Asymptotics of ODE's flow on the torus through a singleton condition and a perturbation result. Applications

Marc Briane & Loïc Hervé

Univ Rennes, INSA Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France

mbriane@insa-rennes.fr & loic.herve@insa-rennes.fr

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Abstract

This paper deals with the long time asymptotics of the flow X solution to the autonomous vector-valued ODE: $X'(t,x) = b(X(t,x))$ for $t \in \mathbb{R}$, with $X(0,x) = x$ a point of the torus Y_d. We assume that the vector field b reads as the product $\rho\,\Phi$, where ρ is a non negative regular function and Φ is a non vanishing regular vector field in Y_d. In this work, the singleton condition means that the Hermann rotation set C_b composed of the average values of b with respect to the invariant probability measures for the flow X is a singleton $\{\zeta\}$. This combined with Liouville's theorem regarded as a divergence-curl lemma, first allows us to obtain the asymptotics of the flow X, when b is a nonlinear current field. Then, we prove a general perturbation result assuming that ρ is the uniform limit in Y_d of a positive sequence $(\rho_n)_{n\in\mathbb{N}}$ satisfying for any $n \in \mathbb{N}$, $\rho \le \rho_n$ and $\mathsf{C}_{\rho_n\Phi}$ is a singleton $\{\zeta_n\}$. It turns out that the limit set C_b either remains a singleton, or enlarges to the closed line segment $[0_{\mathbb{R}^d}, \lim_n \zeta_n]$ of $\mathbb{R}^d$. We provide various corollaries of this perturbation result involving or not the classical ergodic condition, according to the positivity or not of some weighted harmonic means of ρ. These results are illustrated by different examples which highlight the alternative satisfied by the rotation set C_b. Finally, we prove that the singleton condition allows us to homogenize in any dimension the linear transport equation induced by the oscillating velocity $b(x/\varepsilon)$ beyond any ergodic condition satisfied by the flow X.

Keywords: ODE's flow, asymptotics, perturbation, homogenization, conductivity equation, transport equation, invariant measure, rotation set, ergodic

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Contents

1 Introduction

In this paper we study the large time asymptotics of the solution $X(\cdot, x)$ for $x \in \mathbb{R}^d$, to the ODE

$$\begin{cases} \dfrac{\partial X}{\partial t}(t,x) = b(X(t,x)), \quad t \in \mathbb{R} \\ X(0,x) = x, \end{cases} \tag{1.1}$$

where b is a C^1-regular vector field defined in the torus $Y_d := \mathbb{R}^d/\mathbb{Z}^d$ (denoted by $b \in C^1_\sharp(Y_d)^d$) according to the $\mathbb{Z}^d$-periodicity (1.13). The solution $X(\cdot, x)$ is well-defined for any $x \in Y_d$ by virtue of (1.17). More precisely, we focus on the existence of the limit of $X(t,x)/t$ as $t \to \infty$ for $x \in Y_d$. This question naturally arises in ergodic theory, since it involves the *dynamic flow* X induced by ODE (1.1), and the Borel measures μ on the torus Y_d which are *invariant for the flow* X, *i.e.*

$$\forall\, t \in \mathbb{R},\ \forall\, \psi \in C^0_\sharp(Y_d), \qquad \int_{Y_d} \psi\big(X(t,y)\big)\, d\mu(y) = \int_{Y_d} \psi(y)\, d\mu(y). \tag{1.2}$$

A strengthened variant of the famous Birkhoff ergodic theorem [15, Theorem 2, Section 1.8] claims that if the flow is *uniquely ergodic*, *i.e.* there exists a unique probability measure μ on Y_d which is invariant for the flow, then any function $f \in C^0_\sharp(Y_d)$ satisfies

$$\forall\, x \in Y_d, \quad \lim_{t\to\infty} \left[\frac{1}{t}\int_0^t f(X(s,x))\, ds\right] = \int_{Y_d} f(y)\, d\mu(y), \tag{1.3}$$

and the converse actually holds true. In the particular case where $f = b$, limit (1.3) yields

$$\forall\, x \in Y_d, \quad \lim_{t\to\infty} \frac{X(t,x)}{t} = \int_{Y_d} b(y)\, d\mu(y). \tag{1.4}$$

The unique ergodicity condition is a rather restrictive condition on the flow (1.1). Alternatively, define the set

$$\mathscr{I}_b := \big\{\mu \in \mathscr{M}_p(Y_d) : \mu \text{ invariant for the flow } X\big\}, \tag{1.5}$$

where $\mathscr{M}_p(Y_d)$ is the set of probability measures on Y_d, and the subset of $\mathscr{I}_b$

$$\mathscr{E}_b := \big\{\mu \in \mathscr{I}_b : \mu \text{ ergodic for the flow } X\big\}. \tag{1.6}$$

It is known that the ergodic measures for the flow X are the extremal points of the convex set $\mathscr{I}_b$ so that

$$\mathscr{I}_b = \mathrm{conv}(\mathscr{E}_b). \tag{1.7}$$

Also define for any vector field $b \in C^1_\sharp(Y_d)^d$ the two following non empty subsets of $\mathbb{R}^d$:

- The set of all the limit points of the sequences $\big(X(n,x)/n\big)_{n\geq 1}$ for $x \in Y_d$ (denoted by $\rho_{\mathrm{p}}(b)$ in [30])

$$\mathsf{A}_b := \bigcup_{x\in Y_d}\left[\bigcap_{n\geq 1}\overline{\left\{\frac{X(k,x)}{k} : k\geq n\right\}}\right]. \tag{1.8}$$

- The so-called Herman [23] rotation set

$$\mathsf{C}_b := \left\{\int_{Y_d} b(y)\,d\mu(y) : \mu \in \mathscr{I}_b\right\} = \mathrm{conv}\left\{\int_{Y_d} b(y)\,d\mu(y) : \mu \in \mathscr{E}_b\right\}, \tag{1.9}$$

  which is a compact and convex subset of $\mathbb{R}^d$.

An implicit consequence of [30, Theorem 2.4, Remark 2.5, Corollary 2.6] shows that

$$\mathsf{A}_b \subset \mathsf{C}_b = \mathrm{conv}(\mathsf{A}_b) \quad \text{and} \quad \#\mathsf{A}_b = 1 \Leftrightarrow \#\mathsf{C}_b = 1, \tag{1.10}$$

Note that by definition (1.8) the equivalence of (1.10) can be written for any $\zeta \in \mathbb{R}^d$,

$$\mathsf{C}_b = \{\zeta\} \quad \Leftrightarrow \quad \forall\, x \in Y_d,\ \lim_{n\to\infty}\frac{X(n,x)}{n} = \zeta \quad \Leftrightarrow \quad \forall\, x \in Y_d,\ \lim_{t\to\infty}\frac{X(t,x)}{t} = \zeta. \tag{1.11}$$

In the sequel the "singleton condition" means that Herman's rotation set C_b is a singleton $\{\zeta\}$. Proposition 2.1 below provides an alternative proof of (1.11). Then, the "singleton approach" consists in establishing sufficient conditions on the vector field b to ensure the singleton condition. The aim of this paper is to exploit this approach either to get the asymptotics of the flow X for suitable vector fields b, or in the less favorable cases to determine Herman's rotation set C_b as a closed line segment of $\mathbb{R}^d$.

First of all, revisiting Liouville's theorem for invariant probability measures (see, *e.g.*, [15, Theorem 1, Section 2.2]) as a divergence-curl lemma (see Proposition 2.2) we obtain (see Proposition 2.4) a rather surprising null asymptotics of the flow X associated with a nonlinear current field of type $b = F(\cdot,\nabla v)$, where $F(x,\xi)$ is a vector-valued function in $C^1_\sharp(Y_d; C^1(\mathbb{R}^d))^d$ which is strictly monotonic with respect to variable ξ, and where v is a scalar potential in $C^2_\sharp(Y_d)$.

Actually, except the one-dimensional case where b is parallel to a fixed direction so that the flow can be computed explicitly (see Example 4.1), there are very few examples of vector fields b for which the asymptotics of the flow X (1.1) is completely known in dimension $d \geq 2$. There are at least three cases:

- If the two-dimensional flow X has a cross section Σ (see [19, Theorems A,B,D], [5, Appendix A] for cross sections to flows, and see also [14, Theorem 2.1]), and if the return time τ to Σ is of type "coboundary plus constant", *i.e.* $\tau(x) = h(f(x)) - h(x) + T$ for $x \in Y_2$, where f is the return map to Σ and h some bounded function, then the asymptotics of $X(\cdot,x)$ is a vector in $\mathbb{R}^2$ independent of x, involving the constant T and the rotation number of f.

- If b is a non vanishing regular field in dimension two, Peirone [31, Theorem 3.1] proved that the asymptotics of the flow $X(\cdot,x)$ does exist at each point $x \in Y_2$. Moreover, Peirone showed that the asymptotics is actually independent of x, *i.e.* the singleton condition is satisfied, when for any $x \in Y_2$, the flow $X(\cdot,x)$ is not periodic in Y_2 according to (1.18). However, the asymptotics may depend on x in the periodic case, contrary to the statement of [27, Proposition 14.7.1] which was corrected by Peirone in [31, page 922].

- Under the global rectification condition $\nabla\Psi\, b = \zeta$ in Y_d, where Ψ is a C^2-diffeomorphism on the torus Y_d and ζ is a non zero constant vector in $\mathbb{R}^d$, the set C_b is the singleton $\big\{(\int_{Y_d} \nabla\Psi)^{-1}\zeta\big\}$ in any dimension (see [9, Corollary 4.1] and Remark 2.1).

In the two former situations the vector field b does not vanish. In order to extend these two results among others to a vanishing vector field b, a natural question is to know if the singleton condition is stable under a uniform non vanishing perturbation b_n of b for $n \in \mathbb{N}$. We provide a partial answer to this question with the main result Theorem 3.1 of the paper. Restricting ourselves to a perturbation $b_n = \rho_n\,\Phi$, where $(\rho_n)_{n\in\mathbb{N}}$ is a sequence of positive functions in $C^1_\sharp(Y_d)$ converging uniformly in Y_d to some regular function $\rho \le \rho_n$ and where Φ is a fixed vector field, we prove that if $\mathsf{C}_{b_n} = \{\zeta_n\}$ for any $n \in \mathbb{N}$, then the sequence $(\zeta_n)_{n\in\mathbb{N}}$ converges to some $\zeta \in \mathbb{R}^d$. Moreover, we get that the limit set C_b where $b = \rho\,\Phi = \lim_n b_n$, is the singleton $\{\zeta\}$ if ρ is positive, and that C_b is the closed line segment $[0_{\mathbb{R}^d}, \zeta]$ if ρ is only non negative. In Theorem 3.1 it is essential that the vector field Φ in $b_n = \rho_n\,\Phi$ is independent of n, otherwise the perturbation result does not hold in general (see Remark 3.3 and Example 4.1). Moreover, the two-dimensional Example 4.2 shows that the sequence of singletons $(\mathsf{C}_{b_n})_{n\in\mathbb{N}}$ may be actually enlarged to the limit closed line segment $\mathsf{C}_b = [0_{\mathbb{R}^d}, \zeta]$ with $\zeta \neq 0$. From the perturbation result we first deduce (see Corollary 3.1) that for a fixed vector field $\Phi \in C^1_\sharp(Y_d)^d$, the set of the positive functions $\rho \in C^1_\sharp(Y_d)$ with $\#\mathsf{C}_{\rho\Phi} = 1$ is closed for the uniform convergence in $C^1_\sharp(Y_d)^d$. This result does not extend to the larger set composed of the non negative functions ρ as shown in Remark 3.4. Then, we prove various corollaries of Theorem 3.1 in terms of the asymptotics of the ODE's flow (1.1):

- We show (see Corollary 3.2) that the asymptotics of the flow $X(\cdot,x)$ associated with $b = \rho\,\Phi$ does exist at any point x such that the orbit $X(\mathbb{R},x)$ is far enough from the set $\{\rho = 0\}$.

- When the flow associated with the vector field Φ admits an invariant probability measure with a positive density $\sigma \in C^1_\sharp(Y_d)$ with respect to Lebesgue's measure, we prove (see Corollary 3.3) the following alternative:
  - $\mathsf{C}_{\rho\Phi} = \{0_{\mathbb{R}^d}\}$ if the harmonic mean of ρ/σ is equal to 0,
  - $\mathsf{C}_{\rho\Phi}$ is some closed line segment $[0_{\mathbb{R}^d}, \zeta]$ of $\mathbb{R}^d$ if the harmonic mean of ρ/σ is positive (see the two-dimensional Example 4.2).

- As a by-product of Corollary 3.3 we determine (see Corollary 3.4) the set $\mathsf{C}_{\rho\Phi}$ in dimension two when Φ is parallel to an orthogonal gradient satisfying an ergodic condition, extending Peirone's result [31, Theorem 3.1] (see Remark 3.6) to the case where the vector field b does vanish. Corollary 3.4 is illustrated in Example 4.4 by the case where b is a two-dimensional electric field, while dimension three is shown to be quite different. Similarly, we extend (see Corollary 3.5) the non ergodic case of [9, Corollary 4.1] to a vanishing vector field b in any dimension (see Example 4.3).

It turns out that the singleton approach cannot be regarded exclusively as an ergodic approach. It may contain an ergodic condition as in Corollary 3.4. But it may be also independent of any ergodic condition as in Corollary 3.5. This does make this approach an original alternative to the classical ergodic approach.

Finally, we apply the asymptotics of the flow (1.1) to the homogenization of the linear transport equation with an oscillating velocity

$$\frac{\partial u_\varepsilon}{\partial t}(t,x) + b\left(\frac{x}{\varepsilon}\right) \cdot \nabla_x u_\varepsilon(t,x) = 0 \quad \text{for } (t,x) \in [0,\infty) \times \mathbb{R}^d, \tag{1.12}$$

where the vector field b belongs to $C^1_\sharp(Y_d)^d$. Tartar [35] and Amirat *et al* [2, 3, 4] showed that in general the homogenization of equation (1.12) leads to a nonlocal limit problem. Here, we focus on the cases where the homogenized equation remains a linear transport equation with some average velocity $\langle b \rangle$ of the vector field b. In this perspective, the following works may be quoted: First, assuming that b is divergence free and the flow associated with b is ergodic, Brenier [7] proved the convergence of the solution u_ε in any dimension. This result was extended by Golse [21, Theorem 8] (see also [22]) for a more general velocity $b(x, x/\varepsilon)$ with $\mathrm{div}_y b(x,\cdot) = 0$, assuming the ergodicity of the flows associated with the vector fields $b(x,\cdot)$. Hou and Xin [25] performed the homogenization of (1.12) in dimension two with an oscillating initial condition $u_\varepsilon(0,x) = u^0(x, x/\varepsilon)$, assuming that b is a non vanishing divergence free vector field in $\mathbb{R}^2$ and that the flow (1.1) is ergodic. To this end, they used a two-scale convergence approach combined with Kolmogorov's theorem [28] involving some rotation number. This two-scale approach based on the divergence free condition was extended by Jabin and Tzaveras [26] using a kinetic decomposition in the two-scale procedure. Moreover, Tassa [34] extended the two-dimensional homogenization result of [25], assuming that the flow X associated with b has an invariant probability measure with a positive regular density with respect to Lebesgue's measure. More generally, Peirone [31] proved the convergence of the solution u_ε to equation (1.12) in dimension two, under the sole assumption that b does not vanish in Y_2. More recently, the first author proved in [9, Corollary 4.4] (see also [10] for an extension to the non periodic case) the homogenization of (1.12), replacing the classical ergodic condition by the rectification condition $\nabla\Psi\, b = \zeta$ for some C^2-diffeomorphism Ψ on Y_d, and a non null constant vector $\zeta \in \mathbb{R}^d$.

In the present case, extending the previous results in the case where the initial condition $u_\varepsilon(0,\cdot)$ does not oscillate, we prove a new result (see Theorem 5.1) on the homogenization of the linear transport equation (1.12) in any dimension, only assuming the singleton condition (without therefore assuming that the velocity of transport equation (1.12) is divergence free). The results of Section 2 and Section 3 provide various and rather general situations where the singleton condition applies:

- the case of the nonlinear current field in Proposition 2.4,
- some of the cases of Corollary 3.3 and Corollary 3.4,
- Corollary 3.5 illustrated by Example 4.3,
- Example 4.2 when $\alpha \geq 1$,
- the two-dimensional conductivity case of Example 4.4 under the ergodic condition (4.15).

The paper is organized as follows. In Section 2 we revisit the singleton condition and the Liouville theorem, from which we deduce the asymptotics of the flow (1.1) when b is a current field. In Section 3 we establish the perturbation Theorem 3.1 which is the main result of the paper, and we derive various corollaries on the set C_b and on the asymptotics of the flow X. Section 4 presents four examples which illustrate the results of Section 2 and Section 3. Section 5 is devoted to the homogenization of the transport equation (1.12) in connection with the asymptotics of the flow.

Notation

- $(e_1, \ldots, e_d)$ denotes the canonical basis of $\mathbb{R}^d$, and $0_{\mathbb{R}^d}$ denotes the null vector of $\mathbb{R}^d$.
- "$\cdot$" denotes the scalar product in $\mathbb{R}^N$, and $|\cdot|$ denotes the euclidian norm in $\mathbb{R}^N$.
- Y_d for $d \geq 1$, denotes the d-dimensional torus $\mathbb{R}^d/\mathbb{Z}^d$, which is identified to the cube $[0,1)^d$ in $\mathbb{R}^d$.
- $C_c^k(\mathbb{R}^d)$ for $k \in \mathbb{N} \cup \{\infty\}$, denotes the space of the real-valued functions in $C^k(\mathbb{R}^d)$ with compact support.
- $C_\sharp^k(Y_d)$ for $k \in \mathbb{N} \cup \{\infty\}$, denotes the space of the real-valued functions $f \in C^k(\mathbb{R}^d)$ which are $\mathbb{Z}^d$-periodic, *i.e.*
$$\forall\, \kappa \in \mathbb{Z}^d,\ \forall\, x \in \mathbb{R}^d, \quad f(x+\kappa) = f(x). \tag{1.13}$$
- $L_\sharp^p(Y_d)$ for $p \geq 1$, denotes the space of the real-valued functions in $L^p_{\rm loc}(\mathbb{R}^d)$ which are $\mathbb{Z}^d$-periodic.
- $\mathscr{M}(\mathbb{R}_d)$, resp. $\mathscr{M}(Y_d)$, denotes the space of the Radon measures on $\mathbb{R}^d$, resp. Y_d, and $\mathscr{M}_p(Y_d)$ denotes the space of the probability measures on Y_d.
- The notation $\mathscr{I}_b$ in (1.5) will be used throughout the paper.
- $\mathscr{D}'(\mathbb{R}^d)$ denotes the space of the distributions on $\mathbb{R}^d$.
- If a is a non negative measurable function in Y_d, the arithmetic mean $\overline{a}$ and the harmonic mean $\underline{a}$ of a are defined in $[0,\infty]$ by
$$\overline{a} := \int_{Y_d} a(y)\,dy \quad \text{and} \quad \underline{a} := \left(\int_{Y_d} \frac{dy}{a(y)}\right)^{-1}.$$

Definitions and recalls

Let $b : \mathbb{R}^d \to \mathbb{R}^d$ be a vector-valued function in $C_\sharp^1(Y_d)^d$. Consider the dynamical system
$$\begin{cases} \dfrac{\partial X}{\partial t}(t,x) = b(X(t,x)), \quad t \in \mathbb{R} \\ X(0,x) = x \in \mathbb{R}^d. \end{cases} \tag{1.14}$$
The solution $X(\cdot,x)$ to (1.14) which is known to be unique (see, *e.g.*, [24, Section 17.4]) induces the dynamic flow X defined by
$$\begin{array}{rrcl} X: & \mathbb{R} \times \mathbb{R}^d & \to & \mathbb{R}^d \\ & (t,x) & \mapsto & X(t,x), \end{array} \tag{1.15}$$
which satisfies the semi-group property
$$\forall\, s,t \in \mathbb{R},\ \forall\, x \in \mathbb{R}^d, \quad X(s+t,x) = X(s, X(t,x)). \tag{1.16}$$
The flow X is actually well defined in the torus Y_d, since
$$\forall\, t \in \mathbb{R},\ \forall\, x \in \mathbb{R}^d,\ \forall\, \kappa \in \mathbb{Z}^d, \quad X(t,x+\kappa) = X(t,x) + \kappa. \tag{1.17}$$

Property (1.17) follows immediately from the uniqueness of the solution $X(\cdot, x)$ to (1.14) combined with the $\mathbb{Z}^d$-periodicity of b.

For any $x \in Y_d$, the solution $X(\cdot, x)$ to (1.14) is said to be *periodic in the torus* Y_d if there exist $T > 0$ and $\kappa \in \mathbb{Z}^d$ such that

$$\forall\, t \in \mathbb{R}, \quad X(t+T, x) = X(t, x) + \kappa. \tag{1.18}$$

If $\kappa = 0_{\mathbb{R}^d}$ the solution is said to be *periodic in* $\mathbb{R}^d$.

2 Some variants of classical ergodicity results

2.1 The singleton result

Equivalences (1.11) have been obtained in [30] as a consequence of the so-called ergodic decomposition theorem. Here, we provide a simpler and more direct proof of (1.11), which is based on Proposition A.1 (in the Appendix) only involving the weak-$*$ compactness of $\mathscr{M}_p(Y_d)$. Also note that the uniform convergence result below is mentioned in [23, Section 9] with no reference.

Proposition 2.1 *Let $b \in C^1_\sharp(Y_d)^d$. Then, the following equivalence holds for any $\zeta \in \mathbb{R}^d$,*

$$\begin{aligned} \mathsf{C}_b = \{\zeta\} \;&\Leftrightarrow\; \forall\, x \in Y_d,\; \lim_{t\to\infty} \frac{X(t,x)}{t} = \zeta \\ &\Leftrightarrow\; \frac{X(t,\cdot)}{t} \textit{ converges uniformly as } t \to \infty \textit{ to } \zeta \textit{ on } Y_d. \end{aligned} \tag{2.1}$$

Proof of Proposition 2.1. First, assume that $\mathsf{C}_b = \{\zeta\}$. Assume by contradiction that the second right hand-side of (2.1) does not hold. Then, there exists $\varepsilon > 0$ such that for any $n \in \mathbb{N}$, there exist a number $r_n > n$ and a point $y_n \in Y_d$ satisfying

$$\left| \frac{X(r_n, y_n) - y_n}{r_n} - \zeta \right| = \left| \frac{1}{r_n} \int_0^{r_n} b(X(s, y_n))\, ds - \zeta \right| \geq \varepsilon. \tag{2.2}$$

Now, let ν_n for $n \in \mathbb{N}$, be the probability measure on Y_d defined by

$$\int_{Y_d} f(y)\, d\nu_n(y) = \frac{1}{r_n} \int_0^{r_n} f(X(s, y_n))\, ds \quad \text{for } f \in C^0_\sharp(Y_d)^d. \tag{2.3}$$

By virtue of Lemma A.2 there exists a subsequence $(\nu_{n_k})_{k\in\mathbb{N}}$ of $(\nu_n)_{n\in\mathbb{N}}$ which converges weakly $*$ to some probability measure $\mu \in \mathscr{M}_p(Y_d)$ which is invariant for the flow X. Hence, passing to the limit as $n_k \to \infty$ both in (2.2) and (2.3) with $f := b$, we deduce from $\mathsf{C}_b = \{\zeta\}$ that

$$|\zeta - \zeta| = \left| \int_{Y_d} b(y)\, d\mu(y) - \zeta \right| \geq \varepsilon > 0,$$

which yields a contradiction.
It is clear that the second right-hand side of (2.1) implies the first one.

Finally, let us assume that the first right-hand side of (2.1) holds true with ζ, and let us prove that $\mathsf{C}_b = \{\zeta\}$. We thus have

$$\forall\, x \in Y_d, \quad \lim_{t\to\infty} \left(\frac{1}{t} \int_0^t b(X(s, x))\, ds \right) = \zeta.$$

Then, integrating over Y_d the former equality with respect to any probability measure $\mu \in \mathscr{I}_b$, then applying successively Lebesgue's dominated convergence theorem and Fubini's theorem, we get that

$$\begin{aligned}\zeta &= \lim_{t\to\infty} \int_{Y_d} \left(\frac{1}{t}\int_0^t b(X(s,x))\,ds\right) d\mu(x)\\ &= \lim_{t\to\infty} \frac{1}{t}\int_0^t \left(\int_{Y_d} b(X(s,x))\,d\mu(x)\right) ds = \int_{Y_d} b(x)\,d\mu(x),\end{aligned}$$

which shows that $\mathsf{C}_b = \{\zeta\}$. This concludes the proof of (2.1). □

Remark 2.1 *Stability of the singleton condition by a diffeomorphism on the torus.*
A mapping $\Psi \in C^1(\mathbb{R}^d)^d$ is said to be a C^1-diffeomorphism on Y_d if Ψ satisfies the following conditions:

- $\det(\nabla\Psi(x)) \neq 0$ *for any $x \in \mathbb{R}^d$,*
- *there exist a matrix $A \in \mathbb{Z}^{d\times d}$ with $|\det(A)| = 1$, and a mapping $\Psi_\sharp \in C^1_\sharp(Y_d)^d$ such that*

$$\forall\, x \in \mathbb{R}^d, \quad \Psi(x) = Ax + \Psi_\sharp(x). \tag{2.4}$$

Note that the invertibility of A and the $\mathbb{Z}^d$-periodicity of $\Psi_\sharp$ in (2.4) *imply that Ψ is a proper function (*i.e.*, the inverse image by the function of any compact set in $\mathbb{R}^d$ is a compact set). Hence, by virtue of Hadamard-Caccioppoli's theorem [13] (also called Hadamard-Lévy's theorem) the mapping Ψ is actually a C^1-diffeomorphism on $\mathbb{R}^d$. Also note that due to $A^{-1} \in \mathbb{Z}^{d\times d}$, we have*

$$\forall\, \kappa \in \mathbb{Z}^d,\ \forall\, x \in Y_d, \quad \begin{cases} \Psi(x+\kappa) - \Psi(x) = A\kappa & \in \mathbb{Z}^d \\ \Psi^{-1}(x+\kappa) - \Psi^{-1}(x) = A^{-1}\kappa & \in \mathbb{Z}^d, \end{cases}$$

hence Ψ well defines an isomorphism on the torus.

Now, let b be a vector field in $C^1_\sharp(Y_d)^d$ and let Ψ be a C^2-diffeomorphism on Y_d. Define the flow $\tilde{X}$ obtained from Ψ by

$$\tilde{X}(t,x) := \Psi\big(X(t,\Psi^{-1}(x))\big) \quad \text{for } (t,x) \in \mathbb{R}\times Y_d. \tag{2.5}$$

Using the chain rule it is easy to check that the mapping $\tilde{X}$ is the flow associated with the vector field $\tilde{b} \in C^1_\sharp(Y_d)^d$ defined by

$$\tilde{b}(x) = \nabla\Psi(\Psi^{-1}(x))\, b(\Psi^{-1}(x)) \quad \text{for } x \in Y_d. \tag{2.6}$$

Combining (2.4) *and* (2.5) *we clearly have*

$$\forall\, x \in Y_d, \quad \lim_{t\to\infty} \frac{\tilde{X}(t,x)}{t} \text{ exists} \;\Leftrightarrow\; \lim_{t\to\infty} \frac{X(t,\Psi^{-1}(x))}{t} \text{ exists},$$

and in the case of existence of the limit for a given $x \in Y_d$, we get the equality

$$\lim_{t\to\infty} \frac{\tilde{X}(t,x)}{t} = A\left(\lim_{t\to\infty} \frac{X(t,\Psi^{-1}(x))}{t}\right).$$

This combined with equivalence (2.1) *implies that*

$$\#\mathsf{C}_{\tilde{b}} = 1 \;\Leftrightarrow\; \#\mathsf{C}_b = 1, \tag{2.7}$$

and in this case we obtain the equality $\mathsf{C}_{\tilde{b}} = A\,\mathsf{C}_b$. *Therefore, the singleton condition is stable by any* C^2*-diffeomorphism on* Y_d.

In particular, such a diffeomorphism has been used by Tassa [34] in dimension two, assuming that the first coordinate b_1 *of* b *does not vanish in* Y_2 *and that there exists an invariant probability measure for the flow associated with* b *having a density* $\sigma \in C^1_\sharp(Y_2)$ *with respect to Lebesgue's measure. In this case a variant [34, Theorem 2.3] of Kolmogorov's theorem [28] (which holds under the weaker assumption that* b *is non vanishing) provides a diffeomorphism on* Y_2 *which rectifies the vector field* b *to a vector field* $\tilde{b} = a\,\xi$ *with a positive function* $a \in C^1_\sharp(Y_2)$ *and a fixed direction* $\xi \in \mathbb{R}^d$. *Under the additional ergodic assumption that the coordinates of* ξ *are rationally independent, the singleton condition is shown to be satisfied [34, Theorem 4.2]. Corollary 3.4 below provides a more general result in dimension two with a vanishing vector field* b*, without using a rectification of the vector field* b.

2.2 A divergence-curl result

Liouville's theorem provides a criterium for a probability measure on a smooth compact manifold in $\mathbb{R}^d$ (see, *e.g.*, [15, Theorem 1, Section 2.2]) to be invariant for the flow. The next result revisits this theorem in $\mathscr{M}_p(Y_d)$ in association with a divergence-curl result on the torus.

Proposition 2.2 *Let* $b \in C^1_\sharp(Y_d)^d$ *and let* $\mu \in \mathscr{M}_p(Y_d)$. *Define the Borel measure* $\tilde{\mu}$ *on* $\mathbb{R}^d$ *by*

$$\int_{\mathbb{R}^d} \varphi(x)\, d\tilde{\mu}(x) := \int_{Y_d} \varphi_\sharp(y)\, d\mu(y) \quad \textit{where} \quad \varphi_\sharp(\cdot) := \sum_{\kappa\in\mathbb{Z}^d} \varphi(\cdot+\kappa) \quad \textit{for } \varphi \in C^0_c(\mathbb{R}^d). \tag{2.8}$$

Then, the three following assertions are equivalent:

(i) μ *is invariant for the flow* X, i.e. (1.2) *holds,*

(ii) $\tilde{\mu}\, b$ *is divergence free in* $\mathbb{R}^d$, i.e.

$$\operatorname{div}(\tilde{\mu}\, b) = 0 \quad \textit{in } \mathscr{D}'(\mathbb{R}^d), \tag{2.9}$$

(iii) $\mu\, b$ *is divergence free in* Y_d, i.e.

$$\forall\, \psi \in C^1_\sharp(Y_d), \quad \int_{Y_d} b(y)\cdot\nabla\psi(y)\, d\mu(y) = 0. \tag{2.10}$$

Proof of Proposition 2.2.

Proof of (i) $\Rightarrow$ *(ii).* Assume that μ is invariant for the flow, *i.e.* (1.2). Let $\varphi \in C^1_c(\mathbb{R}^d)$. Since by (1.17) we have for any $t \in \mathbb{R}$ and $y \in \mathbb{R}^d$,

$$\big[\varphi(X(t,\cdot))\big]_\sharp(y) = \sum_{\kappa\in\mathbb{Z}^d} \varphi(X(t,y+\kappa)) = \sum_{\kappa\in\mathbb{Z}^d} \varphi(X(t,y)+\kappa) = \varphi_\sharp(X(t,y)), \tag{2.11}$$

it follows from (2.8) and the invariance of μ that

$$\forall\, t \in \mathbb{R}, \quad \int_{\mathbb{R}^d} \varphi(X(t,x))\, d\tilde{\mu}(x) = \int_{Y_d} \big[\varphi(X(t,\cdot))\big]_\sharp(y)\, d\mu(y) = \int_{Y_d} \varphi_\sharp(X(t,y))\, d\mu(y) = \int_{Y_d} \varphi_\sharp(y)\, d\mu(y) = \int_{\mathbb{R}^d} \varphi(x)\, d\tilde{\mu}(x).$$

Taking the derivative of the former expression with respect to t, we get that

$$\forall\, t \in \mathbb{R}, \quad \int_{\mathbb{R}^d} b(X(t,x)) \cdot \nabla\varphi(X(t,x))\, d\tilde{\mu}(x) = 0,$$

which at $t = 0$ yields

$$\forall\, \varphi \in C^1_c(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} b(x) \cdot \nabla\varphi(x)\, d\tilde{\mu}(x) = 0, \tag{2.12}$$

namely the variational formulation of the distributional equation (2.9).

Proof of $(ii) \Rightarrow (i)$. Conversely, assume that equation (2.9) holds true, and let us prove that μ is invariant for the flow X. Let $\varphi \in C^1_c(\mathbb{R}^d)$ and define the function $\phi \in C^1(\mathbb{R} \times \mathbb{R}^d)$ by $\phi(t,x) := \varphi(X(t,x))$. By the semi-group property (1.16) we have for any $s, t \in \mathbb{R}$ and $x \in \mathbb{R}^d$,

$$\begin{aligned} \frac{\partial}{\partial s}\big(\phi(s+t, X(-s,x))\big) &= \frac{\partial}{\partial s}\big(\phi(t,x)\big) = 0 \\ &= \frac{\partial \phi}{\partial s}(s+t, X(-s,x)) - b(X(-s,x)) \cdot \nabla_x \phi(s+t, X(-s,x)), \end{aligned}$$

which at $s = 0$ gives the classical transport equation

$$\forall\, t \in \mathbb{R},\ \forall\, x \in Y_d, \quad \frac{\partial \phi}{\partial t}(t,x) = b(x) \cdot \nabla_x \phi(t,x). \tag{2.13}$$

Hence, since $\varphi(X(t,\cdot))$ is in $C^1(\mathbb{R}^d)$ and has a compact support independent of t when t lies in a compact set of $\mathbb{R}$, we deduce from (2.13) and (2.9) that

$$\forall\, t \in \mathbb{R}, \quad \frac{d}{dt}\left(\int_{\mathbb{R}^d} \varphi(X(t,x))\, d\tilde{\mu}(x)\right) = \int_{\mathbb{R}^d} b(x) \cdot \nabla_x\big(\varphi(X(t,x))\big)\, d\tilde{\mu}(x) = 0,$$

or equivalently,

$$\forall\, t \in \mathbb{R}, \quad \int_{\mathbb{R}^d} \varphi(X(t,x))\, d\tilde{\mu}(x) = \int_{\mathbb{R}^d} \varphi(x)\, d\tilde{\mu}(x).$$

On the other hand, we have the following result.

Lemma 2.3 ([8], Lemma 3.5) *For any smooth function $\psi \in C^\infty_\sharp(Y_d)$ defined in Y_d, there exists a smooth function $\varphi \in C^\infty_c(\mathbb{R}^d)$ with compact support in $\mathbb{R}^d$ such that $\psi = \varphi_\sharp$.*

Hence, using relation (2.11) and definition (2.8) we get that for any $\psi \in C^\infty_\sharp(Y)$,

$$\begin{aligned} \forall\, t \in \mathbb{R}, \quad \int_{Y_d} \psi(X(t,y))\, d\mu(y) &= \int_{Y_d} \varphi_\sharp(X(t,y))\, d\mu(y) = \int_{\mathbb{R}^d} [\varphi \circ X(t,\cdot)]_\sharp(y)\, d\mu(y) = \\ = \int_{\mathbb{R}^d} \varphi(X(t,x))\, d\tilde{\mu}(x) &= \int_{\mathbb{R}^d} \varphi(x)\, d\tilde{\mu}(x) = \int_{Y_d} \varphi_\sharp(y)\, d\mu(y) = \int_{Y_d} \psi(y)\, d\mu(y), \end{aligned}$$

which shows that μ is invariant for the flow X. We have just proved the equivalence between the invariance of μ for the flow and the distributional equation (2.9) satisfied by $\tilde{\mu}$.

Proof of $(ii) \Leftrightarrow (iii)$. The equivalence between (2.9), or equivalently (2.12), and (2.10) is a straightforward consequence of the following relation (which is deduced from $[b \cdot \nabla\varphi]_\sharp = b \cdot \nabla\varphi_\sharp$ and (2.8))

$$\forall\, \varphi \in C^1_c(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} b(x) \cdot \nabla\varphi(x)\, d\tilde{\mu}(x) = \int_{Y_d} b(y) \cdot \nabla\varphi_\sharp(y)\, d\mu(y),$$

combined with Lemma 2.3. This concludes the proof of Proposition 2.2. $\square$

Remark 2.2 *Equation* (2.10) *can be considered as the divergence free of the vector-valued measure* $\mu\, b$ *in the torus* Y_d*, while equation* (2.9) *is exactly the divergence free of the vector-valued measure* $\tilde{\mu}\, b$ *in the space* $\mathbb{R}^d$*. Equation* (2.10) *is also equivalent to*

$$\forall\, (\nabla\psi) \in C^0_\sharp(Y_d)^d, \quad \int_{Y_d} b(y)\cdot\nabla\psi(y)\, d\mu(y) = \left(\int_{Y_d} b(y)\, d\mu(y)\right)\cdot\left(\int_{Y_d}\nabla\psi(y)\, dy\right), \tag{2.14}$$

since

$$\nabla\psi \in C^0_\sharp(Y_d)^d \;\Leftrightarrow\; \left(x \mapsto \psi(x) - x\cdot\int_{Y_d}\nabla\psi(y)\, dy\right) \in C^1_\sharp(Y_d).$$

So, condition (2.14) *may be regarded as a divergence-curl result involving the divergence free vector field* $\mu\, b$ *with the invariant probability measure* μ *and the gradient field* $\nabla\psi$ *with Lebesgue's measure.*

2.3 The case of a nonlinear current field

As a direct consequence of Proposition 2.1 and Proposition 2.2, the following result gives the asymptotics of the ODE's flow (1.14) when b is a (non necessarily divergence free) nonlinear current field.

Proposition 2.4 *Let* $v \in C^2(\mathbb{R}^d)$ *be a function such that*

$$\nabla v \in C^1_\sharp(\mathbb{R}^d)^d \quad \text{and} \quad \#\left(\left\{x \in Y_d : \nabla v(x) = \overline{\nabla v}\right\}\right) < \infty, \tag{2.15}$$

and let $F(x,\xi) \in C^1_\sharp(Y_d; C^1(\mathbb{R}^d))^d$ *be a vector-valued function satisfying*

$$\begin{cases} \forall\, x \in Y_d, \quad F(x,\overline{\nabla v}) = 0_{\mathbb{R}^d} \\ \forall\, (x,\xi,\eta) \in Y_d\times\mathbb{R}^d\times\mathbb{R}^d, \; \xi\neq\eta, \quad \big(F(x,\xi) - F(x,\eta)\big)\cdot(\xi-\eta) > 0. \end{cases} \tag{2.16}$$

Then, the vector field $b \in C^1_\sharp(Y_d)^d$ *defined by*

$$b(x) := F(x,\nabla v(x)) \quad \text{for } x \in Y_d, \tag{2.17}$$

satisfies

$$\mathsf{C}_b = \{0_{\mathbb{R}^d}\}. \tag{2.18}$$

Proof of Proposition 2.4. Let μ be an invariant probability measure on Y_d for the flow X associated with the vector field b (2.17). By virtue of the divergence-curl result (2.10) we have

$$\int_{Y_d} b(x)\cdot\nabla v(x)\, d\mu(x) = \int_{Y_d} F(x,\nabla v(x))\cdot\nabla v(x)\, d\mu(x) = \left(\int_{Y_d} F(x,\nabla v(x))\, d\mu(x)\right)\cdot\overline{\nabla v}.$$

This combined with $F(\cdot,\overline{\nabla v}) = 0_{\mathbb{R}^d}$ and the monotonicity (2.16) yields

$$\int_{Y_d} \underbrace{\big(F(x,\nabla v(x)) - F(x,\overline{\nabla v})\big)\cdot\big(\nabla v(x) - \overline{\nabla v}\big)}_{\geq 0}\, d\mu(x) = 0, \tag{2.19}$$

which implies that

$$\big(F(x,\nabla v(x)) - F(x,\overline{\nabla v})\big)\cdot\big(\nabla v(x) - \overline{\nabla v}\big) = 0 \quad d\mu(x)\text{-a.e.}$$

Hence, since $F(x, \cdot)$ is strictly monotonic in the sense of (2.16), we deduce that

$$\nabla v(x) = \overline{\nabla v} \quad d\mu(x)\text{-a.e.}$$

Therefore, due to (2.15) the measure μ is a convex combination of the Dirac masses

$$\mu = \sum_{x \in \{\nabla v = \overline{\nabla v}\}} c_x \, \delta_x \qquad \text{with} \qquad c_x \geq 0 \quad \text{and} \sum_{x \in \{\nabla v = \overline{\nabla v}\}} c_x = 1.$$

Finally, this combined with $F(\cdot, \overline{\nabla v}) = 0_{\mathbb{R}^d}$ implies that

$$\int_{Y_d} b(y) \, d\mu(y) = \sum_{x \in \{\nabla v = \overline{\nabla v}\}} c_x \, F(x, \nabla v(x)) = \sum_{x \in \{\nabla v = \overline{\nabla v}\}} c_x \, F(x, \overline{\nabla v}) = 0_{\mathbb{R}^d},$$

which leads us to (2.18). $\square$

Exemple 2.1 *A class of functions F satisfying* (2.16) *is given by*

$$F(x, \xi) = \nabla_\xi f(x, \xi) \quad \textit{for } (x, \xi) \in Y_d \times \mathbb{R}^d,$$

where $f \in C^1_\sharp(Y_d; C^2(\mathbb{R}^d))$, and for any $x \in Y_d$, the function $f(x, \cdot)$ is strictly convex in $\mathbb{R}^d$ with $\overline{\nabla v}$ as unique minimizer.

For example, the vector field b is the linear current field

$$b(x) = A(x) \nabla v(x) = \nabla_\xi f(x, \nabla v(x)) \quad \textit{for } x \in Y_d,$$

when f is the non negative quadratic functional defined by

$$f(x, \xi) := \frac{1}{2} A(x) \, \xi \cdot \xi \quad \textit{for } (x, \xi) \in Y_d \times \mathbb{R}^d,$$

for any non negative symmetric matrix-valued function $A \in C^1_\sharp(Y_d)^{d \times d}$.

Note that in this case, the finite set condition of (2.15) *and the strict convexity of f can be replaced by the unique condition $\overline{\nabla v} = 0_{\mathbb{R}^d}$, or equivalently, v is $\mathbb{Z}^d$-periodic. Indeed, let $\mu \in \mathscr{I}_b$ be an invariant probability measure for the flow X. Similarly as* (2.19)*, by the divergence-curl relation* (2.10) *we have*

$$\int_{Y_d} \underbrace{A(y) \nabla v(y) \cdot \nabla v(y)}_{\geq 0} \, d\mu(y) = \int_{Y_d} b(y) \cdot \nabla v(y) \, d\mu(y) = 0,$$

which implies that $A \nabla v \cdot \nabla v = 0$ μ-a.e. in Y_d. However, since the matrix-valued A is symmetric and non negative, from the Cauchy-Schwarz inequality we deduce that $A \nabla v = 0$ μ-a.e. in Y_d, and thus

$$\int_{Y_d} b(y) \, d\mu(y) = \int_{Y_d} A(y) \nabla v(y) \, d\mu(y) = 0_{\mathbb{R}^d}.$$

Therefore, we obtain the desired equality (2.18).

3 Some new results involving the singleton condition

3.1 A perturbation result

The main result of the paper is the following.

Theorem 3.1 *Let $b \in C^1_\sharp(Y_d)^d$ be such that $b = \rho\,\Phi$, where ρ is a non negative not null function in $C^1_\sharp(Y_d)$ and Φ is a non vanishing vector field in $C^1_\sharp(Y_d)^d$. Assume that there exists a sequence $(\rho_n)_{n\in\mathbb{N}} \in C^1_\sharp(Y_d)^{\mathbb{N}}$ such that*

(i) for any $n \in \mathbb{N}$, $\rho \le \rho_n > 0$ in Y_d, and $(\rho_n)_{n\in\mathbb{N}}$ converges uniformly to ρ on Y_d,

(ii) for any $n \in \mathbb{N}$, $\mathsf{C}_{b_n} = \{\zeta_n\}$ for some $\zeta_n \in \mathbb{R}^d$, where $b_n := \rho_n\,\Phi$.

Then, the sequence $(\zeta_n)_{n\in\mathbb{N}}$ converges to some $\zeta \in \mathbb{R}^d$, and we have the following properties:

- *If ρ is positive in Y_d, then $\mathsf{C}_b = \{\zeta\}$. If in addition $d = 2$, then $\zeta \neq 0_{\mathbb{R}^2}$.*
- *If ρ vanishes in Y_d, then $\mathsf{C}_b = [0_{\mathbb{R}^d}, \zeta]$. Moreover, we have $\{0_{\mathbb{R}^d}, \zeta\} \subset \mathsf{A}_b$.*

Remark 3.1 *In dimension two and in the second alternative of Theorem 3.1, it is not surprising to obtain for the rotation set C_b a closed line segment with one end at $0_{\mathbb{R}^2}$. Indeed, Franks and Misiurewicz [17, Theorem 1.2] proved that the rotation set of any two-dimensional continuous flow is always a closed line segment of a line passing through $0_{\mathbb{R}^2}$ or a unit set. Moreover, this segment has one end at $0_{\mathbb{R}^2}$, when it has an irrational slope. However, our perturbation result provides such a closed line segment in any dimension and for any non zero slope.*

The proof of Theorem 3.1 is based on the following lemma.

Lemma 3.1 *Let $b = \rho\,\Phi$ be a vector field satisfying the assumptions of Theorem 3.1. Also assume that there exists an invariant probability measure $\mu \in \mathscr{I}_b$ for the flow associated with b such that*

$$\int_{Y_d} \rho(x)\,d\mu(x) > 0. \tag{3.1}$$

Then, the sequence $(\zeta_n)_{n\in\mathbb{N}}$ converges to some $\zeta \in \mathbb{R}^d$. Moreover, for any $\mu \in \mathscr{I}_b$ satisfying (3.1), the two following properties hold true:

$$\int_{Y_d} b(x)\,d\mu(x) = \beta_\mu\,\zeta \quad \textit{where} \quad \beta_\mu := \int_{\{\rho>0\}} d\mu(x)\ \in (0,1], \tag{3.2}$$

$$\lim_{t\to\infty} \frac{X(t,x)}{t} = \zeta \quad \textit{for } \mu\textit{-a.e. } x \in \{\rho > 0\}. \tag{3.3}$$

Proof of Lemma 3.1. Let $\mu \in \mathscr{I}_b$ be an invariant probability measure satisfying (3.1). For any $n \in \mathbb{N}$, define the probability measure μ_n on Y_d by

$$d\mu_n(x) := C_n\,\frac{\rho(x)}{\rho_n(x)}\,d\mu(x), \quad \text{where} \quad C_n := \left(\int_{Y_d} \frac{\rho(y)}{\rho_n(y)}\,d\mu(y)\right)^{-1} \in (0,\infty) \tag{3.4}$$

due to $\rho_n > 0$ and (3.1). Since $\mu \in \mathscr{I}_b$, it follows from equality (2.10) that

$$\forall\,\varphi \in C^1_\sharp(Y_d), \qquad \int_{Y_d} \rho_n(x)\,\Phi(x)\cdot\nabla\varphi(x)\,d\mu_n(x) = C_n \int_{Y_d} \rho(x)\,\Phi(x)\cdot\nabla\varphi(x)\,d\mu(x) = 0,$$

which implies that $\mu_n \in \mathscr{I}_{b_n}$ by the equivalence (i)-(iii) of Proposition 2.2. Then, from equality $\mathsf{C}_{b_n} = \{\zeta_n\}$ we deduce that

$$\zeta_n = \int_{Y_d} b_n(x)\, d\mu_n(x) = C_n \int_{Y_d} \rho_n(x)\, \Phi(x) \frac{\rho(x)}{\rho_n(x)}\, d\mu(x) = C_n \int_{Y_d} b(x)\, d\mu(x).$$

Moreover, by Lebesgue's theorem and (3.1) we get that

$$\lim_{n\to\infty} C_n = c_\mu := \left(\int_{\{\rho>0\}} d\mu(x) \right)^{-1} \in [1, \infty).$$

Therefore, the sequence $(\zeta_n)_{n\in\mathbb{N}}$ converges to some vector $\zeta \in \mathbb{R}^d$ which is independent of μ. Moreover, equality (3.2) holds with $\beta_\mu := 1/c_\mu \in (0, 1]$.

Now, let us prove (3.3). By Birkhoff's theorem there exists $g : Y_d \to \mathbb{R}^d$ a measurable function which by (1.8) and (1.10) satisfies

$$\lim_{t\to\infty} \frac{X(t,x)}{t} = g(x) \in \mathsf{A}_b \subset \mathsf{C}_b \quad \text{for } \mu\text{-a.e. } x \in Y_d.$$

(Also see Proposition A.1 of the Appendix to get that $g(x) \in \mathsf{C}_b$). Hence, by virtue of (3.2) we have

$$\lim_{t\to\infty} \frac{X(t,x)}{t} = \Delta(x)\, \zeta \quad \text{for } \mu\text{-a.e. } x \in Y_d, \tag{3.5}$$

where $\Delta : Y_d \to [0, 1]$ is a measurable function. Note that, if $\zeta = 0_{\mathbb{R}^d}$, then asymptotics (3.5) clearly implies (3.3). Now, assume that $\zeta \neq 0_{\mathbb{R}^d}$. Applying successively convergence (3.5), Lebesgue's theorem, Fubini's theorem and the invariance of the measure μ, we get that

$$\begin{aligned}
&\left(\int_{Y_d} \Delta(x)\, d\mu(x) \right) \zeta = \lim_{t\to\infty} \int_{Y_d} \frac{X(t,x)}{t}\, d\mu(x) = \lim_{t\to\infty} \int_{Y_d} \left(\frac{1}{t} \int_0^t b(X(s,x))\, ds \right) d\mu(x) \\
&= \lim_{t\to\infty} \frac{1}{t} \int_0^t \left(\int_{Y_d} b(X(s,x))\, d\mu(x) \right) ds = \int_{Y_d} b(x)\, d\mu(x),
\end{aligned}$$

or equivalently,

$$\int_{\{\rho>0\}} b(x)\, d\mu(x) = \left(\int_{\{\rho>0\}} \Delta(x)\, d\mu(x) \right) \zeta,$$

since $\rho(x) = 0$ implies both that $b(x) = 0_{\mathbb{R}^d}$ and $\Delta(x) = 0$ due to $X(\mathbb{R}, x) = \{x\}$. Moreover, equalities (3.2) give

$$\int_{Y_d} b(x)\, d\mu(x) = \int_{\{\rho>0\}} b(x)\, d\mu(x) = \beta_\mu\, \zeta,$$

which due to $\zeta \neq 0_{\mathbb{R}^d}$ implies that

$$\int_{\{\rho>0\}} d\mu(x) = \int_{\{\rho>0\}} \Delta(x)\, d\mu(x),$$

or equivalently,

$$\int_{\{\rho>0\}} \big(1 - \Delta(x)\big)\, d\mu(x) = 0.$$

Since the function Δ takes values in $[0, 1]$, the former equality implies that $\Delta = 1$ for μ-a.e. in $\{\rho > 0\}$, which combined with (3.5) yields the desired limit (3.3). $\square$

Proof of Theorem 3.1. The proof is split according to the two cases $\mathsf{C}_b \neq \{0_{\mathbb{R}^d}\}$ and $\mathsf{C}_b = \{0_{\mathbb{R}^d}\}$.

Case $\mathsf{C}_b \neq \{0_{\mathbb{R}^d}\}$. First of all, note that if $\mu \in \mathscr{I}_b$ satisfies

$$\int_{Y_d} b(x)\, d\mu(x) \neq 0_{\mathbb{R}^d}, \tag{3.6}$$

then, we have

$$\int_{Y_d} \rho(x)\, d\mu(x) = \int_{Y_d} \frac{|b(x)|}{|\Phi(x)|}\, d\mu(x) > 0, \tag{3.7}$$

namely (3.1) holds true.

By Lemma 3.1 the sequence $(\zeta_n)_{n\in\mathbb{N}}$ converges to some $\zeta \in \mathbb{R}^d$.

On the one hand, if ρ is positive in Y_d, then for any $\mu \in \mathscr{I}_b$ satisfying (3.6) and thus (3.7), we have $\beta_\mu = 1$ in (3.2) so that

$$\zeta = \int_{Y_d} b(x)\, d\mu(x) \neq 0_{\mathbb{R}^d}.$$

The convexity of C_b thus implies that $\mathsf{C}_b = \{\zeta\}$.

On the other hand, if ρ vanishes in Y_d, *i.e.* if there exists $\alpha \in Y_d$ such that $\rho(\alpha) = 0$, then $0_{\mathbb{R}^d} \in \mathsf{C}_b$, since the Dirac distribution δ_α at the point α is invariant for the flow associated with b. Moreover, take $\mu \in \mathscr{I}_b$ satisfying (3.6) (recall that $\mathsf{C}_b \neq \{0_{\mathbb{R}^d}\}$). Then, by (3.7) we have

$$\mu\big(\{\rho > 0\}\big) > 0.$$

Hence, by (3.3) and (1.10) we get that $\zeta \in \mathsf{A}_b \subset \mathsf{C}_b$. Therefore, from (3.2) and from the convexity of C_b we deduce that $\mathsf{C}_b = [0_{\mathbb{R}^d}, \zeta]$ with $\zeta \neq 0_{\mathbb{R}^d}$.

Case $\mathsf{C}_b = \{0_{\mathbb{R}^d}\}$. Note that the function ρ may either be positive or vanish in Y_d (except in dimension $d = 2$, see the end of the proof), and that we only have to prove that the sequence $(\zeta_n)_{n\in\mathbb{N}}$ converges to $0_{\mathbb{R}^d}$. To this end consider a subsequence $(\zeta_{n_k})_{k\in\mathbb{N}}$ which converges to some $\hat{\zeta} \in \mathbb{R}^d$. Up to extract a new subsequence, we may assume that the sequence $(\mu_{n_k})_{k\in\mathbb{N}}$ converges weakly $*$ to some probability measure $\hat{\mu}$ on Y_d. Passing to the limit in the divergence-curl relation (2.10) satisfied by $\mu_{n_k} \in \mathscr{I}_{b_{n_k}}$:

$$\forall\, \varphi \in C^1_\sharp(Y_d), \qquad \int_{Y_d} b_{n_k}(x) \cdot \nabla\varphi(x)\, d\mu_{n_k}(x) = 0, \tag{3.8}$$

and using the uniform convergence of b_n to b combined with the fact that μ_n is a probability measure for any $n \in \mathbb{N}$, we get that

$$\forall\, \varphi \in C^1_\sharp(Y_d), \qquad \int_{Y_d} b(x) \cdot \nabla\varphi(x)\, d\hat{\mu}(x) = 0, \tag{3.9}$$

so that $\hat{\mu} \in \mathscr{I}_b$ again by the equivalence (i)-(iii) of Proposition 2.2. Similarly, we have

$$\hat{\zeta} = \lim_{k\to\infty} \zeta_{n_k} = \lim_{k\to\infty} \int_{Y_d} b_{n_k}(x)\, d\mu_{n_k}(x) = \int_{Y_d} b(x)\, d\hat{\mu}(x) = 0_{\mathbb{R}^d},$$

where the last equality follows from $\hat{\mu} \in \mathscr{I}_b$ and $\mathsf{C}_b = \{0_{\mathbb{R}^d}\}$. Therefore, the whole sequence $(\zeta_n)_{n\in\mathbb{N}}$ converges to $0_{\mathbb{R}^d}$.

Finally, let us focus on the specific two-dimensional case when $\rho > 0$ in Y_2 and $\mathsf{C}_b = \{\zeta\}$. Assume by contradiction that $\zeta = 0_{\mathbb{R}^2}$. Hence, by the second equality of (1.9) there exists an ergodic invariant probability measure $\mu \in \mathscr{E}_b$ for the flow X associated with b such that

$$\int_{Y_2} b(x)\, d\mu(x) = 0_{\mathbb{R}^2}.$$

Now, consider the homeomorphism F on Y_2 homothopic to identity whose associated lift in $\mathbb{R}^2$ is given by $\widetilde{F} := X(1,\cdot)$, *i.e.* $F \circ \pi = \pi \circ \widetilde{F}$ where π is the canonical projection from $\mathbb{R}^2$ on Y_2. Then, by Fubini's theorem and by the invariance of μ we have

$$\begin{aligned}\int_{Y_2} \big(\widetilde{F}(x) - x\big)\, d\mu(x) &= \int_{Y_2} \big(X(1,x) - x\big)\, d\mu(x) = \int_{Y_2} \left(\int_0^1 b(X(t,x))\, dt\right) d\mu(x) \\ &= \int_0^1 \left(\int_{Y_2} b(X(t,x))\, d\mu(x)\right) = \int_{Y_2} b(x)\, d\mu(x) = 0_{\mathbb{R}^2},\end{aligned}$$

namely the mean translation of the lift $\widetilde{F}$ with respect to the ergodic invariant probability measure μ is $0_{\mathbb{R}^2}$. Therefore, by virtue of [16, Theorem 3.5] the homeomorphism F has a fixed point x_0 in Y_2, *i.e.* there exists $k \in \mathbb{Z}^2$ such that

$$X(1,x_0) = x_0 + k,$$

which implies that

$$\lim_{n\to\infty} \frac{X(n,x_0)}{n} = k.$$

However, since C_b is the unit set $\{0_{\mathbb{R}^2}\}$, we deduce from (2.1) that $k = 0_{\mathbb{R}^2}$. Hence, we get that $X(1,x_0) = x_0$, *i.e.* the trajectory $X(\cdot,x_0)$ is periodic in $\mathbb{R}^2$. Therefore, by the preliminary remark to the proof of [31, Theorem 3.1] this periodicity implies that the vector field $b = \rho\,\Phi$ does vanish in $\mathbb{R}^2$, a contradiction. We have just proved that $\zeta \neq 0_{\mathbb{R}^2}$.

The proof of Theorem 3.1 is now complete. □

Remark 3.2 *In the setting of Theorem 3.1, Example 4.2 below provides a two-dimensional case of vector field $b = \rho\,\Phi$ with a vanishing function ρ, in which C_b is a closed line segment of $\mathbb{R}^2$ not reduced to a singleton, and thus $\#\mathsf{A}_b \geq 2$.*

Remark 3.3 *Theorem 3.1 cannot be extended to the more general case where the direction Φ of b_n also depends on n, as shown in Example 4.1 below. More precisely, the independence of Φ with respect to n is crucial for the proof of Theorem 3.1 to build in (3.4) the invariant probability measure $\mu_n \in \mathscr{I}_{b_n}$ from a given invariant probability measure $\mu \in \mathscr{I}_b$.*

Corollary 3.1 *Let Φ be a non vanishing vector field in $C^1_\sharp(Y_d)^d$. Then, we have for the uniform convergence topology in $C^0_\sharp(Y_d)$,*

$$\begin{gathered}\big\{\rho \in C^1_\sharp(Y_d) : \rho > 0 \text{ and } \#\,\mathsf{C}_{\rho\Phi} = 1\big\} = \big\{\rho \in C^1_\sharp(Y_d) : \rho > 0 \text{ and } \#\,\mathsf{A}_{\rho\Phi} = 1\big\} \\ \text{is a closed subset of } \big\{\rho \in C^1_\sharp(Y_d) : \rho > 0\big\}.\end{gathered} \tag{3.10}$$

Proof of Corollary 3.1. The equality of the sets in (3.10) follows directly from (1.10). Moreover, the fact that the first set is closed is a straightforward consequence of the first case of Theorem 3.1. □

Remark 3.4 *Example 1 of [29] provides a two-dimensional case where the vector field* $b^* = \Phi^*$, i.e. $\rho^* = 1$, *is such that* $\#\mathsf{A}_{\Phi^*} = 2$. *Therefore, by* (3.10) *there exists an open ball* B^* *centered at* $\rho^* = 1$ *such that* $\#\mathsf{A}_{\rho\,\Phi^*} > 1$ *for any* $\rho \in B^*$.

On the contrary, the assertion (3.10) *of Corollary 3.1 does not hold in general when condition* $\rho > 0$ *is enlarged to condition* $\rho \geq 0$. *Indeed, in the setting of Theorem 3.1 the two-dimensional Example 4.2 provides a sequence of fields* $(b_n = \rho_n\,\Phi)_{n\geq 1}$ *which converges uniformly in* Y_2 *to some field* $b = \rho\,\Phi$, *so that* C_b *is not reduced to a singleton while* C_{b_n} *is a singleton for any* $n \geq 1$. *Therefore, in this case the set*

$$\left\{r \in C^1_\sharp(Y_d) : r \geq 0 \text{ and } \#\,\mathsf{C}_{r\Phi} = 1\right\} = \left\{r \in C^1_\sharp(Y_d) : r \geq 0 \text{ and } \#\,\mathsf{A}_{r\phi} = 1\right\}$$

is not closed in $\left\{r \in C^1_\sharp(Y_d) : r \geq 0\right\}$.

3.2 Applications to the asymptotics of the ODE's flow

The first result provides the asymptotics of the flow $X(\cdot, x)$ at any point x whose orbit does not meet the set $\{\rho = 0\}$ in Y_d.

Corollary 3.2 *Assume that the conditions of Theorem 3.1 hold with a vanishing function* ρ. *Denote by* π *the canonical projection from* $\mathbb{R}^d$ *on the torus* Y_d, *and define for* $x \in \mathbb{R}^d$,

$$F_x := \overline{\pi\big(X(\mathbb{R}, x)\big)}^{Y_d}, \tag{3.11}$$

i.e. *the closure in* Y_d *of the projection on* Y_d *of the orbit* $X(\mathbb{R}, x)$ *of* x. *Then, we have*

$$\forall\, x \in Y_d, \quad F_x \cap \{\rho = 0\} = \varnothing \;\Rightarrow\; \lim_{t\to\infty} \frac{X(t,x)}{t} = \zeta. \tag{3.12}$$

Proof of Corollary 3.2. Let $x \in Y_d$ be such that $F_x \cap \{\rho = 0\} = \varnothing$. First, by virtue of Proposition A.1 with $g = b$ and $x_n = x$, any limit point derived from the asymptotics $X(t,x)/t$ as $t \to \infty$, is of the form

$$\int_{Y_d} b(y)\, d\mu(y) \quad \text{for some } \mu \in \mathscr{I}_b.$$

Let us prove that the support of such an invariant probability measure μ is contained in the closed set F_x. By Lemma A.2 μ is a limit point for the weak-$*$ topology of some sequence $(\nu_n)_{\in\mathbb{N}}$ in $\mathscr{M}_p(Y_d)$ given by

$$\int_{Y_d} f(y)\, d\nu_n(y) = \frac{1}{r_n}\int_0^{r_n} f(X(s,x))\, ds \quad \text{for } f \in C^0_\sharp(Y_d).$$

Let $f \in C^0_\sharp(Y_d)$ which is zero in F_x. Then, since $\pi(X(s,x)) \in F_x$ for any $s \in \mathbb{R}$, we have

$$\int_{Y_d} f(y)\, d\nu_n(y) = 0.$$

Passing to the limit in the previous equality we get that for any $f \in C^0_\sharp(Y_d)$ which is zero in F_x,

$$\int_{Y_d} f(y)\, d\mu(y) = 0,$$

which means that the support of μ is contained in F_x.

Now, let a be a limit point of $X(t,x)/t$ as $t \to \infty$. Then, we have

$$a = \int_{Y_d} b(y)\, d\mu(y) \quad \text{with} \quad \operatorname{supp}(\mu) \subset F_x \subset \{\rho > 0\}.$$

Clearly, we have (3.1) and $\beta_\mu = 1$ in (3.2), which implies that

$$\int_{Y_d} b(y)\, d\mu(y) = \zeta.$$

Therefore, any limit point a is equal to the constant vector ζ obtained in Theorem 3.1, which implies that

$$\lim_{t\to\infty} \frac{X(t,x)}{t} = \zeta.$$

The desired implication (3.12) is thus established. $\square$

The following result provides some conditions on the direction Φ of the vector field $b = \rho\,\Phi$, which allows us to specify the result of Theorem 3.1 when the function ρ vanishes.

Corollary 3.3 *Consider a vector field $b = \rho\,\Phi \in C^1_\sharp(Y_d)^d$, a sequence $(\rho_n)_{n\in\mathbb{N}} \in C^1_\sharp(Y_d)^{\mathbb{N}}$ and $b_n = \rho_n\,\Phi$ satisfying the assumptions of Theorem 3.1. Also assume that there exists a positive function $\sigma \in C^1_\sharp(Y_d)$ such that $\sigma\,\Phi$ is divergence free in $\mathbb{R}^d$, and that ρ vanishes in Y_d.*
We have the following alternative involving the harmonic mean $\underline{\rho/\sigma}$ of ρ/σ:

- *If $\underline{\rho/\sigma} = 0$, then the flow X associated with b satisfies the asymptotics*

$$\forall\, x \in Y_d, \quad \lim_{t\to\infty} \frac{X(t,x)}{t} = 0_{\mathbb{R}^d}. \tag{3.13}$$

- *If $\underline{\rho/\sigma} > 0$, then we have*

$$\mathsf{C}_b = [0_{\mathbb{R}^d}, \zeta] \quad \textit{with} \quad \zeta := \underline{\rho/\sigma} \int_{Y_d} \Phi(y)\,\sigma(y)\,dy. \tag{3.14}$$

Proof of Corollary 3.3. Since $\sigma\,\Phi$ is divergence free in $\mathbb{R}^d$ and σ is $\mathbb{Z}^d$-periodic, by virtue of Proposition 2.2 the probability measure on Y_d: $\sigma(x)/\overline{\sigma}\,dx$, where $\overline{\sigma} > 0$ is the arithmetic mean of σ, is an invariant probability measure for the flow associated with the vector field Φ, *i.e.* $\sigma(x)/\overline{\sigma}\,dx \in \mathscr{I}_\Phi$.
For every $n \in \mathbb{N}$, define the probability measure μ_n on Y_d by

$$d\mu_n(x) := \frac{C_n}{\rho_n(x)}\,\sigma(x)\,dx \quad \text{where} \quad C_n := \left(\int_{Y_d} \frac{1}{\rho_n(y)}\,\sigma(y)\,dy\right)^{-1}. \tag{3.15}$$

Due to $\sigma(x)/\overline{\sigma}\,dx \in \mathscr{I}_\Phi$ we have by Proposition 2.2

$$\forall\, \psi \in C^1_\sharp(Y_d), \quad \int_{Y_d} b_n(x)\cdot\nabla\psi(x)\,d\mu_n(x) = C_n \int_{Y_d} \Phi(x)\cdot\nabla\psi(x)\,\sigma(x)\,dx = 0,$$

which again by Proposition 2.2 implies that $\mu_n \in \mathscr{I}_{b_n}$. This combined with the singleton assumption $\mathsf{C}_{b_n} = \{\zeta_n\}$ yields

$$\zeta_n = \int_{Y_d} b_n(x)\,d\mu_n(x) = C_n \int_{Y_d} \Phi(x)\,\sigma(x)\,dx. \tag{3.16}$$

- If $\underline{\rho/\sigma} = 0$, then by Fatou's lemma we get that

$$\infty = \int_{Y_d} \frac{\sigma(y)}{\rho(y)}\, dy \leq \liminf_{n\to\infty} \left(\int_{Y_d} \frac{\sigma(y)}{\rho_n(y)}\, dy \right),$$

  which implies that the sequence $(C_n)_{n\in\mathbb{N}}$ tends to 0. Hence, by (3.16) the sequence $(\zeta_n)_{n\in\mathbb{N}}$ converges to $\zeta = 0_{\mathbb{R}^d}$. Therefore, by the second case of Theorem 3.1 we have $\mathsf{C}_b = \{0_{\mathbb{R}^d}\}$, or equivalently by (2.1), the null asymptotics (3.13) holds.

- If $\underline{\rho/\sigma} > 0$, then from the convergence of ρ_n to ρ, the inequality $\rho_n \geq \rho$ and Lebesgue's theorem we deduce that

$$\lim_{n\to\infty} \int_{Y_d} \frac{\sigma(y)}{\rho_n(y)}\, dy = \int_{Y_d} \frac{\sigma(y)}{\rho(y)}\, dy = \frac{1}{\underline{\rho/\sigma}} < \infty,$$

  which by (3.15), (3.16) implies that

$$\zeta = \lim_{n\to\infty} \zeta_n = \underline{\rho/\sigma} \int_{Y_d} \Phi(y)\, \sigma(y)\, dy.$$

  Therefore, again by the second case of Theorem 3.1 we obtain the set C_b (3.14).

□

Remark 3.5 *In the two-dimensional case of Corollary 3.3, if ρ is in $C^2_\sharp(Y_2)$ and vanishes at some point $x_0 \in Y_2$, then the harmonic mean $\underline{\rho/\sigma}$ is 0. Indeed, since ρ is non negative, x_0 is a critical point of ρ. Hence, we get that for any x close to x_0,*

$$\rho(x) = \frac{1}{2} \nabla^2\rho(x_0)\,(x - x_0)\cdot(x - x_0) + o(|x - x_0|^2) \quad \textit{and thus} \quad \frac{\sigma(x)}{\rho(x)} \geq \frac{C}{|x - x_0|^2}.$$

which implies that $\sigma/\rho \notin L^1(Y_2)$ and $\underline{\rho/\sigma} = 0$. Therefore, the null asymptotics (3.13) holds.

Otherwise, if

$$\#\big\{x \in Y_2 : \rho(x) = 0\big\} \in (0, \infty),$$

and if for any $x_0 \in Y_2$ with $\rho(x_0) = 0$ we have for any x close to x_0,

$$\rho(x) \geq c_0\, |x - x_0|^{\alpha_0}, \quad \textit{for some } \alpha_0 \in (1/2, 1) \textit{ and } c_0 > 0,$$

(note that the former condition remains compatible with $\rho \in C^1_\sharp(Y_2)$), then $\underline{\rho/\sigma} > 0$. Indeed, we are led by a translation to $x_0 = 0_{\mathbb{R}^2}$, and passing to polar coordinates we deduce that for any $r_0 \in (0, 1/2)$,

$$\int_{\{x\in Y_2 : |x| < r_0\}} \frac{dx}{|x|^{\alpha_0}} = 2\pi \int_0^{r_0} \frac{dr}{r^{2\alpha_0 - 1}} < \infty.$$

Therefore, we get the full closed line segment (3.14) for the limit set C_b.

In Example 4.2 below we will provide a two-dimensional example of such an enlarged limit set C_b obtained from a sequence of singleton sets $(\mathsf{C}_{b_n})_{n\in\mathbb{N}}$ where $b_n = \rho_n\, \Phi$ has a fixed direction Φ. These results also apply to Corollary 3.4 replacing $1/\sigma$ by a.

The next result uses both Theorem 3.1, Corollary 3.3 and the two-dimensional ergodic approach of [31].

Corollary 3.4 *Let b be a two-dimensional vector field in $C^1_\sharp(Y_2)^2$ such that $b = \rho\,\Phi$, with ρ a non zero non negative function in $C^1_\sharp(Y_2)$, and*

$$\Phi = a\,R_\perp \nabla u \quad \text{in } Y_2, \tag{3.17}$$

where a is a positive function in $C^1_\sharp(Y_2)$, ∇u is a non vanishing gradient field in $C^1_\sharp(Y_2)^2$ and $R_\perp$ is the $-\pi/2$ rotation matrix of $\mathbb{R}^{2\times 2}$. Also assume that

$$\forall\,\kappa \in \mathbb{Z}^2 \setminus \{0_{\mathbb{R}^2}\}, \quad \left(\int_{Y_2} \nabla u(y)\,dy\right)\cdot \kappa \neq 0. \tag{3.18}$$

We have the following cases:

- *If ρ is positive in Y_2, then the flow X associated with b satisfies the asymptotics*

$$\forall\,x \in Y_2, \quad \lim_{t\to\infty} \frac{X(t,x)}{t} = \underline{a\rho} \int_{Y_2} R_\perp \nabla u(y)\,dy. \tag{3.19}$$

- *If ρ vanishes in Y_2 and $\underline{a\rho} = 0$, then the flow X satisfies the null asymptotics*

$$\forall\,x \in Y_2, \quad \lim_{t\to\infty} \frac{X(t,x)}{t} = 0_{\mathbb{R}^2}. \tag{3.20}$$

- *If ρ vanishes in Y_2 and $\underline{a\rho} > 0$, then the set C_b is given by*

$$\mathsf{C}_b = [0_{\mathbb{R}^2}, \zeta] \quad \textit{with} \quad \zeta := \underline{a\rho} \int_{Y_2} R_\perp \nabla u(y)\,dy \neq 0_{\mathbb{R}^2}. \tag{3.21}$$

Proof of Corollary 3.4. Consider the sequence $(\sigma_n)_{n\in\mathbb{N}}$ defined by

$$\sigma_n := \frac{C_n}{a\rho_n} \quad \text{where} \quad C_n = \underline{a\rho_n} := \left(\int_{Y_2} \frac{dy}{(a\rho_n)(y)}\right)^{-1}, \tag{3.22}$$

where $(\rho_n)_{n\in\mathbb{N}}$ is a sequence in $C^1_\sharp(Y_2)^{\mathbb{N}}$ satisfying the condition (i) of Theorem 3.1.

First, note that by definition (3.22) the vector field $b_n := \rho_n\,\Phi$ satisfies

$$\sigma_n\,b_n = \frac{C_n}{a}\,\Phi = C_n\,R_\perp \nabla u \tag{3.23}$$

(recall that C_n is a positive constant) so that b_n is orthogonal to ∇u. Hence, the function u is invariant by the flow X_n associated with b_n, *i.e.*

$$\forall\,x \in Y_2,\ \forall\,t \in \mathbb{R}, \quad u(X_n(t,x)) = u(x). \tag{3.24}$$

This combined with the irrationality (or ergodicity) condition (3.18) implies that the flow X_n has no periodic solution in Y_2 according to (1.18). Otherwise, there exists $x \in Y_2$, $T > 0$, and $\kappa \in \mathbb{Z}^2$ such that $X_n(T,x) = x + \kappa$, hence it follows that

$$u(x) = u(X_n(T,x)) = u(x+\kappa). \tag{3.25}$$

Moreover, since ∇u is $\mathbb{Z}^2$-periodic, the function

$$z \longmapsto u(z) - \left(\int_{Y_2} \nabla u(y)\,dy\right)\cdot z \quad \text{is } \mathbb{Z}^2\text{-periodic.}$$

This combined with (3.25) yields

$$\left(\int_{Y_2} \nabla u(y)\, dy\right) \cdot \kappa = 0.$$

Hence, from the incommensurability condition of (3.18) we deduce that $\kappa = 0_{\mathbb{R}^2}$. The flow $X_n(\cdot, x)$ is thus T-periodic, namely $X_n(\mathbb{R}, x)$ is a closed orbit. However, by virtue of the preliminary remark of the proof of [31, Theorem 3.1] this leads us to a contradiction since $b_n = \rho_n \Phi$ is non vanishing. Therefore, the flow X_n associated with b_n has no periodic solution in Y_2. Then, using the second step of the proof of [31, Theorem 3.1] we get the existence of a vector $\zeta_n \in \mathbb{R}^2$ such that

$$\forall\, x \in Y_2, \quad \lim_{t\to\infty} \frac{X_n(t,x)}{t} = \zeta_n, \tag{3.26}$$

or equivalently by (2.1), $\mathsf{C}_{b_n} = \{\zeta_n\}$, namely the condition (ii) of Theorem 3.1 holds. Moreover, due to (3.23) $\sigma_n\, b_n$ is divergence free in $\mathbb{R}^2$, or equivalently, in the torus sense (2.10)

$$\forall\, \psi \in C^1_\sharp(Y_d), \quad \int_{Y_d} b_n(x) \cdot \nabla\psi(x)\, \sigma_n(x)\, dx = 0.$$

Hence, by virtue of Proposition 2.2 $\sigma_n(x)\, dx$ is an invariant probability measure for the flow X_n, which combined with (3.23) implies that

$$\int_{Y_2} \sigma_n(y)\, b_n(y)\, dy = C_n \int_{Y_2} R_\perp \nabla u(y)\, dy \in \mathsf{C}_{b_n} = \{\zeta_n\}. \tag{3.27}$$

Let us conclude:

- If ρ is positive in Y_2, then from equality (3.27) and the uniform convergence of ρ_n to $\rho > 0$ we deduce that

  $$\lim_{n\to\infty} \zeta_n = \zeta := C \int_{Y_2} R_\perp \nabla u(y)\, dy, \quad \text{where} \quad C := \lim_{n\to\infty} C_n = \underline{a\rho}.$$

  Therefore, by the first result of Theorem 3.1 we get that $\mathsf{C}_b = \{\zeta\}$, or equivalently by (2.1), asymptotics (3.19) is satisfied.

- Otherwise, ρ vanishes in Y_2. Moreover, by (3.17) the vector field $a^{-1}\Phi$ is clearly divergence free in $\mathbb{R}^2$. Therefore, by virtue of Corollary 3.3 with $\sigma = a^{-1}$, we deduce the null asymptotics (3.20) if $\underline{a\rho} = 0$, and the set C_b (3.21) if $\underline{a\rho} > 0$. The fact that $\zeta \neq 0_{\mathbb{R}^2}$ in (3.21) follows immediately from the ergodic condition (3.18).

The proof is now complete. $\square$

Remark 3.6 *The condition* (3.17) *on the direction* Φ *of the vector field* $b = \rho\, \Phi$ *may seem to be quite restrictive at the first glance. Actually, we can deduce* (3.17) *from the existence of a function* $v \in C^2(Y_2)$ *and a constant* $c > 0$ *satisfying the inequality*

$$\Phi \cdot \nabla v \geq c \ \text{ in } \mathbb{R}^2. \tag{3.28}$$

This inequality means that the equipotential $\{v = 0\}$ *(or any equipotential of* v*) is transverse to each orbit* $Y(\mathbb{R}, x)$, $x \in \mathbb{R}^2$, *of the flow* Y *associated with* Φ. *In other words, the equipotential* $\{v = 0\}$ *can be regarded as a Siegel's curve [32, Lemma 3] for the flow* Y *in* $\mathbb{R}^2$ *rather than in the torus* Y_2. *Assuming that* b *is non vanishing and that the flow* X *associated with* b *has no* *periodic trajectory in Y_2 according to* (1.18) *(which is an ergodic type condition) and using a Siegel's curve in the torus, Peirone [31, Theorem 3.1] has proved that the asymptotics of the flow $X(\cdot, x)$ does exist for any $x \in Y_2$ and is independent of x, or equivalently by* (2.1)*, that C_b is a singleton. Therefore, condition* (3.17) *and the ergodic condition* (3.18) *play the same role for the vector field Φ as Peirone's conditions for the vector field b through a similar Siegel's curve approach. However, working with the non vanishing vector field Φ rather than b allows us to obtain some new asymptotics when the vector field $b = \rho\,\Phi$ does vanish with the non negative function ρ.*

Now, let us check that condition (3.28) *implies condition* (3.17) *under some (minor) additional assumptions. To this end, we will follow the same procedure as [12, Theorem 2.15] derived for a gradient field. Since we have for any $x \in Y_2$,*

$$\forall\, t \in \mathbb{R}, \quad \frac{\partial}{\partial t}\big(v(Y(t,x))\big) = (\Phi \cdot \nabla v)(Y(t,x)) \geq c > 0,$$

the mapping $t \mapsto v(Y(t,x))$ is a C^1-diffeomorphism on $\mathbb{R}$, hence there exists a unique $\tau(x) \in \mathbb{R}$ such that

$$v\big(Y(\tau(x), x)\big) = 0,$$

namely the trajectory $Y(\cdot, x)$ reaches the equipotential $\{v = 0\}$ at time $\tau(x)$. The uniqueness of τ combined with the semi-group property of the flow Y easily implies that

$$\forall\, x \in Y_2,\ \forall\, t \in \mathbb{R}, \quad \tau(Y(t,x)) = \tau(x) - t. \tag{3.29}$$

Moreover, by the implicit functions theorem and the $C^1(\mathbb{R} \times \mathbb{R}^2)$ regularity of the flow Y, the function τ belongs to $C^1(\mathbb{R}^2)$. Then, define the positive function $\sigma_0 \in C^1(\mathbb{R}^2)$ by

$$\sigma_0(x) := \exp\left(\int_0^{\tau(x)} (\operatorname{div}\Phi)(Y(s,x))\,ds\right) \quad \textit{for } x \in \mathbb{R}^2. \tag{3.30}$$

From now on, assume that $\operatorname{div}(\Phi) \in C^1(\mathbb{R}^2)$. Then, the function σ_0 belongs to $C^1(\mathbb{R}^2)$. By using (3.29) *and the semi-group property of the flow Y, then making the change of variable $r = s + t$, we get that*

$$\forall\, x \in Y_2,\ \forall\, t \in \mathbb{R}, \quad \sigma_0(Y(t,x)) = \exp\left(\int_t^{\tau(x)} (\operatorname{div}\Phi)(Y(r,x))\,dr\right).$$

Next, taking the derivative of the previous equality at $t = 0$, we obtain that

$$\nabla\sigma_0 \cdot \Phi + \sigma_0 \operatorname{div}(\Phi) = \operatorname{div}(\sigma_0\,\Phi) = 0 \ \ \textit{in } \mathbb{R}^2,$$

or equivalently, there exists a function $u_0 \in C^1(\mathbb{R}^2)$ such that

$$\Phi = \frac{1}{\sigma_0}\,R_\perp \nabla u_0 \ \ \textit{in } \mathbb{R}^2.$$

This is nearly the desired condition (3.17) *except that the function σ_0 is not necessarily $\mathbb{Z}^2$-periodic. However, also assuming that the function σ_0 is bounded from below and above by positive constants, the averaging procedure of [12, Theorem 2.17] allows us to build a positive periodic function $\sigma \in L^\infty_\sharp(Y_2)$ satisfying*

$$\operatorname{div}(\sigma\,\Phi) = 0 \ \ \textit{in } \mathbb{R}^2,$$

which is equivalent to condition (3.17) *with $a := 1/\sigma$. But the regularity of σ is not ensured. Finally, to get the regularity $\sigma \in C^1_\sharp(Y_2)$, it is enough to assume in addition that σ_0 and $\nabla\sigma_0$ are uniformly continuous in $\mathbb{R}^2$ (see [12, Remark 2.19]).*

In contrast with Corollary 3.4 the following result uses both Theorem 3.1 and the non-ergodic approach of [9] in any dimension, and provides an alternative approach to the two-dimensional Corollary 3.4.

Corollary 3.5 *For $d \geq 2$ and $n \in \mathbb{N}^*$, let $U_n = (u_1^n, u_2, \ldots, u_d)$ be a sequence of vector fields in $C^2(Y_d)^d$ with $\nabla U_n \in C^1_\sharp(Y_d)^{d\times d}$, and let a be a positive function in $C^1_\sharp(Y_d)$ such that*

$$\begin{cases} \det(\nabla U_n) > 0 & \text{in } Y_d \\ \rho_n := \dfrac{1}{a\det(\nabla U_n)} & \text{converges uniformly in } Y_d \text{ to some } \rho \in C^1_\sharp(Y_d) \\ \rho \leq \rho_n & \text{in } Y_d. \end{cases} \tag{3.31}$$

Then, the sequence of vector fields $(b_n)_{n\in\mathbb{N}}$ defined by

$$b_n = \rho_n\,\Phi, \quad \text{where} \quad \Phi := \begin{cases} a\,R_\perp \nabla u_2 & \text{if } d = 2 \\ a\,(\nabla u_2 \times \cdots \times \nabla u_d) & \text{if } d > 2 \end{cases} \tag{3.32}$$

converges uniformly in Y_d to the vector field $b = \rho\,\Phi$. Moreover, we have:

- *If ρ is positive in Y_d, then the flow X associated with b satisfies the asymptotics*

$$\forall\, x \in Y_d, \quad \lim_{t\to\infty} \frac{X(t,x)}{t} = \zeta := \begin{cases} \underline{a\rho} \displaystyle\int_{Y_2} R_\perp \nabla u_2(y)\,dy & \text{if } d = 2 \\ \underline{a\rho} \displaystyle\int_{Y_d} (\nabla u_2 \times \cdots \times \nabla u_d)(y)\,dy & \text{if } d > 2, \end{cases} \tag{3.33}$$

  where $\underline{a\rho}$ is the harmonic mean of ρ.

- *If ρ vanishes in Y_d and $\underline{a\rho} = 0$, then the flow X associated with b satisfies the asymptotics*

$$\forall\, x \in Y_d, \quad \lim_{t\to\infty} \frac{X(t,x)}{t} = 0_{\mathbb{R}^d}. \tag{3.34}$$

- *If ρ vanishes in Y_d and $\underline{a\rho} > 0$, then we get the set*

$$\mathsf{C}_b = [0_{\mathbb{R}^d}, \zeta] \quad \text{with} \quad \zeta = \begin{cases} \underline{a\rho} \displaystyle\int_{Y_2} R_\perp \nabla u_2(y)\,dy \neq 0_{\mathbb{R}^2} & \text{if } d = 2 \\ \underline{a\rho} \displaystyle\int_{Y_d} (\nabla u_2 \times \cdots \times \nabla u_d)(y)\,dy \neq 0_{\mathbb{R}^d} & \text{if } d > 2. \end{cases} \tag{3.35}$$

Proof of Corollary 3.5. We have

$$b_n \cdot \nabla u_1^n = \left.\begin{cases} a\rho_n\, \nabla u_1^n \cdot R_\perp \nabla u_2 & \text{if } d = 2 \\ a\rho_n\, \nabla u_1^n \cdot (\nabla u_2 \times \cdots \times \nabla u_d) & \text{if } d > 2 \end{cases}\right\} = a\rho_n\, \det(\nabla U_n) = 1 \quad \text{in } Y_d, \tag{3.36}$$

and

$$\frac{1}{a\rho_n}\, b_n = \begin{cases} R_\perp \nabla u_2 & \text{if } d = 2 \\ \nabla u_2 \times \cdots \times \nabla u_d & \text{if } d > 2 \end{cases}$$

is divergence free in $\mathbb{R}^d$. Hence, from [9, Corollary 4.1] we deduce that C_{b_n} is the singleton $\{\zeta_n\}$ with

$$\zeta_n := \frac{\displaystyle\int_{Y_d} (a\rho_n)^{-1}(y)\, b_n(y)\, dy}{\displaystyle\int_{Y_d} (a\rho_n)^{-1}(y)\, dy} = \begin{cases} = \underline{a\rho_n} \displaystyle\int_{Y_2} R_\perp \nabla u_2(y)\, dy & \text{if } d = 2 \\ = \underline{a\rho_n} \displaystyle\int_{Y_d} (\nabla u_2 \times \cdots \times \nabla u_d)(y)\, dy & \text{if } d > 2, \end{cases} \tag{3.37}$$

where $\underline{a\rho_n}$ is the harmonic mean of $a\rho_n$.
Let us conclude:

- If $\rho > 0$ in Y_d, then the sequence $(a\rho_n)^{-1}$ converges uniformly to $(a\rho)^{-1}$ in Y_d. Therefore, by the first case of Theorem 3.1 combined with (3.37) we get that $\mathsf{C}_b = \{\zeta\}$, or equivalently by (2.1), asymptotics (3.33) holds.
- Otherwise, ρ vanishes in Y_2. Moreover, by (3.32) the vector field $a^{-1}\Phi$ is clearly divergence free in $\mathbb{R}^2$. Therefore, by virtue of Corollary 3.3 with $\sigma = a^{-1}$, we deduce the null asymptotics (3.34) if $\underline{a\rho} = 0$, and the set C_b (3.35) if $\underline{a\rho} > 0$. It remains to prove that $\zeta \neq 0_{\mathbb{R}^d}$ in (3.35). By the definition of U_n and (3.31) we have

$$\int_{Y_2} \underbrace{\det(\nabla U_n)(y)}_{>0}\, dy = \begin{cases} \displaystyle\int_{Y_2} \nabla u_1^n(y) \cdot R_\perp \nabla u_2(y)\, dy > 0 & \text{if } d = 2 \\ \displaystyle\int_{Y_d} \nabla u_1^n(y) \cdot (\nabla u_2 \times \cdots \times \nabla u_d)(y) > 0 & \text{if } d > 2. \end{cases}$$

  Hence, from the quasi-affinity of the determinant (see, *e.g.*, [18, Section 4.3.2]), namely:

$$\det\left(\int_{Y_2} \nabla U_n(y)\, dy\right) = \int_{Y_2} \det(\nabla U_n)(y)\, dy > 0,$$

  we deduce that

$$\begin{cases} \displaystyle\int_{Y_2} \nabla u_1^n \cdot \left(\int_{Y_2} R_\perp \nabla u_2\right) = \int_{Y_2} \nabla u_1^n \cdot R_\perp \nabla u_2 > 0 & \text{if } d = 2 \\ \displaystyle\int_{Y_d} \nabla u_1^n \cdot \left(\int_{Y_d} \nabla u_2 \times \cdots \times \int_{Y_d} \nabla u_d\right) = \int_{Y_d} \nabla u_1^n \cdot (\nabla u_2 \times \cdots \times \nabla u_d) > 0 & \text{if } d > 2. \end{cases}$$

  Therefore, again using the quasi-affinity of the determinant (multiplying the second equality by any constant vector of $\mathbb{R}^d$ to get a determinant) we get that

$$\begin{cases} \displaystyle\int_{Y_2} R_\perp \nabla u_2(y)\, dy \neq 0_{\mathbb{R}^2} & \text{if } d = 2 \\ \displaystyle\int_{Y_d} (\nabla u_2 \times \cdots \times \nabla u_d)(y)\, dy = \int_{Y_d} \nabla u_2(y)\, dy \times \cdots \times \int_{Y_d} \nabla u_d(y)\, dy \neq 0 & \text{if } d > 2, \end{cases}$$

  which implies that $\zeta \neq 0_{\mathbb{R}^d}$ in (3.35).

$\square$

Remark 3.7 *In Corollary 3.4 and in the two-dimensional case of Corollary 3.5, the vector field b_n has the same form $b_n = \rho_n\, a\, R_\perp \nabla u$. In Corollary 3.4 the function ρ_n is arbitrary, while ∇u satisfies the ergodic condition* (3.18)*. On the contrary, in the two dimensional case of Corollary 3.5 ρ_n does depend on the functions a and ∇u by* (3.31)*, while ∇u is arbitrary. Therefore, these results provide two quite different approaches on the asymptotics of the flow: an ergodic one using [31] and a non-ergodic one using [9].*

4 Examples

The following counter-example shows that Theorem 3.1 does not extend to a vector-valued perturbation.

Exemple 4.1 *Consider the vector fields b and b_n defined by*

$$b(x) = a(x)\, e_1 \quad \text{and} \quad b_n(x) = a_n(x)\, (e_1 + \gamma_n e_2), \qquad \text{for } x \in Y_2 \text{ and } n \in \mathbb{N},$$

where a is a non negative function in $C^1_\sharp(Y_2)$, $a_n := a + 1/n$, and $(\gamma_n)_{n\in\mathbb{N}}$ is a positive sequence in $\mathbb{R} \setminus \mathbb{Q}$ which converges to 0. In this case, the flows associated with the vector fields b and b_n can be computed explicitly.

On the one hand, by the ergodic case of [34, Section 3.1] with the irrational rotation number γ_n, the flow X_n associated with the vector field b_n satisfies the asymptotics

$$\forall\, x \in Y_2, \quad \lim_{t\to\infty} \frac{X_n(t,x)}{t} = \underline{a_n}\, (e_1 + \gamma_n\, e_2),$$

where $\underline{a_n}$ is the harmonic mean of a_n, or equivalently by (2.1),

$$\mathsf{C}_{b_n} = \{\zeta_n\} \quad \text{with} \quad \zeta_n := \underline{a_n}\, (e_1 + \gamma_n\, e_2). \tag{4.1}$$

On the other hand, by the non-ergodic case of [34, Section 3.1] with the rational rotation number 0 and its extension when a vanishes in Y_2, the flow X associated with the vector field b satisfies the asymptotics

$$\forall\, x \in Y_2, \quad \lim_{t\to\infty} \frac{X(t,x)}{t} = \begin{cases} \underline{a(\cdot\, e_1 + x)}\, e_1 & \text{if } a(\cdot\, e_1 + x) \text{ is positive in } Y_2 \\ 0 & \text{if } a(\cdot\, e_1 + x) \text{ vanishes in } Y_2, \end{cases}$$

where $\underline{a(\cdot\, e_1 + x)}$ is the harmonic mean defined by

$$\underline{a(\cdot\, e_1 + x)} := \left(\int_{Y_1} \frac{dt}{a(t\, e_1 + x)} \right)^{-1} \quad \text{for } x \in Y_2,$$

or equivalently, the set A_b (1.8) is given by

$$\mathsf{A}_b = \left\{ \underline{a(\cdot\, e_1 + x)}\, e_1 : x \in Y_2 \right\}.$$

Moreover, the function $\big(x \mapsto \underline{a(\cdot\, e_1 + x)}\big)$ is continuous on the compact set Y_2. It is clear at any point $x \in Y_2$ such that $a(\cdot\, e_1 + x)$ is positive. Otherwise, if $a(\cdot\, e_1 + x) \in C^1_\sharp(Y_1)$ vanishes in Y_1, by Fatou's lemma we get that for any sequence $(x_n)_{n\in\mathbb{N}}$ converging to x,

$$\infty = \frac{1}{\underline{a(\cdot\, e_1 + x)}} \le \liminf_{n\to\infty} \left(\frac{1}{\underline{a(\cdot\, e_1 + x_n)}} \right) = \lim_{n\to\infty} \left(\frac{1}{\underline{a(\cdot\, e_1 + x_n)}} \right) = \infty.$$

Hence, the set A_b is actually a closed line segment of $\mathbb{R}^2$, which by (1.10) implies that

$$\mathsf{C}_b = \operatorname{conv}(\mathsf{A}_b) = \mathsf{A}_b = \left\{ \underline{a(\cdot\, e_1 + x)}\, e_1 : x \in Y_2 \right\}. \tag{4.2}$$

In particular, when a vanishes in Y_2, we get that

$$\mathsf{C}_b = [0_{\mathbb{R}^2}, \zeta] \quad \text{with} \quad \zeta := \left(\max_{x\in Y_2} \underline{a(\cdot\, e_1 + x)} \right) e_1.$$

Therefore, taking into account (4.1), *contrary to the second case of Theorem 3.1 we may have*

$$\underline{a}\, e_1 = \lim_{n\to\infty} \zeta_n \neq \zeta = \left(\max_{x\in Y_2} \underline{a(\cdot\, e_1 + x)}\right) e_1.$$

For example, take

$$a(x) := \sin^2(\pi x_1) + \sin^2(\pi x_2) \quad \text{for } x = (x_1, x_2) \in \mathbb{R}^2.$$

Then, it follows that

$$\begin{aligned}
\frac{1}{\underline{a}} = \int_{Y_1}\left(\int_{Y_1} \frac{dx_2}{\sin^2(\pi x_1) + \sin^2(\pi x_2)}\right) dx_1 \quad &> \quad \int_{Y_1} \frac{dx_1}{\sin^2(\pi x_1) + 1} \\
(\text{for any } x_1 \in Y_1 \text{ and } x_2 = \tfrac{1}{2}) \quad &= \quad \min_{x\in Y_2}\left(\int_{Y_1} \frac{dt}{\sin^2(\pi t + \pi x_1) + \sin^2(\pi x_2)}\right) \\
&= \quad \min_{x\in Y_2}\left(\frac{1}{\underline{a(\cdot\, e_1 + x)}}\right),
\end{aligned}$$

which implies that

$$\underline{a} < \max_{x\in Y_2} \underline{a(\cdot\, e_1 + x)}.$$

Therefore, Theorem 3.1 does not extend in general to the case where the direction Φ *of the vector field* $b_n = \rho_n\, \Phi$ *also depends on* n.

Finally, note that the inclusion $\{0_{\mathbb{R}^2}, \zeta\} \subset \mathsf{A}_b$ *of the second case of Theorem 3.1 is not in general an equality, since in the particular case* (4.2) A_b *is the closed line segment* $[0_{\mathbb{R}^2}, \zeta]$.

The second example shows that the singleton condition is not in general asymptotically preserved under the assumptions of Theorem 3.1.

Exemple 4.2 *Let* $\nabla u \in C^1_\sharp(Y_2)^2$ *be satisfying the ergodic condition* (3.18), *let* $(\rho_n)_{n\in\mathbb{N}}$ *be the sequence of positive functions in* $C^1_\sharp(Y_2)$ *defined by*

$$\rho_n(x) := \left(\sin^2(\pi x_1) + \sin^2(\pi x_2) + 1/n\right)^\alpha \quad \text{for } x \in Y_2, \quad \text{with } \alpha \in (1/2, 1).$$

and let $(b_n)_{n\in\mathbb{N}}$ *be the sequence of vector fields defined by* $b_n := \rho_n\, R_\perp \nabla u$. *Hence, by the asymptotics* (3.19) *of Corollary 3.4 the rotation set* C_{b_n} *is a unit set.*

On the other hand, since the function $(t \mapsto t^\alpha)$ *is uniformly continuous in* $[0, \infty)$, *the sequence* $(\rho_n)_{n\in\mathbb{N}}$ *converges uniformly in* Y_d *to the function*

$$\rho(x) := \left(\sin^2(\pi x_1) + \sin^2(\pi x_2)\right)^\alpha \quad \text{for } x = (x_1, x_2) \in Y_2,$$

which belongs to $C^1_\sharp(Y_2)$ *due to* $\alpha > 1/2$, *and vanishes at the sole point* $(0,0)$ *in the torus* Y_2. *Moreover, we have for any* x *close to* $(0,0)$,

$$\frac{c^{-1}}{|x|^{2\alpha}} \leq \frac{1}{\rho(x)} \leq \frac{c}{|x|^{2\alpha}}, \quad \text{for some } c > 1,$$

so that $\underline{\rho} > 0$ *due to* $\alpha < 1$ *(see Remark 3.5). Therefore, by the asymptotics* (3.21) *of Corollary 3.4 we obtain that*

$$\mathsf{C}_b = [0_{\mathbb{R}^2}, \zeta] \quad \text{with} \quad \zeta := \left(\int_{Y_2} \frac{dx}{\left(\sin^2(\pi x_1) + \sin^2(\pi x_2)\right)^\alpha}\right)^{-1} \int_{Y_d} R_\perp \nabla u(y)\, dy \neq 0_{\mathbb{R}^2}.$$

Note that, if $\alpha \geq 1$, *then* $\underline{\rho} = 0$. *Therefore, by the asymptotics* (3.20) *of Corollary 3.4 we get that* $\mathsf{C}_b = \{0_{\mathbb{R}^2}\}$.

The third example illustrates Corollary 3.5.

Exemple 4.3 *Let $U_n = (u_1^n, u_2, \ldots, u_d) \in C^2(Y_d)^d$, $d \geq 2$ and $n \geq 1$, be such that ∇U_n is $\mathbb{Z}^d$-periodic, the functions $u_2, \ldots, u_d$ only depend on the variables $x' = (x_2, \ldots, x_d)$,*

$$u_1^n(x) := \int_0^{x_1} \frac{dt}{f_n(t, x')} \quad \text{and} \quad \Delta(x') := \det\left(\left[\frac{\partial u_i}{\partial x_j}(x')\right]_{2\leq i,j\leq d}\right) > 0 \quad \text{for } x \in Y_d,$$

where $(f_n)_{n\in\mathbb{N}}$ is a positive sequence in $C^1_\sharp(Y_d)^{\mathbb{N}}$ which converges uniformly to $f \leq f_n$ in Y_d. Expanding the determinant with respect to its first column we have

$$\forall\, x \in Y_d, \quad \det(\nabla U_n)(x) = \frac{\Delta(x')}{f_n(x)} > 0 \quad \text{and} \quad \rho_n(x) := \frac{1}{\det(\nabla U_n)(x)} \to \rho(x) := \frac{f(x)}{\Delta(x')} \leq \rho_n(x)$$

uniformly in Y_d, so that condition (3.31) is fulfilled with $a = 1$. Define the vector field b_n in $C^1_\sharp(Y_d)^d$ by (3.32). Therefore, the sequence $(b_n)_{n\in\mathbb{N}}$ converges uniformly in Y_d to the function b given by

$$b(x) = \begin{cases} \dfrac{f(x)}{\Delta(x')}\, R_\perp \nabla u_2(x) & \text{if } d = 2 \\[2ex] \dfrac{f(x)}{\Delta(x')}\, (\nabla u_2 \times \cdots \times \nabla u_d)(x) & \text{if } d > 2, \end{cases} \qquad \text{for } x \in Y_d.$$

Moreover, due to the 1-periodicity of $\nabla_{x'} u_1^n$ with respect to the variable x_1, we have

$$\forall\, x' \in \mathbb{R}^{d-1}, \qquad \int_0^1 \nabla_{x'}\left(\frac{1}{f_n(t, x')}\right) dt = 0,$$

which implies the existence of a positive constant c_n such that

$$\forall\, x' \in \mathbb{R}^{d-1}, \qquad \int_0^1 \frac{dt}{f_n(t, x')} = c_n. \tag{4.3}$$

Hence, from inequality $f \leq f_n$ and Fatou's lemma we deduce that

$$\forall\, x' \in \mathbb{R}^{d-1}, \quad \limsup_{n\to\infty} c_n \leq \int_0^1 \frac{dt}{f(t, x')} \leq \liminf_{n\to\infty} \int_0^1 \frac{dt}{f_n(t, x')} = \liminf_{n\to\infty} c_n,$$

which implies that

$$\forall\, x' \in \mathbb{R}^{d-1}, \qquad \int_0^1 \frac{dt}{f(t, x')} = \lim_{n\to\infty} c_n. \tag{4.4}$$

Then, we have the following alternative:

- *If f is positive in Y_d, then by virtue of the first case of Corollary 3.5 we obtain the asymptotics of the flow associated with b*

$$\forall\, x \in Y_d, \quad \lim_{t\to\infty} \frac{X(t, x)}{t} = \begin{cases} \underline{\rho} \displaystyle\int_{Y_2} R_\perp \nabla u_2(y)\, dy & \text{if } d = 2 \\[2ex] \underline{\rho} \displaystyle\int_{Y_d} (\nabla u_2 \times \cdots \times \nabla u_d)(y)\, dy & \text{if } d > 2, \end{cases}$$

  where

$$\underline{\rho} = \left(\int_{Y_d} \frac{\Delta(x')}{f(x)}\, dx\right)^{-1}.$$

- *If f vanishes at some point $x_0 \in Y_d$, then since $f(\cdot, x_0')$ is in $C^1_\sharp(Y_1)$ and vanishes at $t = (x_0)_1$, we have by* (4.4)

$$\forall\, x' \in \mathbb{R}^{d-1}, \quad \lim_{n\to\infty} c_n = \lim_{n\to\infty} \int_0^1 \frac{dt}{f_n(t,x')} = \int_0^1 \frac{dt}{f(t,x_0')} = \infty. \tag{4.5}$$

Assume by contradiction that $\underline{f} > 0$. Then, using successively Lebesgue's theorem with inequality $f \le f_n$, Fubini's theorem and equality (4.3)*, we get that*

$$\infty > \int_{Y_d} \frac{dx}{f(x)} = \lim_{n\to\infty} \int_{Y_d} \frac{dx}{f_n(x)} = \lim_{n\to\infty} \left[\int_{Y_{d-1}} dx' \left(\int_0^1 \frac{dt}{f_n(t,x')} \right) \right] = \lim_{n\to\infty} c_n,$$

which contradicts (4.5)*. Hence, we deduce that $\underline{f} = 0$, and due to the positivity of Δ we get that*

$$\underline{\rho} = \left(\int_{Y_d} \frac{\Delta(x')}{f(x)}\, dx \right)^{-1} = 0.$$

Therefore, by virtue of the second case of Corollary 3.5 we obtain the null asymptotics (3.34).

Note that the third case of Corollary 3.5 cannot arise when the functions $u_2, \ldots, u_d$ are independent of the variable x_1.

The fourth example deals with the case of an electric field. It is based on the divergence-curl Proposition 2.2, and illustrates the framework of Theorem 3.1. We cannot characterize precisely the set C_b except in the two-dimensional ergodic case. However, the two-dimensional case and the three-dimensional case are shown to be quite different.

Exemple 4.4 *Let $\sigma \in C^3_\sharp(Y_d)$ be a positive function with $\int_{Y_d} \sigma(y)\, dy = 1$. Consider the vector-valued function $U \in C^2(\mathbb{R}^d)^d$ (see,* e.g.*, [20, Theorem 8.13]) unique solution (up to an additive constant vector) to the conductivity problem*

$$\begin{cases} \mathrm{Div}(\sigma DU) = 0_{\mathbb{R}^d} & \textit{in } \mathbb{R}^d \\ y \mapsto U(y) - y & \textit{is } \mathbb{Z}^d\textit{-periodic}, \end{cases} \tag{4.6}$$

where $DU = (\nabla U)^T$ and the vector-valued operator Div *consists in the divergence of the columns of σDU. The variational formulation of* (4.6) *reads as*

$$DU \in C^1_\sharp(Y_d)^d \quad \textit{and} \quad \forall\, \Psi \in C^1_\sharp(Y_d)^d, \ \int_{Y_d} \sigma(y) DU(y) : D\Psi(y)\, dy = 0. \tag{4.7}$$

In (4.7) *" : " denotes the scalar product in $\mathbb{R}^{d\times d}$ defined by*

$$M : N := \mathrm{tr}(M^T N) \quad \textit{for } M, N \in \mathbb{R}^{d\times d}.$$

The so-called homogenized matrix (see, e.g.*, [6, Chapter I, Section 2.3]) associated with the conductivity σ is defined by*

$$A^* := \int_{Y_d} \sigma(y)\, DU(y)\, dy, \tag{4.8}$$

which is known to be symmetric positive definite. Also define the associated electric field

$$b_\lambda := \nabla u_\lambda = \nabla(U\lambda) = DU\lambda \quad \textit{for } \lambda \in \mathbb{R}^d \setminus \{0_{\mathbb{R}^d}\}. \tag{4.9}$$

Case $d = 2$: Alessandrini and Nesi [1, Theorems 1,2] have proved that U is a C^1-diffeomorphism of $\mathbb{R}^2$ with

$$\det(DU) > 0 \quad \textit{in } Y_2. \tag{4.10}$$

Let $\lambda \in \mathbb{R}^2 \setminus \{0_{\mathbb{R}^2}\}$. As a consequence (see [1, Proposition 2]) the gradient field b_λ defined by (4.9) does not vanish in Y_2. Hence, the set of invariant probability measures $\mathscr{I}_{b_\lambda}$ for the flow associated with b_λ does not contain any Dirac measure.

Moreover, by (4.7) and (4.9) we have

$$\forall\, \psi \in C^1_\sharp(Y_2), \quad \int_{Y_2} b_\lambda(y) \cdot \nabla\psi(y)\, \sigma(y)\, dy = \int_{Y_2} \sigma(y) \nabla u_\lambda(y) \cdot \nabla\psi(y)\, dy = 0, \tag{4.11}$$

which by virtue of the equivalence (i)-(iii) of Proposition 2.2 implies that $\sigma(x)\, dx$ is an invariant probability measure with positive density. Therefore, by definition (4.8) we can only conclude that (recall that A^* is positive definite and λ is non zero)

$$A^*\lambda = \int_{Y_2} b_\lambda(y)\, \sigma(y)\, dy \in \mathsf{C}_{b_\lambda}. \tag{4.12}$$

On the other hand, let $\mu \in \mathscr{I}_{b_\lambda}$ be an invariant probability measure for the flow X_λ associated with b_λ. By the divergence-curl relation (2.14) and (4.6) we have

$$\int_{Y_2} |b_\lambda(y)|^2\, \mu(dy) = \int_{Y_2} b_\lambda(y) \cdot \nabla u_\lambda\, d\mu(y) = \left(\int_{Y_2} b_\lambda(y)\, d\mu(y) \right) \cdot \lambda. \tag{4.13}$$

Due to inequality (4.10) the gradient field $b_\lambda = DU\lambda$ does not vanish in Y_2. Hence, since μ is a probability measure, we deduce that $\mu(\{b_\lambda \neq 0\}) > 0$, which implies that the first term of (4.13) is positive. Therefore, we get that

$$0_{\mathbb{R}^2} \neq \int_{Y_2} b_\lambda(y)\, d\mu(y) \in \mathsf{C}_{b_\lambda}.$$

This combined with (4.12) yields

$$\{A^*\lambda\} \subset \mathsf{C}_{b_\lambda} \subset \mathbb{R}^2 \setminus \{0_{\mathbb{R}^2}\}. \tag{4.14}$$

We may improve the former result under the extra ergodic condition

$$\forall\, \kappa \in \mathbb{Z}^2 \setminus \{0_{\mathbb{R}^2}\}, \quad (A^*\lambda) \cdot \kappa \neq 0. \tag{4.15}$$

Indeed, since by (4.11) $\sigma\, b_\lambda$ is divergence free, by a classical duality argument there exists a potential v_λ with $\nabla v_\lambda \in C^1_\sharp(Y_2)^2$, such that $\sigma\, b_\lambda = R_\perp \nabla v_\lambda$ in Y_2. Moreover, since by (4.12)

$$\int_{Y_2} \nabla v_\lambda(y)\, dy = -\, R_\perp \int_{Y_2} \sigma(y)\, b_\lambda(y)\, dy = -\, R_\perp A^*\lambda,$$

condition (4.15) means that the gradient field ∇v_λ satisfies the ergodic condition (3.18). Hence, the vector field $b_\lambda = 1/\sigma\, R_\perp \nabla v_\lambda$ satisfies the first result of Corollary 3.4 with $u := v_\lambda$, $a := 1/\sigma$ and $\rho := 1$. Therefore, the flow X_λ associated with b_λ satisfies asymptotics (3.19) which reads as

$$\forall\, x \in Y_2, \quad \lim_{t \to \infty} \frac{X_\lambda(t,x)}{t} = \underline{1/\sigma} \int_{Y_2} R_\perp \nabla v(y)\, dy = A^*\lambda,$$

or equivalently by (2.1)*, we obtain that*

$$\mathsf{C}_{b_\lambda} = \{A^*\lambda\}. \tag{4.16}$$

Case $d = 3$*: Contrary to the two-dimensional case with the positivity* (4.10)*, by virtue of [11, Theorem 4.1] there exists a positive conductivity* $\sigma_\gamma \in L^\infty_\sharp(Y_3)$ *such that the vector-valued function* U_γ *solution to the equation* (4.6) *with* σ_γ *has a determinant which changes sign.*

More precisely, the conductivity of [11] rescaled by its Y_3*-average value denoted by* σ_γ *(recall that* $\sigma_\gamma(x)\,dx$ *has to be a probability measure) takes two values:* $\sigma_\gamma = 1$ *in a cubic symmetric lattice of interlocking rings which do not intersect, and* $\sigma_\gamma = \gamma \ll 1$ *elsewhere in* $\mathbb{R}^3$*. The conductivity* σ_γ *is thus not regular. However, enlarging each ring with a width* $d_\gamma \ll 1$*, we can build a new conductivity* $\sigma \in C^3_\sharp(Y_3)$ *(also depending on* γ*) whose values pass from* 1 *on the boundary of each ring to* γ *on the boundary of the corresponding enlarged ring. Then, it is easy to check that for* γ *and* d_γ *small enough, the matrix-valued fonction* DU *defined by* (4.6) *with the regular conductivity* σ *has a determinant which also changes sign. Thus, by a continuity continuity argument we get that*

$$\exists\, y_0 \in Y_3, \quad \det(DU)(y_0) = 0, \tag{4.17}$$

which implies that there exists $\lambda \in \mathbb{R}^3 \setminus \{0_{\mathbb{R}^3}\}$ *such that* $DU(y_0)\lambda = 0_{\mathbb{R}^3}$*. Hence, the gradient field* b_λ *defined by* (4.9) *vanishes at point* y_0*. Therefore, in contrast with the two-dimensional result* (4.12) *we obtain the more complete result*

$$A^*\lambda = \int_{Y_3} b_\lambda(y)\,\sigma(y)\,dy \in \mathsf{C}_{b_\lambda} \setminus \{0_{\mathbb{R}^3}\} \quad \textit{and} \quad [0_{\mathbb{R}^3}, A^*\lambda] \subset \mathsf{C}_{b_\lambda}, \tag{4.18}$$

since the Dirac mass δ_{y_0} *belongs to* $\mathscr{I}_{b_\delta}$ *and*

$$\int_{Y_3} b_\lambda(y)\,d\delta_{y_0}(dy) = b_\lambda(y_0) = 0_{\mathbb{R}^3}.$$

Result (4.18) *corresponds to the second case of Theorem 3.1. In contrast with the result* (4.14) *of the two-dimensional case, we obtain that*

$$[0_{\mathbb{R}^3}, A^*\lambda] \subset \mathsf{C}_{b_\lambda}. \tag{4.19}$$

5 Homogenization of linear transport equations

The following theorem is an extension of various homogenization results [7, 21, 22, 25, 34] (and the references therein) of linear transport equations with an oscillating velocity, which are based on the classical ergodic approach. Here, in a regular and periodic framework the ergodic approach is replaced by the singleton approach of Section 2.1, whose a very particular case has been first obtained in [8, Corollary 4.4].

Theorem 5.1 *Let* b *be a vector field in* $C^1_\sharp(Y_d)^d$ *and let* $u_0 \in C^1(\mathbb{R}^d)$*. Consider the transport equation with the oscillating velocity* $b(x/\varepsilon)$*:*

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - b(x/\varepsilon)\cdot\nabla u_\varepsilon = 0 & \textit{in } (0,\infty)\times\mathbb{R}^d \\ u_\varepsilon(0,x) = u_0(x) & \textit{for } x \in \mathbb{R}^d. \end{cases} \tag{5.1}$$

Assume that there exists a vector $\zeta \in \mathbb{R}^d$ such that

$$\forall\, x \in Y_d, \quad \lim_{t\to\infty} \frac{X(t,x)}{t} = \zeta, \tag{5.2}$$

where X is the flow (1.15) *associated with the vector field b. Then, the solution u_ε to transport equation* (5.1) *converges strongly in $L^p_{\rm loc}([0,\infty)\times\mathbb{R}^d)$ for any $p \in [1,\infty)$, to $u_0(x + t\,\zeta)$ which is solution to the transport equation* (5.1) *with the constant velocity ζ in place of $b(x/\varepsilon)$.*

Proof of Theorem 5.1. Let X be the flow associated to the vector field b. By ε-rescaling the flow X, let us define the flow X_ε associated with the oscillating vector field $b(x/\varepsilon)$ by

$$\forall\,(t,x) \in (0,\infty)\times Y_d, \quad X_\varepsilon(t,x) := \varepsilon\, X\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right) = x + \varepsilon \int_0^{\frac{t}{\varepsilon}} b\left(X\big(s,\frac{x}{\varepsilon}\big)\right) ds. \tag{5.3}$$

Taking into account the regularity conditions the characteristics method induced by the flow X_ε implies that the solution u_ε to (5.1) is given by

$$\forall\,(t,x) \in [0,\infty)\times Y_d, \quad u_\varepsilon(t,x) = u_0(X_\varepsilon(t,x)) = u_0\left(x + \varepsilon\int_0^{\frac{t}{\varepsilon}} b\left(X\big(s,\frac{x}{\varepsilon}\big)\right) ds\right). \tag{5.4}$$

On the other hand, let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a positive sequence converging to 0. Let $(t,x) \in (0,\infty)\times Y_d$, set $t_n := t/\varepsilon_n$ and $x_n := x/\varepsilon_n$. Then, by virtue of Proposition A.1 the limit points of the sequence

$$v_n := \varepsilon_n \int_0^{\frac{t}{\varepsilon_n}} b\left(X\big(s,\frac{x}{\varepsilon_n}\big)\right) ds = t \times \frac{1}{t_n}\int_0^{t_n} b(X(s,x_n))\,ds$$

belong to $t\,\mathsf{C}_b$. However, by (5.2) combined with equivalence (2.1) we have $\mathsf{C}_b = \{\zeta\}$. Hence, for any positive sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ converging to 0, the whole sequence $(v_n)_{n\in\mathbb{N}}$ converges to $t\,\zeta$, which combined with (5.3) implies that

$$\forall\,(t,x) \in (0,\infty)\times Y_d, \quad \lim_{\varepsilon\to 0} X_\varepsilon(t,x) = x + t\,\zeta. \tag{5.5}$$

Moreover, making the change of variable $r = \varepsilon\, s$ in (5.3) we have

$$\forall\,(t,x) \in [0,\infty)\times Y_d, \quad X_\varepsilon(t,x) = x + \int_0^t b\left(X\big(\frac{r}{\varepsilon},\frac{x}{\varepsilon}\big)\right) dr.$$

This combined with the boundedness of b and Lebesgue's theorem implies that the pointwise convergence (5.5) of X_ε holds actually in $L^p_{\rm loc}([0,\infty)\times\mathbb{R}^d)$ for any $p \in [1,\infty)$. Therefore, by the expression (5.4) with $u_0 \in C^1(\mathbb{R}^d)$, $u_\varepsilon(t,x)$ converges strongly in $L^p_{\rm loc}([0,\infty)\times\mathbb{R}^d)$ for any $p \in [1,\infty)$, to the function $u_0(x + t\,\zeta)$, which concludes the proof. □

A Derivation of invariant probability measures

Let T_t for $t \in \mathbb{R}$, be the mapping from $C^0_\sharp(Y_d)$ into itself defined by

$$(T_t f)(x) := f\big(X(t,x)\big) \quad \text{for } f \in C^0_\sharp(Y_d) \text{ and } x \in Y_d. \tag{A.1}$$

When a flow preserves the set of the continuous functions on a compact metric space, the existence of an invariant probability measure for the flow is a classical statement which can be derived thanks to a weak compactness argument applied to sequences of probability measures defined from the Birkhoff time averages in (1.5) (see, *e.g.*, [15, Theorem 1, Section 1.8] in the discrete time case). The following result adapts this statement restricting it to the limit points of the Birkhoff time averages for a given fixed function, adding possible variations of the spatial parameter x in the averages.

Proposition A.1 *Let $b \in C^1_\sharp(Y_d)^d$. There exists an invariant probability measure on Y_d for the flow X (1.15) associated with b. Moreover, let $g \in C^0_\sharp(Y_d)^d$, let $(x_n)_{n\in\mathbb{N}} \in (\mathbb{R}^d)^\mathbb{N}$, and let $(t_n)_{n\in\mathbb{N}} \in \mathbb{R}^\mathbb{N}$ be such that $\lim_n t_n = \infty$. Then, for any limit point $a \in \mathbb{R}^d$ of the sequence $(u_n)_{n\in\mathbb{N}} \in (\mathbb{R}^d)^\mathbb{N}$ defined by*

$$u_n := \frac{1}{t_n}\int_0^{t_n} g(X(s,x_n))\,ds, \quad n \in \mathbb{N}, \tag{A.2}$$

there exists a probability measure $\mu \in \mathscr{M}_p(Y_d)$ (independent of g) which is invariant for the flow X and which satisfies

$$a = \int_{Y_d} g(y)\,d\mu(y). \tag{A.3}$$

Proof of Proposition A.1. We will use the following result which is proved below.

Lemma A.2 *Let $(y_n)_{n\in\mathbb{N}} \in (\mathbb{R}^d)^\mathbb{N}$, let $(r_n)_{n\in\mathbb{N}} \in \mathbb{R}^\mathbb{N}$ be such that $\lim_n r_n = \infty$, and let ν_n, $n \in \mathbb{N}$, be the probability measure defined by*

$$\int_{Y_d} f(y)\,d\nu_n(y) = \frac{1}{r_n}\int_0^{r_n} f(X(s,y_n))\,ds \quad \text{for } f \in C^0_\sharp(Y_d). \tag{A.4}$$

Then, there exists a subsequence $(\nu_{n_k})_{k\in\mathbb{N}}$ of $(\nu_n)_{n\in\mathbb{N}}$ which converges weakly $*$ to some probability measure $\mu \in \mathscr{M}_p(Y_d)$ which is invariant for the flow X.

Let a be a limit point of the sequence $(u_n)_{n\in\mathbb{N}}$ (A.2), namely

$$a = \lim_{n\to\infty} \frac{1}{t_{\theta(n)}}\int_0^{t_{\theta(n)}} g(X(t,x_{\theta(n)}))\,dt,$$

for some strictly increasing sequence $(\theta(n))_{n\in\mathbb{N}}$ of integer numbers. Set $r_n := t_{\theta(n)}$, $y_n := x_{\theta(n)}$, and consider the associated sequence $(\nu_n)_{n\in\mathbb{N}}$ of probability measures on Y_d given by (A.4). By Lemma A.2 we can extract a subsequence $(\nu_{n_k})_{k\in\mathbb{N}}$ which converges weakly $*$ to some invariant probability measure $\mu \in \mathscr{I}_b$ for the flow X. We thus have

$$\forall\, f \in C^0_\sharp(Y_d), \quad \lim_{k\to\infty}\int_{Y_d} f(y)\,d\nu_{n_k}(y) = \int_{Y_d} f(y)\,d\mu(y),$$

which applied to each coordinate $f = g \cdot e_i$ of the vector-valued g, yields

$$a = \lim_{k\to\infty}\frac{1}{t_{\theta(n_k)}}\int_0^{t_{\theta(n_k)}} g(X(s,x_{\theta(n_k)}))\,ds = \lim_{k\to\infty}\int_{Y_d} g(y)\,d\nu_{n_k}(y) = \int_{Y_d} g(y)\,d\mu(y).$$

□

Proof of Lemma A.2. Since Y_d is a compact metrizable space, there exists a subsequence $(\nu_{n_k})_{k\in\mathbb{N}}$ of $(\nu_n)_{n\in\mathbb{N}}$ which converges weakly $*$ to some probability measure $\mu \in \mathscr{M}_p(Y_d)$, namely for any $f \in C^0_\sharp(Y_d)$,

$$\int_{Y_d} f(y)\,d\nu_{n_k}(y) = \frac{1}{r_{n_k}}\int_0^{r_{n_k}} f(X(s,y_{n_k}))\,ds \underset{k\to\infty}{\longrightarrow} \int_{Y_d} f(y)\,d\mu(y). \tag{A.5}$$

Let us prove that μ is invariant for the flow X. For the sake of simplicity denote $\tau_k := r_{n_k}$, $z_k := y_{n_k}$ and $\mu_k := \nu_{n_k}$. Let $t \in \mathbb{R}$ and $f \in C^0_\sharp(Y_d)$. By the semi-group property of the flow (1.16) we have

$$\int_{Y_d} (T_t f)(y)\,d\mu_k(y) = \frac{1}{\tau_k}\int_0^{\tau_k} f(X(s+t,z_k))\,ds.$$

By the change of variable $r = s + t$, it follows that

$$\int_{Y_d} (T_t f)(y)\, d\mu_k(y) = \frac{1}{\tau_k} \int_t^{t+\tau_k} f(X(r, z_k))\, dr$$
$$= \frac{1}{\tau_k} \int_0^{\tau_k} f(X(r, z_k))\, dr + \frac{1}{\tau_k} \int_{\tau_k}^{t+\tau_k} f(X(r, z_k))\, dr - \frac{1}{\tau_k} \int_0^{t} f(X(r, z_k))\, dr$$

Since f is bounded and $t \in \mathbb{R}$ is fixed, we deduce from (A.5) that

$$\lim_{k\to\infty} \int_{Y_d} (T_t f)(y)\, d\mu_k(y) = \int_{Y_d} f(y)\, d\mu(y).$$

However, by the definition of μ we also have

$$\lim_{k\to\infty} \int_{Y_d} (T_t f)(y)\, d\mu_k(y) = \int_{Y_d} (T_t f)(y)\, d\mu(y).$$

Hence, we get that

$$\forall\, t \in \mathbb{R},\ \forall\, f \in C^0_\sharp(Y_d), \quad \int_{Y_d} (T_t f)(y)\, d\mu(y) = \int_{Y_d} f(y)\, d\mu(y),$$

which implies that μ is invariant for the flow X. □

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References

[1] G. Alessandrini & V.Nesi: "Univalent σ-harmonic mappings", *Arch. Rational Mech. Anal.*, **158** (2001), 155-171.

[2] Y. Amirat, K. Hamdache & A. Ziani: "Homogénéisation d'équations hyperboliques du premier ordre et application aux écoulements miscibles en milieu poreux" (French) [Homogenization of a system of first-order hyperbolic equations and application to miscible flows in a porous medium], *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **6** (5) (1989), 397-417.

[3] Y. Amirat, K. Hamdache & A. Ziani: "Homogénéisation par décomposition en fréquences d'une équation de transport dans $\mathbb{R}^n$" (French) [Homogenization by decomposition into frequencies of a transport equation in $\mathbb{R}^n$], *C. R. Acad. Sci. Paris Sér. I Math.*, **312** (1) (1991), 37-40.

[4] Y. Amirat, K. Hamdache & A. Ziani: "Homogenisation of parametrised families of hyperbolic problems", *Proc. Roy. Soc. Edinburgh Sect. A*, **120** 3-4 (1992), 199-221.

[5] C. Baesens, J. Guckenheimer, S. Kim & R.S. MacKay: "Three coupled oscillators: mode-locking, global bifurcations and toroidal chaos", *Phys. D*, **49** (3) (1991), 387-475.

[6] A. Bensoussan, A., J.-L Lions & G. Papanicolaou: *Asymptotic Analysis for Periodic Structures*, Studies in Mathematics and its Applications **5**, North-Holland Publishing Co., Amsterdam-New York, 1978, 700 pp.

[7] Y. Brenier: "Remarks on some linear hyperbolic equations with oscillatory coefficients", *Proceedings of the Third International Conference on Hyperbolic Problems* (Uppsala 1990) Vol. I, II, Studentlitteratur, Lund (1991), 119-130.

[8] M. Briane: "Isotropic realizability of a strain field for the two-dimensional incompressible elasticity system", *Inverse Problems*, **32** (6) (2016), 22 pp.

[9] M. Briane: "Isotropic realizability of fields and reconstruction of invariant measures under positivity properties. Asymptotics of the flow by a nonergodic approach", *SIAM J. App. Dyn. Sys.*, **18** (4) (2019), 1846-1866.

[10] M. Briane: "Homogenization of linear transport equations. A new approach,", *J. École Polytechnique - Mathématiques*, **7** (2020), 479-495.

[11] M. Briane, G.W. Milton & V. Nesi : "Change of sign of the corrector's determinant for homogenization in three-dimensional conductivity", *Arch. Rat. Mech. Anal.*, **173** (1) (2004), 133-150.

[12] M. Briane, G.W. Milton et A. Treibergs : "Which electric fields are realizable in conducting materials?", *ESAIM: Math. Model. Numer. Anal.*, **48** (2) (2014), 307-323.

[13] R. Caccioppoli: "Sugli elementi uniti delle trasformazioni funzionali: un teorema di esistenza e unicitá alcune sue applicazioni", *Rend. Sem. Mat. Padova*, **3** (1932), 1-15.

[14] J. Carrand: "Logarithmic bounds for ergodic averages of constant type rotation number flows on the torus: a short proof", arXiv: 2012.07481 (2020), pp. 9.

[15] I.P. Cornfeld, S.V. Fomin & Ya.G. Sinaĭ: *Ergodic Theory*, translated from the Russian by A.B. Sosinskii, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] **245**, Springer-Verlag, New York, 1982, 486 pp.

[16] J. Franks: "Recurrence and fixed points of surface homeomorphisms", *Ergodic Theory Dynam. Systems*, **8** (1988), Charles Conley Memorial Issue, 99-107.

[17] J. Franks & M. Misiurewicz: "Rotation sets of toral flows", *Proc. Amer. Math. Soc.*, **109** (1) (1990), 243-249.

[18] B. Dacorogna: *Direct Methods in the Calculus of Variations*, Second Edition, Applied Mathematical Sciences **78**, Springer, New York, 2008, 619 pp.

[19] D. Fried: "The geometry of cross sections to flows", *Topology*, **21** (4) (1982), 353-371.

[20] D. Gilbarg & N.S. Trudinger: *Elliptic Partial Differential Equations of Second Order*, Reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2001, 531 pp.

[21] F. Golse: "Moyennisation des champs de vecteurs et EDP" (French), [The averaging of vector fields and PDEs], *Journées Équations aux Dérivées Partielles*, Saint Jean de Monts 1990, Exp. no. **XVI**, École Polytech. Palaiseau, 1990, 17 pp.

[22] F. Golse: “Perturbations de systèmes dynamiques et moyennisation en vitesse des EDP” (French), [On perturbations of dynamical systems and the velocity averaging method for PDEs], *C. R. Acad. Sci. Paris Sér. I Math.*, **314** (2) (1992), 115-120.

[23] M.R. Herman: “Existence et non existence de tores invariants par des difféomorphismes symplectiques” (French), [Existence and nonexistence of tori invariant under symplectic diffeomorphisms], *Séminaire sur les Équations aux Dérivées Partielles 1987-1988*, **XIV**, École Polytech. Palaiseau, 1988, 24 pp.

[24] M.W. Hirsch, S. Smale & R.L. Devaney: *Differential equations, Dynamical Systems, and an Introduction to Chaos*, Second edition, *Pure and Applied Mathematics* **60**, Elsevier Academic Press, Amsterdam, 2004, 417 pp.

[25] T.Y. Hou & X. Xin: “Homogenization of linear transport equations with oscillatory vector fields”, *SIAM J. Appl. Math.*, **52** (1) (1992), 34-45.

[26] P.-E. Jabin, A.-E. Tzavaras: “Kinetic decomposition for periodic homogenization problems", *SIAM J. Math. Anal.*, **41** (1) (2009), 360-390.

[27] A. Katok & B. Hasselblatt: *Introduction to the Modern Theory of Dynamical Systems*, Encyclopedia of Mathematics and its Applications **54**, Cambridge University Press, Cambridge, 1995, 802 pp.

[28] A.N. Kolmogorov: “On dynamical systems with an integral invariant on the torus” (Russian), *Doklady Akad. Nauk SSSR (N.S.)*, **93** (1953), 763-766.

[29] J. Llibre & R.S. MacKay: “Rotation vectors and entropy for homeomorphisms of the torus isotopic to the identity”, *Ergodic Theory Dynam. Systems*, **11** (1) (1991), 115-128.

[30] M. Misiurewicz & K. Ziemian: “Rotation sets for maps of tori”, *J. London Math. Soc. (2)*, **40** (3) (1989), 490-506.

[31] R. Peirone: “Convergence of solutions of linear transport equations”, *Ergodic Theory Dynam. Systems*, **23** (3) (2003), 919-933.

[32] C.L. Siegel: “Note on Differential Equations on the Torus”, *Annals of Mathematics*, **46** (3) (1945), pp. 423-428.

[33] *Dynamical Systems II, Ergodic Theory with Applications to Dynamical Systems and Statistical Mechanics* (Translated from the Russian), Edited by E.Ya Sinaĭ, Encyclopaedia of Mathematical Sciences **2**, Springer-Verlag Berlin 1989, 281 pp.

[34] T. Tassa: “Homogenization of two-dimensional linear flows with integral invariance”, *SIAM J. Appl. Math.*, **57** (5) (1997), 1390-1405.

[35] L. Tartar: “Nonlocal effects induced by homogenization”, *Partial Differential Equations and the Calculus of Variations Vol. II*, F. Colombini et al. (eds.), 925-938, Progr. Nonlinear Differential Equations Appl. **2**, Birkhäuser Boston, Boston, MA, 1989.