d \\ u(0, x) = \overline{u_0(x, \cdot)} & \text{for } x \in \mathbb{R}^d. \end{cases} \qquad (3.5)$$

Moreover, in both cases we have $\zeta = \bar{b}$.

Proof of Theorem 3.2.

$(i) \Rightarrow (ii)$. First of all, note that, since b is divergence free in $\mathbb{R}^d$, by Proposition 1.1 Lebesgue's measure is an invariant probability measure for the flow X associated with b, which implies that $\bar{b} \in \mathsf{D}_b = \{\zeta\}$ and $\zeta = \bar{b}$.

Now, let $u_0(x, y) \in C_c^0(\mathbb{R}^d; L^2_\sharp(Y_d))$ be an admissible function with compact support in $[0, T] \times \mathbb{R}^d$, and let $f(t, x, y) \in C_c^0([0, T] \times \mathbb{R}^d; L^\infty_\sharp(Y_d))$ be an admissible function whose support is contained in $[0, T] \times K$, K being a compact set of $\mathbb{R}^d$.

Denote $b_\varepsilon(x) := b(x/\varepsilon)$ which is divergence free in $\mathbb{R}^d$, and denote $f_\varepsilon(t, x) := f(t, x, x/\varepsilon)$ which is uniformly bounded in $[0, T] \times \mathbb{R}^d$ and is compactly supported in $[0, T] \times K$. Formally multiplying (3.3) by $u_\varepsilon(t, x)$, integrating by parts over $\mathbb{R}^d$ and using Cauchy-Schwarz inequality, we get that for any $t \in (0, T)$,

$$\frac{1}{2} \frac{d}{dt} \left(\int_{\mathbb{R}^d} u_\varepsilon^2(t, x) \, dx \right) = \frac{1}{2} \frac{d}{dt} \left(\int_{\mathbb{R}^d} u_\varepsilon^2(t, x) \, dx \right) - \frac{1}{2} \int_{\mathbb{R}^d} \mathrm{div}(b_\varepsilon)(x) \, u_\varepsilon^2(t, x) \, dx$$

$$= \int_K f_\varepsilon(t, x) \, u_\varepsilon(t, x) \, dx \le C_f \left(\int_{\mathbb{R}^d} u_\varepsilon^2(t, x) \, dx \right)^{1/2},$$

where C_f is a non negative constant only depending on f. This can be justified following the proof of [10, Proposition II.1]. Hence, we deduce the estimate

$$\|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R}^d)} \le \|u_\varepsilon(0, \cdot)\|_{L^2(\mathbb{R}^d)} + C_f T \quad \text{a.e. } t \in (0, T). \tag{3.6}$$

Therefore, estimate (3.6) combined with (recall that the admissible function $\psi(t, x, y) = u_0(x, y)$ satisfies (3.2))

$$\lim_{\varepsilon \to 0} \|u_\varepsilon(0, \cdot)\|_{L^2(\mathbb{R}^d)} = \|u_0(x, y))\|_{L^2(\mathbb{R}^d \times Y_d)},$$

implies that the sequence u_ε is bounded in $L^\infty((0, T); L^2(\mathbb{R}^d))$. Then, up to a subsequence, $u_\varepsilon(t, x)$ two-scale converges to some function $U(t, x, y) \in L^2((0, T) \times \mathbb{R}^d; L^2_\sharp(Y_d))$, and $u_\varepsilon(t, x)$ converges weakly in $L^2((0, T) \times \mathbb{R}^d)$ to the mean value

$$u(t, x) := \overline{U(t, x, \cdot)} = \int_{Y_d} U(t, x, y) \, dy \quad \text{for a.e. } (t, x) \in (0, T) \times \mathbb{R}^d. \tag{3.7}$$

Next, we follow the two-scale procedure of the proof of [15, Theorem 2.1]. Putting the test function $\varphi(t, x) \in C_c^1([0, T] \times \mathbb{R}^d)$ in the weak formulation of (3.3), and integrating by parts we have

$$-\int_{(0,T) \times \mathbb{R}^d} \frac{\partial \varphi}{\partial t}(t, x) \, u_\varepsilon(t, x) \, dtdx - \int_{\mathbb{R}^d} \varphi(0, x) \, u_0(x, x/\varepsilon) \, dx$$

$$+ \int_{(0,T) \times \mathbb{R}^d} b(x/\varepsilon) \cdot \nabla_x \varphi(t, x) \, u_\varepsilon(t, x) \, dtdx = \int_{\mathbb{R}^d} \varphi(t, x) \, f(t, x, x/\varepsilon) \, dtdx.$$

Then, passing to the two-scale limit and using that $u_0(x, y)$ and $f(t, x, y)$ are admissible functions for two-scale convergence, we get that

$$-\int_{(0,T) \times \mathbb{R}^d \times Y_d} \frac{\partial \varphi}{\partial t}(t, x) \, U(t, x, y) \, dtdxdy - \int_{\mathbb{R}^d \times Y_d} \varphi(0, x) \, u_0(x, y) \, dxdy$$

$$+ \int_{(0,T) \times \mathbb{R}^d \times Y_d} b(y) \cdot \nabla_x \varphi(t, x) \, U(t, x, y) \, dtdxdy = \int_{(0,T) \times \mathbb{R}^d \times Y_d} \varphi(t, x) \, f(t, x, y) \, dtdxdy,$$

10

or, equivalently, by Fubini's theorem

$$-\int_{(0,T)\times\mathbb{R}^d}\frac{\partial\varphi}{\partial t}(t,x)\,u(t,x)\,dx-\int_{\mathbb{R}^d}\varphi(0,x)\,\overline{u_0(x,\cdot)}\,dx$$

$$+\int_{(0,T)\times\mathbb{R}^d}\overline{U(t,x,\cdot)\,b}\cdot\nabla_x\varphi(t,x)\,dtdx=\int_{\mathbb{R}^d\times Y_d}\varphi(t,x)\,\overline{f(t,x,\cdot)}\,dtdx. \tag{3.8}$$

Similarly, passing to the two-scale limit with the admissible test function $\varepsilon\,\varphi(t,x)\,\psi(x/\varepsilon)$ for any $\varphi(t,x)\in C_c^1([0,T]\times\mathbb{R}^d)$ and any $\psi\in C_\sharp^1(Y_d)$, we get that

$$\int_{(0,T)\times\mathbb{R}^d\times Y_d}\varphi(t,x)\,b(y)\cdot\nabla_y\psi(y)\,U(t,x,y)\,dtdxdy$$

$$=\int_{(0,T)\times\mathbb{R}^d}\varphi(t,x)\left(\int_{Y_d}U(t,x,y)\,b(y)\cdot\nabla_y\psi(y)\,dy\right)dtdx=0,$$

which by Proposition 1.1 implies that

$$\operatorname{div}_y(U(t,x,\cdot)\,b)=0\ \ \text{in}\ \mathscr{D}'(\mathbb{R}^d),\quad\text{a.e.}\ (t,x)\in(0,T)\times\mathbb{R}^d. \tag{3.9}$$

Then, applying Lemma 2.2 with $\sigma_0=1$, for a.e. $(t,x)\in(0,T)\times\mathbb{R}^d$, the function $U(t,x,\cdot)$ is an invariant function for the flow X associated with b related to Lebesgue's measure, and so are the positive and negative parts $U^\pm(t,x,\cdot)$ of $U(t,x,\cdot)$. Hence, again by Lemma 2.2 the measures $U^\pm(t,x,y)\,dy$ are invariant for X, which by the definition (1.20) of $\mathsf{D}_b=\{\zeta\}$, implies that

$$\overline{U^\pm(t,x,\cdot)\,b}=\int_{Y_d}b(y)\,U^\pm(t,x,y)\,dy=\left(\int_{Y_d}U^\pm(t,x,y)\,dy\right)\zeta\quad\text{a.e.}\,(t,x)\in(0,T)\times\mathbb{R}^d. \tag{3.10}$$

From (3.10) and (3.7) we deduce that

$$\overline{U(t,x,\cdot)\,b}=\overline{U(t,x,\cdot)}\,\zeta=u(t,x)\,\zeta\quad\text{a.e.}\ (t,x)\in(0,T)\times\mathbb{R}^d. \tag{3.11}$$

Putting this equality in the weak formulation (3.8), we get that for any $\varphi(t,x)\in C_c^1([0,T]\times\mathbb{R}^d)$,

$$-\int_{(0,T)\times\mathbb{R}^d}\frac{\partial\varphi}{\partial t}(t,x)\,u(t,x)\,dx-\int_{\mathbb{R}^d}\varphi(0,x)\,\overline{u_0(x,\cdot)}\,dx$$

$$+\int_{(0,T)\times\mathbb{R}^d}u(t,x)\,\zeta\cdot\nabla_x\varphi(t,x)\,dtdx=\int_{(0,T)\times\mathbb{R}^d}\varphi(t,x)\,\overline{f(t,x,\cdot)}\,dtdx,$$

which is the weak formulation of the homogenized transport equation (3.5).

$(ii)\Rightarrow(i)$. First of all, note that the set D_b contains the mean $\bar{b}$, since by the free divergence of b and by Proposition 1.1, Lebesgue's measure is an invariant probability measure for the flow X associated with b.

Now, let us prove that any invariant probability measure $\sigma(x)\,dx$ with $\sigma\in L_\sharp^1(Y_d)$, for the flow X satisfies the equality $\overline{\sigma\,b}=\zeta$, which will yield the desired equality $\mathsf{D}_b=\{\zeta\}$. To this end, let us first show this for any invariant probability measure $v(x)/\overline{v}\,dx$ with $v\in L_\sharp^\infty(Y_d)$. By virtue of Proposition 1.1 such a function v is solution to the equation

$$\operatorname{div}(v\,b)=b\cdot\nabla v=0\quad\text{in}\ \mathscr{D}'(\mathbb{R}^d). \tag{3.12}$$

11

Let $\theta \in C_c^1(\mathbb{R}^d)$, and define for $\varepsilon > 0$ the function $u_\varepsilon \in C^1([0,T]; L^\infty(\mathbb{R}^d))$ by

$$u_\varepsilon(t,x) := \theta(x + t\,\zeta)\,v(x/\varepsilon) \quad \text{for } (t,x) \in [0,T] \times \mathbb{R}^d, \tag{3.13}$$

where ζ is the vector involving in the homogenized equation (3.5). By (3.12) we have

$$\frac{\partial u_\varepsilon}{\partial t}(t,x) - b(x/\varepsilon) \cdot \nabla_x u_\varepsilon(t,x)$$
$$= v(x/\varepsilon)\,\zeta \cdot \nabla_x \theta(x + t\,\zeta) - v(x/\varepsilon)\,b(x/\varepsilon) \cdot \nabla_x \theta(x + t\,\zeta) - 1/\varepsilon\,\theta(x + t\,\zeta)\,(b \cdot \nabla_y v)(x/\varepsilon)$$
$$= \big(v(x/\varepsilon)\,\zeta - (v\,b)(x/\varepsilon)\big) \cdot \nabla_x \theta(x + t\,\zeta) = f(t,x,x/\varepsilon),$$

where

$$f(t,x,y) := \big(v(y)\,\zeta - (v\,b)(y)\big) \cdot \nabla_x \theta(x + t\,\zeta) \quad \text{for } (t,x,y) \in [0,T] \times \mathbb{R}^d \times Y_d, \tag{3.14}$$

is an admissible function in $C_c^0([0,T] \times \mathbb{R}^d; L_\sharp^\infty(Y_d))$ with compact support in $[0,T] \times \mathbb{R}^d \times Y_d$. Moreover, we have $u_\varepsilon(0,x) = \theta(x)\,v(x/\varepsilon)$ for $x \in \mathbb{R}^d$, where $\theta(x)\,v(y) \in C_c^0(\mathbb{R}^d; L_\sharp^2(Y_d))$ with compact support in $[0,T] \times \mathbb{R}^d$, is also an admissible function. Therefore, the function u_ε defined by (3.13) is solution to the transport equation (3.3) with the right-hand side (3.14). Hence, by the homogenization assumption the sequence $u_\varepsilon(t,x)$ converges weakly in $L^2((0,T) \times \mathbb{R}^d)$ to the function $u(t,x) = \theta(x + t\,\zeta)\,\overline{v}$ which is solution to the homogenized equation (3.5), *i.e.*

$$\forall\,(t,x) \in [0,T] \times \mathbb{R}^d, \quad \frac{\partial u}{\partial t}(t,x) - \zeta \cdot \nabla_x u(t,x) = \overline{f(t,x,\cdot)} = \big(\overline{v}\,\zeta - \overline{v\,b}\big) \cdot \nabla_x \theta(x + t\,\zeta).$$

But, directly from the expression $u(t,x) = \theta(x + t\,\zeta)\,\overline{v}$, we also deduce that

$$\forall\,(t,x) \in [0,T] \times \mathbb{R}^d, \quad \frac{\partial u}{\partial t}(t,x) - \zeta \cdot \nabla_x u(t,x) = 0.$$

Equating the two former equations we get that for any $\theta \in C_c^1(\mathbb{R}^d)$,

$$\forall\,(t,x) \in [0,T] \times \mathbb{R}^d, \quad \big(\overline{v}\,\zeta - \overline{v\,b}\big) \cdot \nabla_x \theta(x + t\,\zeta) = 0,$$

which implies that

$$\overline{v\,b} = \overline{v}\,\zeta. \tag{3.15}$$

Now, let σ be a non negative function in $L_\sharp^1(Y_d)$ with $\overline{\sigma} = 1$, such that $\sigma(x)\,dx$ is an invariant measure for the flow X, or, equivalently, by Lemma 2.2 applied with $\sigma_0 = 1$, the function σ is invariant for X with respect to Lebesgue's measure. Hence, for any $n \in \mathbb{N}$, the truncated function $\sigma \wedge n$ is also invariant for X. Equality (3.15) applied with $v = \sigma \wedge n \in L_\sharp^\infty(Y_d)$, yields

$$\overline{(\sigma \wedge n)\,b} = \overline{\sigma \wedge n}\,\zeta \underset{n \to \infty}{\longrightarrow} \overline{\sigma\,b} = \overline{\sigma}\,\zeta = \zeta.$$

Thus, we obtain the desired equality $\mathsf{D}_b = \{\zeta\} = \{\overline{b}\}$, which owing to Proposition 2.1 concludes the proof of Theorem 3.2. $\qquad\square$

Remark 3.1 *From equation* (3.9) *Hou and Xin [15] used the ergodicity of the flow X to deduce that $U(t,x,\cdot)$ is constant a.e. $(t,x) \in (0,T) \times \mathbb{R}^d$. However, this condition is not necessary. Indeed, the less restrictive condition used in the above proof is that D_b is reduced to the unit set $\{\zeta\}$. This combined with Lemma 2.2 on invariant measures and functions leads us to equality (3.11), and allows us to conclude.*

4 A picture of comparisons between the seven conditions

In the sequel we denote:

Rec	if there exist a C^2-diffeomorphism Ψ on Y_d and $\xi \in \mathbb{R}^d$ such that $\nabla \Psi\, b = \xi$ in Y_d.
Erg	if the ergodic condition (1.17) holds with an invariant probability measure for X, which is absolutely continuous with respect to Lebesgue's measure,
Asy-a.e.	if there exist $\zeta \in \mathbb{R}^d$ such that $\lim\limits_{t\to\infty} X(t,x)/t = \zeta$, a.e. $x \in Y_d$.
Asy-e.	if there exist $\zeta \in \mathbb{R}^d$ such that $\lim\limits_{t\to\infty} X(t,x)/t = \zeta$, $\forall\, x \in Y_d$.
$\#\mathsf{C}_b = 1$	if the unit set condition holds for Herman's set C_b.
$\#\mathsf{D}_b = 1$	if the unit set condition holds for the set D_b.
Hom	if the homogenized equation (3.5) holds when b is divergence free in $\mathbb{R}^d$.

Theorem 4.1 *Let $b \in C^1_\sharp(Y_d)^d$ be a non null but possibly vanishing vector field such that there exists an invariant probability measure $\sigma_0(x)\, dx$ with $\sigma_0 \in L^1_\sharp(Y_d)$, for the flow X associated with b, or, equivalently, $\mathsf{D}_b \neq \emptyset$. Then, we have a complete array (see Figure 1 below) of all the logical connections between the above seven conditions, in which:*

- *A grey square means a tautology.*
- *A square with $\Leftarrow$ means that the condition of the top line implies the condition of the left column, but not the converse in general.*
- *A square with $\Uparrow$ means that the condition of the left column implies the condition of the top line, but not the converse in general.*
- *A square with $\Leftrightarrow$ or $\Updownarrow$ means that the conditions of the top line and of the left column are equivalent.*
- *A dark square means that the conditions of the top line and the left column cannot be compared in general.*
- *Finally, if a square involves condition Hom, then the other condition must be considered under the assumption that b is divergence free in $\mathbb{R}^d$.*

Remark 4.1 *We may have both $\#\mathsf{C}_b = 1$ and $\mathsf{D}_b = \emptyset$.*
To this end, consider a gradient field $b = \nabla u$ with $u \in C^2_\sharp(Y_d)$, such that $\nabla u \neq 0$ a.e. in Y_d. On the one hand, by virtue of [8, Proposition 2.4] we have $\mathsf{C}_b = \{0\}$. On the other hand, assume that there exists an invariant probability measure $\sigma(x)\, dx$ with $\sigma \in L^1_\sharp(Y_d)$, for the flow associated with ∇u. Then, by virtue of Proposition 1.1 we have

$$\int_{Y_d} \sigma(x)\, |\nabla u(x)|^2\, dx = \int_{Y_d} \sigma(x)\, \nabla u(x) \cdot \nabla u(x)\, dx = 0,$$

which implies that $\sigma = 0$ a.e. in Y_d, a contradiction with $\overline{\sigma} = 1$. Therefore, we get that $\mathsf{D}_b = \emptyset$.

Proof of Theorem 4.1.

Condition Rec. By virtue of [6, Corollary 4.1] condition *Rec* implies condition $\#\mathsf{C}_b = 1$ which by Proposition 1.2 is equivalent to condition *Asy-e.*. Moreover, condition *Asy-e.* clearly implies condition *Asy-a.e.* which by Proposition 2.1 is equivalent to condition $\#\mathsf{D}_b = 1$, and by Theorem 3.2 is equivalent to condition *Hom*. Therefore, condition *Rec* implies condition *Asy-a.e.*, condition *Asy-e.*, condition $\#\mathsf{C}_b = 1$, condition $\#\mathsf{D}_b = 1$, and condition *Hom*.

On the other hand, note that if the vector field b vanishes, then condition *Rec* cannot hold true. Otherwise, in equality $\nabla \Psi\, b = \zeta$ the constant vector ζ is necessarily nul, hence due to the

	Rec	Erg	Asy-a.e.	Asy-e.	#C$_b$=1	#D$_b$=1	Hom
Rec			$\Uparrow$	$\Uparrow$	$\Uparrow$	$\Uparrow$	$\Uparrow$
Erg			$\Uparrow$			$\Uparrow$	$\Uparrow$
Asy-a.e.	$\Leftarrow$	$\Leftarrow$		$\Leftarrow$	$\Leftarrow$	$\Leftrightarrow$	$\Leftrightarrow$
Asy-e.	$\Leftarrow$		$\Uparrow$		$\Leftrightarrow$	$\Uparrow$	$\Uparrow$
#C$_b$=1	$\Leftarrow$		$\Uparrow$	$\Updownarrow$		$\Uparrow$	$\Uparrow$
#D$_b$=1	$\Leftarrow$	$\Leftarrow$	$\Updownarrow$	$\Leftarrow$	$\Leftarrow$		$\Leftrightarrow$
Hom	$\Leftarrow$	$\Leftarrow$	$\Updownarrow$	$\Leftarrow$	$\Leftarrow$	$\Updownarrow$	

Figure 1: Logical connections between the seven conditions

invertibility of $\nabla\Psi$, b is the nul vector field, which yields a contradiction. Therefore, since all other conditions may be satisfied with a vanishing vector field b according to the examples of [8, Section 4] combined with Proposition 2.1 and Theorem 3.2, condition *Rec* cannot be deduced in general from any of the other six conditions.

Conditions Rec and Erg cannot be compared. [6, Corollary 4.1] provides a two-dimensional and a three-dimensional example in which condition *Rec* holds true, but not condition *Erg*.

Condition Erg. By virtue of Birkhoff's theorem condition *Erg* implies condition *Asy-a.e.* which is equivalent to condition #D$_b$=1 (by Proposition 2.1) and is equivalent to condition *Hom* (by Theorem 3.2).

Conditions Erg and #C$_b$ = 1 *cannot be compared.* Since condition *Rec* implies #C$_b$ = 1, but condition *Rec* does not imply in general condition *Erg* (by [6, Corollary 4.1]), by a transitivity argument condition #C$_b$=1 does not imply in general condition *Erg*.

On the other hand, extending the two-dimensional results of Oxtoby [20] and Marchetto [17] to any dimension by a different approach, Example 4.1 and Proposition 5.1 below deal with a d-dimensional Stepanoff flow [23, Section 4] defined by

$$\begin{cases} \dfrac{\partial S}{\partial t}(t,x) = b_S(S(t,x)) = \rho_S(S(t,x))\,\xi, & t \in \mathbb{R} \\ S(0,x) = x \in \mathbb{R}^d, \end{cases} \tag{4.1}$$

where ρ_S is a non negative function in $C^1_\sharp(Y_d)$ with a finite positive number of roots in Y_d and $\sigma_S := 1/\rho_S \in L^1_\sharp(Y_d)$, and where ξ is a constant vector of $\mathbb{R}^d$ with incommensurable coordinates. Under these conditions $\sigma_S(x)/\overline{\sigma_S}\, dx$ is the unique invariant probability measure on Y_d for the flow S, which does not load the zero set of ρ_S, and S is ergodic with respect to the measure $\sigma_S(x)/\overline{\sigma_S}\, dx$. Hence, condition *Erg* holds true with the probability measure $\sigma_S(x)/\overline{\sigma_S}\, dx$. Moreover, Proposition 5.1 shows that $\mathsf{D}_{b_S} = \{\zeta\}$ and $\mathsf{C}_{b_S} = [0, \zeta]$ with $\zeta = (1/\overline{\sigma_S})\xi \neq 0$. Therefore, condition *Erg* does not imply in general condition $\#\mathsf{C}_b = 1$, or, equivalently, condition *Asy-e.*.

Conditions $\#\mathsf{C}_b = 1$ *and* $\#\mathsf{D}_b = 1$. Since D_b is assumed to be non empty, condition $\#\mathsf{C}_b = 1$ clearly implies condition $\#\mathsf{D}_b = 1$.

In contrast, as above mentioned the Stepanoff flow induces that $\mathsf{D}_{b_S} = \{\zeta\}$ and $\mathsf{C}_{b_S} = [0, \zeta]$ with $\zeta \in \mathbb{R}^d \setminus \{0\}$. Alternatively, Example 5.1 below provides a different class of two-dimensional vanishing vector fields b such that D_b is a singleton, while C_b is a closed line segment not reduced to a singleton. Therefore, condition $\#\mathsf{D}_b = 1$ does not imply in general $\#\mathsf{C}_b = 1$.

Condition $\#\mathsf{D}_b = 1$. Since condition $\#\mathsf{C}_b = 1$ implies condition $\#\mathsf{D}_b = 1$, but $\#\mathsf{C}_b = 1$ does not imply in general condition *Erg*, by a transitivity argument condition $\#\mathsf{D}_b = 1$ does not imply in general condition *Erg*. Moreover, since condition $\#\mathsf{C}_b = 1$ is equivalent to condition *Asy-e.*, but condition $\#\mathsf{D}_b = 1$ does not imply in general $\#\mathsf{C}_b = 1$, condition $\#\mathsf{D}_b = 1$ does not imply in general condition *Asy-e.*.

Condition Hom. Here, we assume that the vector field b is divergence free in $\mathbb{R}^d$.

On the one hand, consider the constant vector field $b = e_1$ in $\mathbb{R}^d$, which induces the flow

$$X(t, x) = x + t\, e_1 \quad \text{for } (t, x) \in \mathbb{R} \times \mathbb{R}^d.$$

Then, any function $f \in L^1_\sharp(Y_d)$ independent of variable x_1 is invariant for the flow X with respect to any invariant probability measure which is absolutely continuous with respect to Lebesgue's measure. Hence, the flow X is not ergodic with respect to such an invariant probability measure. Moreover, we have immediately $\mathsf{C}_b = \mathsf{D}_b = \{e_1\}$. Therefore, condition *Hom* which is equivalent to condition $\#\mathsf{D}_b = 1$ (by Theorem 3.2), does not imply in general condition *Erg*.

On the other hand, the two-dimensional divergence free Oxtoby example [20, Section 2] combined with the uniqueness result of [20, Theorem 1] (see Example 4.1) provides a flow which is ergodic with respect to Lebesgue's measure, and such that C_b is not a unit set (see Proposition 5.1). Therefore, since condition *Erg* implies condition *Hom* (see, *e.g.*, [15, Theorem 3.2]) condition *Hom* does not imply in general condition $\#\mathsf{C}_b = 1$. Finally, condition *Hom* does not imply in general either condition *Erg*, or condition $\#\mathsf{C}_b = 1$, or, equivalently, condition *Asy-e.*.

The rest of the implications can be easily deduced from the former arguments. □

Exemple 4.1 Oxtoby [20] provided an example of a free divergence analytic two-dimensional vector field b with $(0, 0)$ as unique stationary point in Y_2, such that the associated flow X is ergodic with respect to Lebesgue's measure, and such that Lebesgue's measure is the unique invariant measure for the flow X among all the invariant probability measures which do not load the point $(0, 0)$. Oxtoby's example is actually based on a Stepanoff flow (4.1), where ρ_S is a non negative function in $C^1_\sharp(Y_2)$ with $(0, 0)$ as unique stationary point, and where ξ is a constant vector of $\mathbb{R}^2$ with incommensurable coordinates. Stepanoff [23, Section 4] proved that Birkhoff's theorem applies if $\sigma_S := 1/\rho_S$ is in $L^1_\sharp(Y_2)$, which is not incompatible with the analyticity for ρ_S. A suitable candidate for ρ_S is then the function (see [8, Example 4.2] for another application)

$$\rho_S(x) := \left(\sin^2(\pi x_1) + \sin^2(\pi x_2)\right)^{\beta_0} \quad \text{for } x \in Y_2, \quad \text{with } \beta_0 \in (1/2, 1). \tag{4.2}$$

More generally, Oxtoby [20, Theorem 1] proved that any two-dimensional flow homeomorphic to a Stepanoff flow with a unique stationary point x_0, admits a unique invariant probability measure μ for the flow S (4.1) satisfying $\mu(\{x_0\}) = 0$, and that S is ergodic with respect to μ. Twenty five years later, Marchetto [17, Proposition 1.2] extended this result to any flow homeomorphic to a Stepanoff flow with a finite number of stationary points in Y_2.

5 A new approach of Stepanoff flow

In this section, we extend the two-dimensional results of [20, 17] to any dimension $d \geq 2$, using a non ergodic and elementary approach based on some classical tools of PDE's analysis (mollification, truncation) combined with the characterization of invariant functions of Lemma 2.2.

Proposition 5.1 *Consider a d-dimensional, $d \geq 2$, Stepanoff flow S (4.1) where $\rho_S \in C^1_\sharp(Y_d)$ is non negative with a finite positive number of roots (the stationary points for S) in Y_d and $\sigma_S := 1/\rho_S \in L^1_\sharp(Y_d)$, and where $\xi \in \mathbb{R}^d$ has incommensurable coordinates. Then, the measure $\sigma_S(x)/\overline{\sigma_S}\, dx$ is the unique invariant probability measure on Y_d for the flow S, which does not load the zero set of ρ_S. The flow S is also ergodic with respect to the measure $\sigma_S(x)/\overline{\sigma_S}\, dx$. Moreover, we have $\mathsf{D}_{b_S} = \{\zeta\}$ and $\mathsf{C}_{b_S} = [0, \zeta]$, where $\zeta := 1/\overline{\sigma_S}\, \xi$.*

Remark 5.1 *Similarly to [20, 17] the result of Proposition 5.1 actually extends to any flow which is homeomorphic to a Stepanoff flow.*
Indeed, let Ψ be a C^2-diffeomorphism on Y_d (see [8, Remark 2.1]). Define the flow $\hat{X}$ obtained through the homeomorphism Ψ from the flow X associated with a vector field $b \in C^1_\sharp(Y_d)^d$, by

$$\hat{X}(t, x) := \Psi\big(X(t, \Psi^{-1}(x))\big) \quad \text{for } (t, x) \in \mathbb{R} \times Y_d. \tag{5.1}$$

According to [8, Remark 2.1] the homeomorphic flow $\hat{X}$ is the flow associated with the vector field $\hat{b} \in C^1_\sharp(Y_d)^d$ defined by

$$\hat{b}(x) = \nabla\Psi(\Psi^{-1}(x))\, b(\Psi^{-1}(x)) \quad \text{for } x \in Y_d. \tag{5.2}$$

Now, let μ be a probability mesure on Y_d, and let $\hat{\mu}$ be the image measure of μ by Ψ defined by

$$\int_{Y_d} \varphi(x)\, d\hat{\mu}(x) = \int_{Y_d} \varphi(\Psi(y))\, d\mu(y) \quad \text{for } \varphi \in C^0_\sharp(Y_d).$$

By (5.1) we have

$$\begin{cases} \forall\, \varphi \in C^0_\sharp(Y_d), \quad \int_{Y_d} \varphi\big(\hat{X}(t, x)\big)\, d\hat{\mu}(x) = \int_{Y_d} \varphi\big(\Psi(X(t, y))\big)\, d\mu(y), \\[2mm] \forall\, \rho \in C^0_\sharp(Y_d), \quad \hat{\mu}(\{\rho = 0\}) = \mu(\{\rho \circ \Psi = 0\}), \\[2mm] \forall\, f \in L^1_\sharp(Y_d), \ \hat{f} := f \circ \Psi^{-1}, \ \forall\, t \in \mathbb{R}, \quad \hat{f}\big(\hat{X}(t, x)\big) = f\big(X(t, \Psi^{-1}(x))\big) \ \text{ a.e. } x \in Y_d. \end{cases} \tag{5.3}$$

Also note that, if μ is invariant for X, so is $\hat{\mu}$ for $\hat{X}$. Therefore, if the homeomorphic flow $\hat{X}$ is a Stepanoff flow S satisfying the assumptions of Proposition 5.1, we easily deduce from (5.3) that Proposition 5.1 holds true for the flow X. Namely, there exists a unique invariant probability measure μ on Y_d for the flow X, which does not load the zero set of $\rho_S \circ \Psi$. Moreover, the measure μ is absolutely continuous with respect to Lebesgue's measure with an a.e. positive density, and the flow X is ergodic with respect to μ.

16

Remark 5.2 *Assuming the uniqueness result of Proposition 5.1, the ergodicity of $\sigma_S(x)/\overline{\sigma_S}\, dx$ and equality $\mathsf{C}_{b_S} = [0, \zeta]$ can also be proved using standard arguments of ergodic theory. Indeed, let $\mathscr{E}_{b_S}$ be the set of all the ergodic invariant probability measures for the flow S. Recall (see, e.g., [9, Theorem 2, Chapter 1]) that $\mathscr{I}_{b_S} = \mathrm{conv}\,(\mathscr{E}_{b_S})$, and that two elements in $\mathscr{E}_{b_S}$ are either equal, or mutually singular. Now, if $\mu \in \mathscr{E}_{b_S}$ satisfies $\mu(\{x_0\}) > 0$ for some zero of ρ_S, then $\mu = \delta_{x_0}$ due to $\delta_{x_0} \in \mathscr{E}_{b_S}$. Next, since $\sigma_S(x)/\overline{\sigma_S}\, dx$ is the unique invariant probability measure on Y_d for the flow S, which does not load the zero set of ρ_S, it follows from equality $\mathscr{I}_{b_S} = \mathrm{conv}\,(\mathscr{E}_{b_S})$ that the flow S is ergodic with respect to $\sigma_S(x)/\overline{\sigma_S}\, dx$. Hence, $\mathscr{E}_{b_S}$ is the finite set*

$$\mathscr{E}_{b_S} = \left\{\sigma_S(x)/\overline{\sigma_S}\, dx\right\} \cup \left\{\delta_x : \rho_S(x) = 0\right\}. \tag{5.4}$$

Therefore, $\overline{\sigma_S\, b_S}/\overline{\sigma_S} = \zeta$ provides the unique non zero contribution in C_{b_S} through $\mathscr{E}_{b_S}$, which by convex combination implies that $\mathsf{C}_{b_S} = [0, \zeta]$. Equality $\mathscr{I}_{b_S} = \mathrm{conv}\,(\mathscr{E}_{b_S})$ and property (5.4) also give $\mathsf{D}_{b_S} = \{\zeta\}$.

Proof of Proposition 5.1. Assume that μ is an invariant probability measure for the flow S (4.1), which does not load the zero set of ρ_S. Then, by virtue of Proposition 1.1 the Borel measure $\tilde{\mu}$ on $\mathbb{R}^d$ defined by (1.21) is solution to the equation

$$\mathrm{div}(\tilde{\mu}\, b_S) = \mathrm{div}(\rho_S\, \tilde{\mu}\, \xi) = 0 \quad \text{in } \mathscr{D}'(\mathbb{R}^d).$$

Hence, applying Lemma 5.2 below with the measure $\nu = \rho_S\, \mu$ which is connected to the measure $\tilde{\nu} = \rho_S\, \tilde{\mu}$ by (1.21), there exists a constant $c \in \mathbb{R}$ such that $\rho_S(x)\, d\mu(x) = c\, dx$ on Y_d, i.e.

$$\forall\, \varphi \in C^0_\sharp(Y_d), \quad \int_{Y_d} \varphi(x)\, \rho_S(x)\, d\mu(x) = \int_{Y_d} c\, \varphi(x)\, dx.$$

Then, we get that for any $n \geq 1$,

$$\forall\, \varphi \in C^0_\sharp(\mathbb{R}^d), \quad \int_{Y_d} \frac{\varphi(x)}{\rho_S(x) + 1/n}\, \rho_S(x)\, d\mu(x) = \int_{Y_d} c\, \frac{\varphi(x)}{\rho_S(x) + 1/n}\, dx. \tag{5.5}$$

However, since measure μ does not load the finite zero set of ρ_S in Y_d (at this point this assumption is crucial), we have

$$\begin{cases} \dfrac{\varphi\, \rho_S}{\rho_S + 1/n} \underset{n \to \infty}{\longrightarrow} \varphi \quad d\mu(x)\text{-a.e. in } Y_d, \quad \text{with } \left|\dfrac{\varphi\, \rho_S}{\rho_S + 1/n}\right| \leq \|\varphi\|_\infty \in L^1_\sharp(Y_d, \mu) \\[4mm] \dfrac{\varphi}{\rho_S + 1/n} \underset{n \to \infty}{\longrightarrow} \dfrac{\varphi}{\rho_S} \quad dx\text{-a.e. in } Y_d, \quad\quad \text{with } \left|\dfrac{\varphi}{\rho_S + 1/n}\right| \leq \dfrac{\|\varphi\|_\infty}{\rho_S} \in L^1_\sharp(Y_d). \end{cases}$$

Therefore, passing to the limit as $n \to \infty$ owing to Lebesgue's theorem in (5.5), we get that

$$\forall\, \varphi \in C^0_\sharp(Y_d), \quad \int_{Y_d} \varphi(x)\, d\mu(x) = \int_{Y_d} c\, \varphi(x)\, \sigma_S(x)\, dx.$$

We thus obtain the equality $d\mu(x) = \sigma_S(x)/\overline{\sigma_S}\, dx$, which shows the uniqueness of an invariant probability measure for the flow S, which does not load the zero set of ρ_S. Conversely, $\sigma_S\, b_S = \xi$ is clearly divergence free, which by Proposition 1.1 implies that μ is an invariant probability measure for the flow S. We have just proved that μ is the unique invariant probability measure for the flow S, which does not load the zero set of ρ_S.

Now, let us prove that the flow S is ergodic with respect to the measure $d\mu(x) = \sigma_S(x)/\overline{\sigma_S}\, dx$. To this end, let $f \in L^1_\sharp(Y_d)$ be an invariant function for the flow S with respect to measure μ. Let $\mathsf{T}_n^\pm$ be the truncation functions at level $n \in \mathbb{N}$, defined by

$$\mathsf{T}_n^\pm(t) := \big((\pm t) \vee 0)\big) \wedge n \quad \text{for } t \in \mathbb{R}.$$

Then, the functions $T_n^\pm(f) \in L^\infty_\sharp(Y_d)$ are also invariant functions for the flow S with respect to measure μ. We know that $\sigma_S(x)\, dx$ is an invariant measure for the flow S. Hence, by virtue of Lemma 2.2 the Radon measures $d\mu_n^\pm(x) := (T_n^\pm(f)\,\sigma_S)(x)\, dx$ are invariant for S, which by relation (1.21) and Proposition 1.1 implies that

$$\widetilde{\mu_n^\pm}\, b_S = \mu_n^\pm\, b_S = (T_n^\pm(f)\,\sigma_S\, b_S)(x)\, dx = T_n^\pm(f)(x)\, \xi\, dx$$

are divergence free in $\mathbb{R}^d$. Therefore, applying Lemma 5.2 with measures $d\nu(y) = T_n^\pm(f)(y)\, dy$ which satisfy $\tilde{\nu} = \nu$, the functions $T_n^\pm(f)$ agree with constants $c_n^\pm \in \mathbb{R}$ a.e. in Y_d. However, since the sequences $T_n^\pm(f)$ converge strongly in $L^1_\sharp(Y_d)$ to the non negative and the non positive parts $f^\pm$ of f, the sequences $c_n^\pm$ converge to some constants $c_\pm$ in $\mathbb{R}$. Hence, the function $f = f^+ - f^-$ agrees with the constant $c_+ - c_-$ a.e. in Y_d. This proves the desired property.

Next, since $\sigma_S(x)/\overline{\sigma_S}\, dx$ is the unique invariant probability measure on Y_d for the flow S, among the invariant probability measures which are absolutely continuous with respect to Lebesgue's measure, we have

$$\mathsf{D}_{b_S} = \left\{ \int_{Y_d} \rho_S(x)\, \xi\, \sigma_S(x)/\overline{\sigma_S}\, dx \right\} = \{1/\overline{\sigma_S}\, \xi\}.$$

Note that the former equality can be alternatively deduced from the ergodicity of the flow S combined with Theorem 4.1.

On the other hand, set $b_n := b_S + 1/n$ for $n \geq 1$. Since ξ has incommensurable coordinates, we have (see [8, Example 4.1])

$$\mathsf{C}_{b_n} = \{\zeta_n\} \quad \text{where} \quad \zeta_n := \left(\int_{Y_d} \frac{dx}{\rho_S(x) + 1/n}\, dx \right)^{-1} \zeta.$$

Finally, since the function ρ_S vanishes in Y_d, by virtue of [8, Theorem 3.1] we obtain that

$$\mathsf{C}_{b_S} = [0, \zeta], \quad \text{where} \quad \zeta = \lim_{n \to \infty} \zeta_n = 1/\overline{\sigma_S}\, \xi.$$

Note that the ergodic approach of Remark 5.2 alternatively shows that $\mathsf{C}_{b_S} = [0, \zeta]$. The proof of Proposition 5.1 is now complete. $\qquad\square$

Lemma 5.2 *Let $\nu \in \mathcal{M}_\sharp(Y_d)$, let $\tilde{\nu} \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^d)$ be the Borel measure on $\mathbb{R}^d$ connected to the measure ν by relation (1.21), and let $\xi \in \mathbb{R}^d$ be a vector with incommensurable coordinates. Assume that $\tilde{\nu}\, \xi$ is divergence free in $\mathbb{R}^d$, i.e.*

$$\forall \varphi \in C_c^\infty(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} \xi \cdot \nabla\varphi(x)\, d\tilde{\nu}(x) = 0. \tag{5.6}$$

Then, there exists a constant $c \in \mathbb{R}$ such that $d\nu(y) = c\, dy$ on Y_d.

Remark 5.3 *In Lemma 5.2 the incommensurability of $\xi's$ coordinates is also a necessary condition to get (5.6). Indeed, assume that there exists a non nul integer vector $k \in \mathbb{Z}^d \setminus \{0\}$ such that $k \cdot \xi = 0$. Then, for any non constant $\mathbb{Z}$-periodic function $\theta \in C_\sharp^1(Y_1)$, the function $\big(\tau : x \mapsto \theta(k \cdot x)\big)$ belongs to $C_\sharp^1(Y_d)$, $\tilde{\tau}(x)\, dx = \tau(x)\, dx$, and*

$$\forall x \in \mathbb{R}^d, \quad \mathrm{div}(\tau\, \xi)(x) = \theta'(k \cdot x)\, k \cdot \xi = 0,$$

so that the conclusion of Lemma 5.2 does not hold true.

Proof of Lemma 5.2. Let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence of mollifiers in $C_c^\infty(\mathbb{R}^d)$ with $\overline{\phi_n} = 1$. Applying successively Fubini's theorem twice and (5.6), the convolution $\phi_n * \tilde{\nu} \in C^\infty(\mathbb{R}^d)$ satisfies for any $n \in \mathbb{N}$ and for any $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \phi_n(x-y)\, d\tilde{\nu}(y) \right) \xi \cdot \nabla\varphi(x)\, dx = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \phi_n(x-y)\, \xi \cdot \nabla\varphi(x)\, dx \right) d\tilde{\nu}(y)$$

$$= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \phi_n(x)\, \xi \cdot \nabla_x\varphi(x+y)\, dx \right) d\tilde{\nu}(y) = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \xi \cdot \nabla_y\varphi(x+y)\, d\tilde{\nu}(y) \right) \phi_n(x)\, dx = 0,$$

or, equivalently,

$$\mathrm{div}\big((\phi_n * \tilde{\nu})\,\xi\big) = \nabla(\phi_n * \tilde{\nu}) \cdot \xi = 0 \quad \text{in } \mathbb{R}^d. \tag{5.7}$$

Now, consider $\xi^1, \ldots, \xi^{d-1}$ $(d{-}1)$ vectors in $\mathbb{R}^d$ such that $(\xi^1, \ldots, \xi^{d-1}, \xi)$ is an orthogonal basis of $\mathbb{R}^d$, and let Λ be the matrix in $\mathbb{R}^{(d-1)\times d}$ whose lines are the vectors $\xi^1, \ldots, \xi^{d-1}$, *i.e.* its entries are given by $\Lambda_{ij} = \xi_j^i$ for $(i,j) \in \{1, \ldots, d{-}1\} \times \{1, \ldots, d\}$. Then, make the linear change of variables

$$\begin{aligned} \mathbb{R}^d &\to \mathbb{R}^d \\ x &\mapsto y = (\Lambda x, \xi \cdot x) = (\xi^1 \cdot x, \ldots, \xi^{d-1} \cdot x, \xi \cdot x). \end{aligned}$$

Since (5.7) means that $(\phi_n * \tilde{\nu})(x)$ is independent of the variable $y_d = \xi \cdot x$, it follows that there exists a function $\theta_n \in C^\infty(\mathbb{R}^{d-1})$ such that

$$\forall\, x \in \mathbb{R}^d, \quad (\phi_n * \tilde{\nu})(x) = \theta_n(\Lambda x).$$

Moreover, due to (1.21) and the $\mathbb{Z}^d$-periodicity of $(\phi_n)_\sharp$, we have for any $x \in \mathbb{R}^d$ and $k \in \mathbb{Z}^d$,

$$(\phi_n * \tilde{\nu})(x+k) = \int_{\mathbb{R}^d} \phi_n(x+k-y)\, d\tilde{\nu}(y) = \int_{Y_d} (\phi_n)_\sharp(x+k-y)\, d\nu(y)$$

$$= \int_{Y_d} (\phi_n)_\sharp(x-y)\, d\nu(y) = \int_{\mathbb{R}^d} \phi_n(x-y)\, d\tilde{\nu}(y) = (\phi_n * \tilde{\nu})(x),$$

which implies that the function $\phi_n * \tilde{\nu}$ is also $\mathbb{Z}^d$-periodic. As a consequence, the regular function θ_n satisfies the periodicity condition

$$\forall\, k \in \mathbb{Z}^d,\ \forall\, x \in \mathbb{R}^{d-1}, \quad \theta_n(x + \Lambda k) = \theta_n(x).$$

Hence, by virtue of the density Lemma 5.3 below we get that θ_n is a constant $c_n \in \mathbb{R}$, and thus $\phi_n * \tilde{\nu} = c_n$ in $\mathbb{R}^d$. Therefore, by Fubini's theorem we have for any $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} c_n\, \varphi(x)\, dx = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \phi_n(x-y)\, d\tilde{\nu}(y) \right) \varphi(x)\, dx = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \phi_n(x-y)\, \varphi(x)\, dx \right) d\tilde{\nu}(y),$$

where the function $\big(y \mapsto \int_{\mathbb{R}^d} \phi_n(x-y)\, \varphi(x)\, dx \big)$ converges uniformly to φ on $\mathbb{R}^d$ as $n \to \infty$, whose support is included in a fixed compact set of $\mathbb{R}^d$, and which is bounded uniformly by $\|\varphi\|_\infty$. Therefore, passing to the limit as $n \to \infty$ thanks to Lebesgue's theorem with respect to measure $\tilde{\nu}$, we get that the sequence $(c_n)_{n\in\mathbb{N}}$ converges to some $c \in \mathbb{R}$, and that

$$\forall\, \varphi \in C_c^\infty(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} c\, \varphi(x)\, dx = \int_{\mathbb{R}^d} \varphi(y)\, d\tilde{\nu}(y).$$

Hence, we deduce the equality $d\tilde{\nu}(x) = c\, dx$ on $\mathbb{R}^d$, or, equivalently, $d\nu(y) = c\, dy$ on Y_d by virtue of Remark 1.1. This concludes the proof of Lemma 5.2. $\qquad\square$

Lemma 5.3 *Let ξ be a vector in $\mathbb{R}^d$ for $d \geq 2$, with incommensurable coordinates, let $\xi^1, \ldots, \xi^{d-1}$ be $(d-1)$ vectors in $\mathbb{R}^d$ such that $(\xi^1, \ldots, \xi^{d-1}, \xi)$ is an orthogonal basis of $\mathbb{R}^d$, and let Λ be the matrix in $\mathbb{R}^{(d-1)\times d}$ whose lines are the vectors $\xi^1, \ldots, \xi^{d-1}$. Then, the lattice $\Lambda \mathbb{Z}^d$ is dense in $\mathbb{R}^{d-1}$.*

Proof. Lemma 5.3 follows easily from [3, Proposition 6 & Corollary, Section VII.7] which leads one to Kronecker's approximation theorem [3, Proposition 7, Section VII.7]. For the reader's convenience we propose a more direct proof.

Since matrix Λ has rank $(d-1)$ and $\mathrm{Ker}(\Lambda) = \mathbb{R}\,\xi$, we may assume, up to reorder the vectors, that the vectors $\Lambda e_1, \ldots, \Lambda e_{d-1}$ are linearly independent and that there exist d real numbers $\alpha_1, \ldots, \alpha_{d-1}, \alpha$ satisfying

$$\Lambda e_d = \sum_{i=1}^{d-1} \alpha_i \, \Lambda e_i \quad \text{and} \quad e_d - \sum_{i=1}^{d-1} \alpha_i \, e_i = \alpha \, \xi. \tag{5.8}$$

Replacing the vector e_d in the first equality of (5.8) and using that $\Lambda \xi = 0$, we get that

$$\Lambda \mathbb{Z}^d = \sum_{i=1}^{d} \mathbb{Z} \, \Lambda e_i = \sum_{i=1}^{d-1} (\mathbb{Z} + \alpha_i \, \mathbb{Z}) \, \Lambda e_i.$$

Assume that there exists $j \in \{1, \ldots, d-1\}$ such that the set $(\mathbb{Z} + \alpha_j \, \mathbb{Z})$ is not dense in $\mathbb{R}$, or, equivalently, $\alpha_j \in \mathbb{Q}$. Taking the j-th and d-th coordinates in the second equality of (5.8), it follows that $\xi_j + \alpha_j \, \xi_d = 0$, which contradicts the incommensurability of ξ's coordinates. Therefore, the set $\Lambda \mathbb{Z}^d$ is dense in $\mathbb{R}^{d-1}$, which concludes the proof. $\qquad\square$

Exemple 5.1 Consider a two-dimensional vector field $b = \rho_0 \, R_\perp \nabla u$ such that $\rho_0 \in C^1_\sharp(Y_2)$ is a.e. positive in Y_2 and does vanish in Y_2, and such that $\nabla u \in C^1_\sharp(Y_2)^2$ does not vanish in Y_2 and $\overline{\nabla u}$ has incommensurable coordinates. Also assume that $\sigma_0 := 1/\rho_0 \in L^1_\sharp(Y_2)$. An example of such a function is given by (4.2). Note that, by virtue of Proposition 1.1 the probability measure $\sigma_0(x)/\overline{\sigma_0} \, dx$ is invariant for the flow X associated with b.

Now, let $\sigma \in L^1_\sharp(Y_2)$ be a non negative function with $\overline{\sigma} = 1$, such that $\mathrm{div}(\sigma b) = 0$ in $\mathbb{R}^2$. By Proposition 1.1 $\sigma(x) \, dx$ is an invariant probability measure for the flow X. Hence, by Fubini's theorem we have for any $T > 0$,

$$\int_{Y_2} \sigma(x) \, b(x) \, dx = \frac{1}{T} \int_0^T \left(\int_{Y_2} \sigma(x) \, b(X(t,x)) \, dx \right) dt = \int_{Y_2} \left(\frac{X(T,x) - x}{T} \right) \sigma(x) \, dx. \tag{5.9}$$

On the other hand, since the function ρ_0 does vanish in Y_2 together with $\rho_0 > 0$ a.e. in Y_2, from [8, Lemma 3.1] applied with the invariant probability measure $d\mu(x) := \sigma_0(x)/\overline{\sigma_0} \, dx$, we deduce that

$$\lim_{T \to \infty} \frac{X(T,x)}{T} = \zeta := \frac{\overline{\sigma_0 \, b}}{\overline{\sigma_0}} = \frac{R_\perp \overline{\nabla u}}{\overline{\sigma_0}} \neq (0,0) \quad \text{a.e. } x \in Y_2.$$

Therefore, passing to the limit $T \to \infty$ in equality (5.9) thanks to Lebesgue's theorem, we get that for any invariant probability measure $\sigma(x) \, dx$ with $\sigma \in L^1_\sharp(Y_2)$,

$$\int_{Y_2} \sigma(x) \, b(x) \, dx = \zeta \neq (0,0),$$

which thus implies that $\mathsf{D}_b = \{\zeta\}$. However, by virtue of [8, Corollary 3.4] we obtain that $\mathsf{C}_b = [0, \zeta]$. Therefore, we have $\#C_b = \infty$, while $\#\mathsf{D}_b = 1$.

Remark 5.4 *The result of Example 5.1 can be deduced from the Proposition 5.1 combined with Remark 5.1, using Kolmogorov's theorem [16] (see, e.g., [22, Lecture 11], and see also [24, Theorem 2.1] for an elementary proof when one of the coordinates of the vector field does not vanish). Indeed, since the divergence free field $R_\perp \nabla u$ of Example 5.1 does not vanish in Y_2, by virtue of Kolmogorov's theorem there exists a C^1-diffeomorphism on Y_2 which transforms the flow X associated with the vector field $b = \rho_0 R_\perp \nabla u$, to a Stepanoff flow satisfying the assumptions of Proposition 5.1 provided the zero set of ρ_0 is finite. Therefore, Remark 5.1 allows us to conclude.*

We can extend Example 5.1 to the following variant of [8, Corollary 3.4], which provides a general framework where the sets C_b and D_b may differ.

Proposition 5.4 *Let $b = \rho \, \Phi \in C^1_\sharp(Y_2)^2$ be a vector field, where $\rho \in C^1_\sharp(Y_2)$ is a non negative function with a positive finite number of roots, and where $\Phi \in C^1_\sharp(Y_2)^2$ is a non vanishing vector field. Also assume that there exists a function $u \in C^1(Y_2)$ with $\nabla u \in C^0_\sharp(Y_2)^2$, such that $\overline{\nabla u}$ has incommensurable coordinates and $\Phi \cdot \nabla u = 0$ in Y_2. Then, the exists a vector $\zeta \in \mathbb{R}^2$ such that $\mathsf{C}_b = [0, \zeta]$, together with $\mathsf{D}_b = \varnothing$ or $\mathsf{D}_b = \{\zeta\}$.*

Proof. First of all, define for $n \geq 1$, the function $\rho_n := \rho + 1/n > 0$, and the vector field $b_n := \rho_n \, \Phi$. By the equality $\Phi \cdot \nabla u = 0$ in Y_2, we get that u is an invariant function for the flow X_n associated with the vector field b_n, with respect to Lebesgue's measure. Then, following the proof of [8, Corollary 3.4], from the ergodic case of [21, Theorem 3.1] and the incommensurability of $\overline{\nabla u}$'s coordinates, we deduce that there exists a vector $\zeta_n \in \mathbb{R}^2$ such that $\mathsf{C}_{b_n} = \{\zeta_n\}$.

On the one hand, since the function ρ vanishes in Y_2, by the second case of [8, Theorem 3.1] it turns out that the sequence $(\zeta_n)_{n \geq 1}$ converges to some $\zeta \in \mathbb{R}^2$, and that $\mathsf{C}_b = [0, \zeta]$.

On the other hand, assume that the set D_b is non empty. Then, there exists an invariant probability measure $\sigma(x) \, dx$ with $\sigma \in L^1_\sharp(Y_2)$, for the flow X associated with the vector field b, i.e. $\sigma(x)/\overline{\sigma} \, dx \in \mathscr{I}_b$. Following the proof of [8, Corollary 3.3] define the probability measure μ_n by

$$d\mu_n(x) := C_n \, \frac{\rho(x)}{\rho_n(x)} \, \sigma(x) \, dx \quad \text{where} \quad C_n := \left(\int_{Y_d} \frac{\rho(x)}{\rho_n(y)} \, \sigma(y) \, dy \right)^{-1}.$$

Note that $C_n < \infty$, since $\rho \, \sigma$ is non negative and not nul a.e. in Y_2. Due to $\sigma(x)/\overline{\sigma} \, dx \in \mathscr{I}_b$, by Proposition 1.1 we have

$$\forall \, \varphi \in C^1_\sharp(Y_2), \quad \int_{Y_d} b_n(x) \cdot \nabla \varphi(x) \, d\mu_n(x) = C_n \int_{Y_d} b(x) \cdot \nabla \varphi(x) \, \sigma(x) \, dx = 0,$$

which again by Proposition 1.1 implies that $\mu_n \in \mathscr{I}_{b_n}$. This combined with $\mathsf{C}_{b_n} = \{\zeta_n\}$ yields

$$\zeta_n = \int_{Y_d} b_n(x) \, d\mu_n(x) = C_n \int_{Y_d} b(x) \, \sigma(x) \, dx = C_n \, \overline{\sigma \, b}$$

which is actually independent of σ. Due $\rho > 0$ a.e. in Y_2, by Lebesgue's theorem we get that the sequence $(C_n)_{n \geq 1}$ converges to $\overline{\sigma} = 1$. Hence, we deduce that

$$\zeta = \lim_{n \to \infty} \zeta_n = \overline{\sigma \, b}$$

which is also independent of σ. Therefore, we obtain that $\mathsf{D}_b = \{\zeta\}$, which concludes the proof of Proposition 5.4. □

References

[1] G. ALLAIRE: "Homogenization and two-scale convergence", *SIAM J. Math. Anal.*, **23** (6) (1992), 1482-1518.

[2] Y. AMIRAT, K. HAMDACHE & A. ZIANI: "Homogénéisation d'équations hyperboliques du premier ordre et application aux écoulements miscibles en milieu poreux" (French) [Homogenization of a system of first-order hyperbolic equations and application to miscible flows in a porous medium], *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **6** (5) (1989), 397-417.

[3] N. BOURBAKI: *Éléments de mathématique. Topologie générale. Chapitres 5 à 10*, (French) [Elements of mathematics. General topology. Chapters 5-10], Hermann, Paris, 1974, 330 pp.

[4] Y. BRENIER: "Remarks on some linear hyperbolic equations with oscillatory coefficients", *Proceedings of the Third International Conference on Hyperbolic Problems* (Uppsala 1990) Vol. I, II, Studentlitteratur, Lund (1991), 119-130.

[5] M. BRIANE. "Isotropic realizability of a strain field for the two-dimensional incompressible elasticity system", *Inverse Problems*, **32** (6) (2016), 22 pp.

[6] M. BRIANE: "Isotropic realizability of fields and reconstruction of invariant measures under positivity properties. Asymptotics of the flow by a nonergodic approach", *SIAM J. App. Dyn. Sys.*, **18** (4) (2019), 1846-1866.

[7] M. BRIANE: "Homogenization of linear transport equations. A new approach", *J. École Polytechnique - Mathématiques*, **7** (2020), 479-495.

[8] M. BRIANE & L. HERVÉ. "Asymptotics of ODE's flow on the torus through a singleton condition and a perturbation result. Applications", *ArXiv*, arXiv:2009.13121 (2020), pp. 32.

[9] I.P. CORNFELD, S.V. FOMIN & YA.G. SINAĬ: *Ergodic Theory*, translated from the Russian by A.B. Sosinskii, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] **245**, Springer-Verlag, New York, 1982, 486 pp.

[10] R.J. DIPERNA & P.-L. LIONS: "Ordinary differential equations, transport theory and Sobolev spaces", *Invent. Math.*, **98** (3) (1989), 511-547.

[11] F. GOLSE: "Moyennisation des champs de vecteurs et EDP" (French), [The averaging of vector fields and PDEs], *Journées Équations aux Dérivées Partielles*, Saint Jean de Monts 1990, Exp. no. **XVI**, École Polytech. Palaiseau, 1990, 17 pp.

[12] F. GOLSE: "Perturbations de systèmes dynamiques et moyennisation en vitesse des EDP" (French), [On perturbations of dynamical systems and the velocity averaging method for PDEs], *C. R. Acad. Sci. Paris Sér. I Math.*, **314** (2) (1992), 115-120.

[13] M.R. HERMAN: "Existence et non existence de tores invariants par des difféomorphismes symplectiques" (French), [Existence and nonexistence of tori invariant under symplectic diffeomorphisms], *Séminaire sur les Équations aux Dérivées Partielles 1987-1988*, **XIV**, École Polytech. Palaiseau, 1988, 24 pp.

[14] M.W. Hirsch, S. Smale & R.L. Devaney: *Differential equations, Dynamical Systems, and an Introduction to Chaos*, Second edition, *Pure and Applied Mathematics* **60**, Elsevier Academic Press, Amsterdam, 2004, 417 pp.

[15] T.Y. Hou & X. Xin: "Homogenization of linear transport equations with oscillatory vector fields", *SIAM J. Appl. Math.*, **52** (1) (1992), 34-45.

[16] A.N. Kolmogorov: "On dynamical systems with an integral invariant on the torus" (Russian), *Doklady Akad. Nauk SSSR (N.S.)*, **93** (1953), 763-766.

[17] D. Marchetto "Stepanoff flows on orientable surfaces", *Rocky Mountain J. Math*, **9** (2) (1979), 273-281.

[18] M. Misiurewicz & K. Ziemian: "Rotation sets for maps of tori", *J. London Math. Soc. (2)*, **40** (3) (1989), 490-506.

[19] G. Nguetseng: "A general convergence result for a functional related to the theory of homogenization", *SIAM J. Math. Anal.*, **20** (3) (1989), 608-623.

[20] J.C. Oxtoby: "Stepanoff flows on the torus", *Proc. Amer. Math. Soc.*, **4** (1953), 982-987.

[21] R. Peirone: "Convergence of solutions of linear transport equations", *Ergodic Theory Dynam. Systems*, **23** (3) (2003), 919-933.

[22] Ya.G. Sinaǐ: *Introduction to Ergodic Theory* (Translated from the Russian), Mathematical Notes, Princeton University Press, 1976, 152 pp.

[23] W. Stepanoff: "Sur une extension du théorème ergodique", *Compositio Math.*, **3** (1936), pp. 239-253.

[24] T. Tassa: "Homogenization of two-dimensional linear flows with integral invariance", *SIAM J. Appl. Math.*, **57** (5) (1997), 1390-1405.

[25] L. Tartar: "Nonlocal effects induced by homogenization", *Partial Differential Equations and the Calculus of Variations Vol. II*, F. Colombini et al. (eds.), 925-938, Progr. Nonlinear Differential Equations Appl. **2**, Birkhäuser Boston, Boston, MA, 1989.