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To cite this version:

Marc Briane, Loïc Hervé. Fine asymptotic expansion of the ODE's flow. 2023. hal-03923357v3

HAL Id: hal-03923357
https://hal.science/hal-03923357v3

Preprint submitted on 5 Aug 2023

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Fine asymptotic expansion of the ODE's flow

Marc Briane & Loïc Hervé

Univ Rennes, INSA Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France

mbriane@insa-rennes.fr & loic.herve@insa-rennes.fr

Monday 17$^{\text{th}}$ July, 2023

Contents

Abstract

We study the dynamics of the flow X solution to the ODE: $X'(t,x) = b\big(X(t,x)\big)$ with $X(0,x) = x \in \mathbb{R}^d$, where b is a regular $\mathbb{Z}^d$-periodic vector field in $\mathbb{R}^d$. We provide conditions on b to get the fine asymptotic expansion: $|X(t,x) - x - t\,\zeta(x)| \leq M < \infty$. To this end, we try to express $X(t,x) - x - t\,\zeta(x)$ as $\Phi\big(X(t,x)\big) - \Phi(x)$, which yields the desired expansion when Φ is bounded. Then, assuming that the 2D Kolmogorov theorem and some extension for $d > 2$ hold, we establish several regimes depending on the commensurability of the rotation vectors $\zeta(x)$ for which the expansion of X is valid. Moreover, we prove that for any 2D flow with a non vanishing smooth b inducing a unique incommensurable rotation vector ξ, the fine expansion holds in $\mathbb{R}^2$ if, and only if, ξ_1/ξ_2 is a Diophantine number. The case where ξ is commensurable is also investigated. Finally, several examples illustrate the different results, including the case of a vanishing b which blows up the asymptotic expansion in some direction. In particular, the case of some Euler flows is investigated.

Keywords: ODE's flow, asymptotic expansion, rotation number, incommensurable vector, Diophantine and Liouville numbers, Euler flow

Mathematics Subject Classification: 34E05, 34E10, 37C10, 37C40

1 Introduction

Let b be a C^1-regular vector field in $\mathbb{R}^d$ defined on the torus $Y_d := \mathbb{R}^d/\mathbb{Z}^d$. In this paper, we study the ODE's flow $X(\cdot, x)$ for $x \in Y_d$, defined by

$$\begin{cases} \dfrac{\partial X}{\partial t}(t,x) = b(X(t,x)), \quad t \geq 0 \\ X(0,x) = x. \end{cases} \tag{1.1}$$

Here, we are interested by the asymptotics of the flow $X(t,x)$ as $t \to \infty$ for a given $x \in \mathbb{R}^d$. In dimension two the nice result due to Peirone [22] (see also [24]) claims that if the vector field b does not vanish in Y_2, then one has

$$\forall\, x \in \mathbb{R}^2, \quad \lim_{t\to\infty} \frac{X(t,x)}{t} = \zeta(x) \in \mathbb{R}^2, \tag{1.2}$$

where the limit vector $\zeta(x)$ may depend on x. On the contrary, when either b does vanish in Y_2 (see [24, Theorem 6.1]), or when dimension d is greater than 2 (see [22, Theorem 4.10]), limit (1.2) does not hold necessarily for any $x \in Y_d$. More recently, using the two-dimensional Peirone's result among others, the authors have obtained various asymptotic results for the flow (1.1) in any dimension with applications to the homogenization of linear transport equations [4, 5, 6]. Dimension two is very specific in ergodic theory, since Franks and Misiurewicz [11] have proved that for any continuous flow $X(t,x)$ the Herman rotation set [13] – derived from [21, Corollary 2.6] as the convex combination of the limit points of all the sequences $\big(X(n,x)/n\big)_{n\in\mathbb{N}}$ for $x \in Y_2$ – is actually a closed segment line of $\mathbb{R}^2$. In the case of a two-dimensional ODE's flow, the closed segment C_b is carried by a line passing through $0_{\mathbb{R}^2}$. For the ODE's flow X associated with the vector field b by (1.1), Herman's rotation set may be equivalently defined by

$$\mathsf{C}_b := \left\{ \int_{Y_d} b(x)\,\mu(dx) : \mu \in \mathscr{M}_{\rm p}(Y_d) \text{ s.t. for any } t \geq 0,\ \mu \circ X(t,\cdot) = \mu \right\}, \tag{1.3}$$

i.e. μ in (1.3) is a probability measure on Y_d which is invariant for the flow X. In dimension three the situation is again completely different, since [6, Theorem 4.1] shows that the rotation set (1.3) may be any convex polyhedron of $\mathbb{R}^3$ with rational vertices.

In this paper, we focus on a more precise asymptotics of the flow X (1.1). It is rather natural to study beyond the limits of type (1.2) when they do exist, the asymptotic behavior of the expansions

$$X(t,x) - x - t\,\zeta(x) \quad \text{as } t \to \infty \text{ and for } x \in \mathbb{R}^d. \tag{1.4}$$

In the framework of ergodic theory, the problem of the dynamics of the iterates F^n, $n \in \mathbb{N}$, of the lift F ([1]) obtained from some homeomorphism f homotopic to the identity on the torus Y_d (see, *e.g.*, [21]), is extremely delicate. Indeed, only dimension two is investigated, the estimates of the vector-valued expansion (1.4) for a general lift are only obtained in one direction, and moreover the last developments are quite recent. More precisely (see, *e.g.*, the introduction of [16] and the references therein), the two following results hold:

- By virtue of [17] and [19, Theorem 1] there exists a homeomorphism f on Y_2 homotopic to the identity with a lift F on $\mathbb{R}^2$, such that the Herman rotation set R_f is reduced to the unit set $\{\rho_f\}$ and

$$\forall\, v \in \mathbb{S}_1, \quad \sup_{x\in\mathbb{R}^2,\, n\in\mathbb{N}} \Big[\big(F^n(x) - x - n\,\rho_f\big)\cdot v\Big] = \infty. \tag{1.5}$$

[1] In the context of the ODE's flow X defined by (1.1), we have $F = X(1,\cdot)$, and due to the semi-group property of X we get that $F^n = X(n,\cdot)$ for any $n \in \mathbb{N}$.

In [19, Theorem 1] ρ_f is actually chosen to be $0_{\mathbb{R}^2}$.

- By virtue of [10, Theorem A], for any homeomorphism f on Y_2 homotopic to the identity with a lift F on $\mathbb{R}^2$ and the Herman rotation set R_f of which is a closed line segment of $\mathbb{R}^2$ with an irrational slope containing several points of $\mathbb{Q}^2$, there exist a unit vector v in $(\mathsf{R}_f)^\perp$ and a constant $M > 0$ such that

$$\forall\, \rho \in \mathsf{R}_f, \qquad \sup_{x\in\mathbb{R}^2,\, n\in\mathbb{Z}} \left| \big(F^n(x) - x - n\,\rho\big)\cdot v \right| \le M. \tag{1.6}$$

In our setting, we have obtained an example of a two-dimensional flow X (1.1) associated with a vanishing vector field b parallel to a fixed incommensurable vector ξ, which satisfies the large deviation (1.5) except in the direction $v := \xi^\perp$ (see Proposition 5.1), but whose Herman rotation set C_b is a non degenerate closed line segment of $\mathbb{R}^2$ (see Remark 5.1). In contrast, due to the differential structure we can hope better results than the two-dimensional bounded deviation (1.6) in some direction. More precisely, assuming the existence of the limit (1.2) for any point x in a subset A of $\mathbb{R}^d$, we will prove in several situations a fine asymptotic expansion of the type

$$\sup_{x\in A,\, t\ge 0} \left| X(t,x) - x - t\,\zeta(x) \right| \le M_A < \infty. \tag{1.7}$$

In Section 2 we prove a criterium (see Proposition 2.2) for which expression (1.4) reads as

$$\forall\, t \ge 0,\ \forall\, x \in \mathbb{R}^d, \quad X(t,x) - x - t\,\zeta(x) = \Phi\big(X(t,x)\big) - \Phi(x), \tag{1.8}$$

so that the boundedness of the vector-valued function Φ in $\mathbb{R}^d$ implies immediately the fine asymptotic expansion (1.7) in the whole set $\mathbb{R}^d$. The right-hand side of (1.8) can be regarded as a continuous sum of coboundary terms (see Remark 2.1). In return, from expression (1.8) we deduce (see Proposition 2.3) a general class of vector fields b such that (1.7) holds in $\mathbb{R}^d$. Finally, assuming that there exists a $\mathbb{Z}^d$-periodic regular gradient ∇u satisfying the positivity property $b\cdot\nabla u > 0$ in Y_d, Theorem 2.1 provides sufficient conditions for which the asymptotic expansion (1.7) is satisfied in $\mathbb{R}^d$.

Section 3 deals with the case of a non vanishing vector field b in $\mathbb{R}^2$ such that Herman's rotation set (1.3) is a unit set $\{\xi\}$ of $\mathbb{R}^2$, where the rotation vector $\xi = (\xi_1, \xi_2)$ is incommensurable in $\mathbb{R}^2$ (see (1.9)). This corresponds to the second case of the proof of [22, Theorem 3.1]. Assuming in addition the existence of an invariant probability measure for the flow with a positive regular Lebesgue's density, we prove (see Theorem 3.1) using the celebrated Kolmogorov theorem [18] that if the irrational number ξ_1/ξ_2 is a Diophantine number (see (1.10)), then the fine asymptotic fine expansion (1.7) is fulfilled in $\mathbb{R}^2$. In contrast, given a vector ξ in $\mathbb{R}^2$ such that ξ_1/ξ_2 is a Liouville number (see (1.11)), we can construct a two-dimensional Stepanoff's flow [25], *i.e.* a flow associated with the unidirectional vector field $b = a\,\xi$, such that the fine asymptotic expansion does not hold in $\mathbb{R}^2$.
At this point, note that the alternative between "commensurable and incommensurable" for the rotation vector is well-known in ergodic theory to guarantee the uniqueness of the asymptotics (1.2) of the flow (see, *e.g.*, [22]). Moreover, the alternative between "Diophantine and Liouville" is essential in the conjugacy Denjoy theorem related to the dynamical properties of the diffeomorphisms on the circle $\mathbb{S}_1$ with an irrational rotation number (see Remark 3.1 and the references therein). In the present context of the fine asymptotic expansion (1.7) of a two-dimensional ODE's flow, the same alternative on the irrational number ξ_1/ξ_2 can be regarded, up to our best knowledge, as a new example of the crucial role played by the Diophantine property of the rotation number in a dynamical system. Finally, using the rather restrictive

extension [20, Theorems 1,2] (see also [3, Theorem 3.3] which was obtained and used in an independent way) of Kolmogorov's theorem to dimension $d > 2$) the previous two-dimensional result can be also extended to higher dimension (see Remark 3.2).

In contrast with Section 3, Section 4 is devoted to the commensurable case in any dimension, which is based on the existence of periodic solutions in the torus Y_d to the ODE (1.1). Again assuming that Kolmogorov's theorem in dimension two and its extension [20, Theorems 1,2] in higher dimension hold true, we get (see Theorem 4.1) the fine asymptotic expansion (1.7) in $\mathbb{R}^2$, with an explicit non constant vector-valued function ζ in $\mathbb{R}^d$.

The results stated above are based on the condition that the vector field b does not vanish in Y_d. When b does vanish, the fine asymptotic expansion (1.7) may fail in $\mathbb{R}^d$. Indeed, Proposition 5.1 shows that the two-dimensional Stepanoff flow associated with the vector field $b = a\,\xi$, where a vanishes at one point in Y_2 and ξ is any incommensurable vector in $\mathbb{R}^2$, does not satisfy the fine asymptotic expansion (1.7) in the set $A = \mathbb{R}\,\xi + \mathbb{Z}^2$. In contrast, Example 5.4 provides a two-dimensional Stepanoff's flow which satisfies the fine asymptotic expansion in $\mathbb{R}^2$ for any vector ξ in $\mathbb{R}^2$, but the function a then has an infinite number of roots in Y_2.
Other examples illustrate the results of the paper in Section 5.

We have not succeeded for the moment to derive a fine asymptotic expansion (1.7) of the flow either without using the bounded coboundary sum of (1.8), or without the conditions supporting Kolmogorov's theorem in dimension two and its extension in higher dimension. For instance, when b is only a non vanishing regular two-dimensional vector field, namely the framework of [22], we do not know if the fine asymptotic expansion (1.7) holds in the whole set $\mathbb{R}^2$, while however the asymptotics (1.2) is satisfied at each point of $\mathbb{R}^2$.

Finally, in section 5.3 we illustrate this question with a two-dimensional Euler flow related to atmospheric flows [8, 28], and with the three-dimensional Arnold-Beltrami-Childress (ABC) flow which was first introduced by Arnold [2].

Definitions and notations

- $d \in \mathbb{N}$ denotes the space dimension.
- $\mathbb{S}_1$ denotes the unit sphere of $\mathbb{R}^2$.
- A vector ξ in $\mathbb{R}^d$ is said to be *incommensurable* in $\mathbb{R}^d$ if
$$\forall\, k \in \mathbb{Z}^d \setminus \{0_{\mathbb{R}^d}\}, \quad \xi \cdot k \neq 0. \tag{1.9}$$
Otherwise, the vector ξ is said to be *commensurable* in $\mathbb{R}^d$.
- A *Diophantine* number is an irrational real number λ with the property that there exists $m \in \mathbb{N}$ satisfying
$$\#\left(\left\{(p,q) \in \mathbb{Z} \times \mathbb{N} : \left|\lambda - \frac{p}{q}\right| \leq \frac{1}{q^m}\right\}\right) < \infty, \tag{1.10}$$
i.e. λ is badly approximated by rational numbers.
- On the contrary, a *Liouville number* is an irrational number λ with the property that for any $n \in \mathbb{N}$, there exists a pair of integers (p_n, q_n) with $q_n > 1$, such that
$$0 < \left|\lambda - \frac{p_n}{q_n}\right| < \frac{1}{(q_n)^n}\,, \tag{1.11}$$
i.e. λ is closely approximated by a sequence of rational numbers.

- $(e_1, \ldots, e_d)$ denotes the canonical basis of $\mathbb{R}^d$, and $0_{\mathbb{R}^d}$ denotes the null vector of $\mathbb{R}^d$.
- I_d denotes the unit matrix of $\mathbb{R}^{d\times d}$.
- "$\cdot$" denotes the scalar product and $|\cdot|$ the euclidean norm in $\mathbb{R}^d$.
- $\times$ denotes the cross product in $\mathbb{R}^3$.
- $|A|$ denotes the Lebesgue measure of any measurable set in $\mathbb{R}^d$ or Y_d.
- Y_d denotes the d-dimensional torus $\mathbb{R}^d/\mathbb{Z}^d$ (which may be identified to the unit cube $[0,1)^d$ in $\mathbb{R}^d$), and 0_{Y_d} denotes the null vector of Y_d.
- Π denotes the canonical surjection from $\mathbb{R}^d$ on Y_d.
- $C^k_c(\mathbb{R}^d)$, $k \in \mathbb{N} \cup \{\infty\}$, denotes the space of the real-valued functions in $C^k(\mathbb{R}^d)$ with compact support in $\mathbb{R}^d$.
- $C^k_\sharp(Y_d)$, $k \in \mathbb{N} \cup \{\infty\}$, denotes the space of the real-valued functions $f \in C^k(\mathbb{R}^d)$ which are $\mathbb{Z}^d$-periodic, *i.e.*
$$\forall\, k \in \mathbb{Z}^d,\ \forall\, x \in \mathbb{R}^d, \quad f(x+k) = f(x). \tag{1.12}$$
- The jacobian matrix of a C^1-mapping $F : \mathbb{R}^d \to \mathbb{R}^d$ is denoted by the matrix-valued function ∇F with entries $\dfrac{\partial F_i}{\partial x_j}$ for $i, j \in \{1, \ldots, d\}$.
- The abbreviation "a.e." for almost everywhere, will be used throughout the paper. The simple mention "a.e." refers to the Lebesgue measure on $\mathbb{R}^d$.
- dx or dy denotes the Lebesgue measure on $\mathbb{R}^d$.
- For a Borel measure μ on Y_d, extended by $\mathbb{Z}^d$-periodicity to a Borel measure $\tilde{\mu}$ on $\mathbb{R}^d$, a $\tilde{\mu}$-measurable function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be $\mathbb{Z}^d$-periodic $\tilde{\mu}$-a.e. in $\mathbb{R}^d$, if
$$\forall\, k \in \mathbb{Z}^d, \quad f(\cdot + k) = f(\cdot) \ \ \tilde{\mu}\text{-a.e. in } \mathbb{R}^d. \tag{1.13}$$
- For a Borel measure μ on Y_d, $L^p_\sharp(Y_d, \mu)$, $p \geq 1$, denotes the space of the μ-measurable functions $f : Y_d \to \mathbb{C}$ such that
$$\int_{Y_d} |f(x)|^p\, \mu(dx) < \infty.$$
- $L^p_\sharp(Y_d)$, $p \geq 1$, simply denotes the space of the Lebesgue measurable functions f in $L^p_{\mathrm{loc}}(\mathbb{R}^d)$, which are $\mathbb{Z}^d$-periodic dx-a.e. in $\mathbb{R}^d$.
- $\mathscr{M}_{\mathrm{loc}}(\mathbb{R}^d)$ denotes the space of the non negative Borel measures on $\mathbb{R}^d$, which are finite on any compact set of $\mathbb{R}^d$.
- $\mathscr{M}_\sharp(Y_d)$ denotes the space of the non negative Radon measures on Y_d, and $\mathscr{M}_p(Y_d)$ denotes the space of the probability measures on Y_d.
- $\mathscr{D}'(\mathbb{R}^d)$ denotes the space of the distributions on $\mathbb{R}^d$.

- For a Borel measure μ on Y_d and for $f \in L^1_\sharp(Y_d, \mu)$, we denote

$$\mu(f) := \int_{Y_d} f(x)\, \mu(dx), \tag{1.14}$$

which is simply denoted by $\overline{f}$ when μ is Lebesgue's measure. The same notation is used for a vector-valued function in $L^1_\sharp(Y_d, \mu)^d$. If f is non negative, its harmonic mean $\underline{f}$ is defined by

$$\underline{f} := \left(\int_{Y_d} \frac{dy}{f(y)} \right)^{-1}.$$

- For a given measure $\lambda \in \mathscr{M}_\sharp(Y_d)$, the Fourier coefficients of λ are defined by

$$\hat{\lambda}(n) := \int_{Y_d} e^{-2i\pi\, n\cdot x}\, \lambda(dx) \quad \text{for } n \in \mathbb{Z}^d.$$

The same notation is used for a vector-valued measure in $\mathscr{M}_\sharp(Y_d)^d$.

- c denotes a positive constant which may vary from line to line.

2 Fine asymptotic expansion

Definition 2.1 *A flow X associated with a vector field $b \in C^1_\sharp(Y_d)^d$ by* (1.1) *is said to admit a* fine asymptotic expansion *if there exists a $\mathbb{Z}^d$-periodic vector-valued function ζ such that*

$$\forall\, t \geq 0,\ \forall\, x \in \mathbb{R}^d, \quad X(t,x) = x + t\, \zeta(x) + O(1), \tag{2.1}$$

where $O(1)$ denotes a vector-valued function which is bounded uniformly with respect to t and x. More precisely, the flow X is said to admit a fine asymptotic expansion in the subset A of $\mathbb{R}^d$ *if there exists a constant $C_A > 0$ only depending on A, such that*

$$\forall\, t \geq 0,\ \forall\, x \in A, \quad \big|X(t,x) - x - t\, \zeta(x)\big| \leq C_A. \tag{2.2}$$

The following result gives a way for a flow to admit a fine asymptotic expansion (2.1).

Proposition 2.2 *Let b be a vector field in $C^1_\sharp(Y_d)^d$, and let ζ, Φ be two vector-valued functions in $C^1(\mathbb{R}^d)^d$. Then, the following assertions are equivalent :*

$$\forall\, t \geq 0,\ \forall\, x \in \mathbb{R}^d, \quad X(t,x) = x + t\, \zeta(x) + \Phi\big(X(t,x)\big) - \Phi(x), \tag{2.3}$$

$$(I_d - \nabla\Phi)\, b = \zeta \ \text{ in } \mathbb{R}^d \quad \text{and} \quad \forall\, t \geq 0,\ \ \zeta\big(X(t,\cdot)\big) = \zeta \text{ in } Y_d, \tag{2.4}$$

The last property in (2.4) *means that ζ is invariant for the flow X. If one of these two assertions is satisfied and Φ is bounded in $\mathbb{R}^d$, then ζ is $\mathbb{Z}^d$-periodic, the Herman rotation set is given by*

$$\mathsf{C}_b = \begin{cases} \operatorname{conv}\big(\zeta(Y_d)\big) & \text{if } d \geq 3 \\ \zeta(Y_2) & \text{if } d = 2, \end{cases} \tag{2.5}$$

and the flow X admits a fine asymptotic expansion in the sense of (2.1).

Remark 2.1 *If the flow X satisfies the expression* (2.3)*, then the function Φ is not necessarily periodic. However, for any $t \geq 0$, the function $\Phi\big(X(t,\cdot)\big) - \Phi(\cdot)$ is $\mathbb{Z}^d$-periodic, since the functions $\big(x \mapsto X(t,x) - x\big)$ and ζ are $\mathbb{Z}^d$-periodic. The function $\Phi\big(X(t,\cdot)\big) - \Phi(\cdot)$ can be regarded as a "continuous coboundary sum", since we have*

$$\Phi\big(X(n,\cdot)\big) - \Phi(\cdot) = \sum_{i=0}^{n-1} \left[\Phi\big(X(i+1,\cdot)\big) - \Phi\big(X(i,\cdot)\big)\right] \quad \textit{for } n \in \mathbb{N},$$

where each term of the sum is a coboundary term.
In the sequel we will construct such continuous coboundary sums possibly uniformly bounded in various situations, so that the fine asymptotic expansion (2.1) *will follow immediately.*

Based on Proposition 2.2 the following result allows us to construct a general family of flows which satisfy the fine asymptotic expansion (2.1).

Proposition 2.3 *Let Ψ be a C^2-diffeomorphism on Y_d satisfying the conditions*

$$\Phi : \big(x \in \mathbb{R}^d \mapsto x - \Psi(x)\big) \in C^2_\sharp(Y_d)^d \quad \textit{and} \quad \det(\nabla\Psi) \neq 0 \;\; \textit{in } Y_d. \tag{2.6}$$

Let ζ be a vector field in $C^1_\sharp(Y_d)^d$ satisfying the equality

$$\nabla\zeta\,(\nabla\Psi)^{-1}\,\zeta = 0 \;\; \textit{in } Y_d. \tag{2.7}$$

Then, the flow X associated with the vector field $b \in C^1_\sharp(Y_d)^d$ defined by

$$b := (\nabla\Psi)^{-1}\,\zeta = (I_d - \nabla\Phi)^{-1}\,\zeta \;\; \textit{in } Y_d, \tag{2.8}$$

fulfills both the expression (2.3) *and the fine asymptotic expansion* (2.1)*.*

Proof of Proposition 2.2. First, assume that assertion (2.3) holds. Then, by the boundedness of the vector field Φ and by the semi-group property of the flow X, we deduce from (2.3) that for any $t \geq 0$ and any $x \in \mathbb{R}^d$,

$$\lim_{s\to\infty} \frac{X(s,x)}{s} = \zeta(x) = \lim_{s\to\infty} \frac{X(s+t,x)}{s} = \lim_{s\to\infty} \frac{X(s,X(t,x))}{s} = \zeta\big(X(t,x)\big), \tag{2.9}$$

which shows that the vector-valued function ζ is invariant for the flow X. Moreover, we have

$$\forall\, x \in \mathbb{R}^d,\ \forall\, k \in \mathbb{R}^d, \quad \zeta(x+k) = \lim_{t\to\infty} \frac{X(t,x+k)}{t} = \lim_{t\to\infty} \frac{X(t,x)+k}{t} = \zeta(x),$$

which shows that ζ is $\mathbb{Z}^d$-periodic.

Now, let us determine the Herman rotation set C_b. By [21, Corollary 2.6] combined with (2.9) we have

$$\mathsf{C}_b = \operatorname{conv}\left(\bigcup_{x\in\mathbb{R}^d} \left[\bigcap_{n\in\mathbb{N}} \overline{\left\{\frac{X(k,x)-x}{k} : k \geq n\right\}}\right]\right) = \operatorname{conv}\big(\zeta(Y_d)\big). \tag{2.10}$$

In dimension two the first equality of (2.5) can be refined. Indeed, by virtue of [11, Theorem 1.2] for two-dimensional continuous flows, Herman's rotation set C_b is a closed line segment of $\mathbb{R}^2$, and by the continuity of ζ the subset $\zeta(Y_2)$ of $\mathbb{R}^2$ is a connected compact set. Therefore, it is enough to prove that the extremal points of C_b belong to $\zeta(Y_2)$. To this end, by [21, Remark 2.5] (see [6, Section 6.1] for a proof) each extremal point of C_b is a vector $\nu(b)$ for some ergodic invariant probability measure ν. Then, by Birkhoff's ergodic theorem there exists a point $x \in Y_2$ such that

$$\zeta(x) = \lim_{t\to\infty} \frac{X(t,x)}{t} = \nu(b) \in \zeta(Y_2),$$

which thus implies the second equality of (2.5).

Next, we have for any $t \geq 0$ and any $x \in \mathbb{R}^d$,

$$\frac{\partial}{\partial t}\big[X(t,x) - x - t\,\zeta(x) - \Phi\big(X(t,x)\big) + \Phi(x)\big] = \big(b - \nabla\Phi\, b\big)\big(X(t,x)\big) - \zeta(x). \tag{2.11}$$

Since the assertion (2.3) holds and ζ is invariant for X, the equality (2.11) is reduced to

$$\forall\, t \geq 0,\ \forall\, x \in \mathbb{R}^d, \quad \big(b - \nabla\Phi\, b\big)\big(X(t,x)\big) = \zeta\big(X(t,x)\big).$$

Therefore, taking $t = 0$ in the previous equality we get the relation (2.4).

Conversely, if the assertion (2.4) is satisfied, then the right hand side of (2.11) is zero, which implies that or any $t \geq 0$ and any $x \in \mathbb{R}^d$,

$$X(t,x) - x - t\,\zeta(x) - \Phi\big(X(t,x)\big) + \Phi(x) = X(0,x) - x - \Phi\big(X(0,x)\big) + \Phi(x) = 0,$$

which yields assertion (2.3).

Finally, note that the expression (2.3) of the flow X combined with the boundedness of the vector field Φ provides immediately the fine asymptotic expansion (2.1) of X, which concludes the proof of Proposition 2.2. □

Proof of Proposition 2.3. Define the mapping X by

$$X(t,x) := \Psi^{-1}\big(t\,\zeta(x) + \Psi(x)\big) \quad \text{for } (t,x) \in [0,\infty) \times \mathbb{R}^d. \tag{2.12}$$

First of all, let us prove that the vector-valued function ζ is invariant for X. Using the equalities (2.12) and

$$I_d = \nabla(\Psi^{-1} \circ \Psi) = \big(\nabla(\Psi^{-1}) \circ \Psi\big)\nabla\Psi \quad \text{in } \mathbb{R}^d, \tag{2.13}$$

we have for any $(t,x) \in [0,\infty) \times \mathbb{R}^d$,

$$\begin{aligned}
\frac{\partial}{\partial t}\big[\zeta\big(X(t,x)\big)\big] &= (\nabla\zeta)\big(X(t,x)\big)\,\frac{\partial}{\partial t}\big(X(t,x)\big) \\
&= (\nabla\zeta)\big(X(t,x)\big)\nabla(\Psi^{-1})\big(t\,\zeta(x) + \Psi(x)\big)\zeta(x) \\
&= (\nabla\zeta)\big(X(t,x)\big)(\nabla\Psi)^{-1}\big(X(t,x)\big)\zeta(x).
\end{aligned}$$

This combined with equality (2.7) yields that for a fixed $x \in \mathbb{R}^d$ and any $t \geq 0$,

$$f_x'(t) = -\big(\nabla\zeta\,(\nabla\Psi)^{-1}\big)\big(X(t,x)\big)\,f_x(t) \quad \text{where} \quad f_x(t) := \zeta\big(X(t,x)\big) - \zeta(x). \tag{2.14}$$

Hence, by the continuity of the $\mathbb{Z}^d$-periodic matrix-valued function $\nabla\zeta\,(\nabla\Psi)^{-1}$ in $\mathbb{R}^d$, for any $T \in (0,\infty)$ there exists a constant $c_T \geq 0$ such that

$$\forall\, t \in [0,T], \quad |f_x(t)| \leq c_T \int_0^t |f_x(s)|\,ds,$$

which by Grönwall's inequality applied in $[0,T]$ implies that $f_x = 0$ in $[0,T]$. Therefore, the vector field ζ is invariant for the mapping X.

Now, consider the vector field $b \in C^1_\sharp(Y_d)^d$ defined by (2.8). Hence, due to (2.13) and the invariance of ζ combined with equality (2.8), we have for any $(t,x) \in [0,\infty) \times \mathbb{R}^d$,

$$\begin{aligned}\frac{\partial}{\partial t}\big(X(t,x)\big) &= \nabla(\Psi^{-1})\big(t\,\zeta(x) + \Psi(x)\big)\,\zeta(x)\\ &= \big(\nabla(\Psi^{-1}) \circ \Psi\big)\big(X(t,x)\big)\,\zeta\big(X(t,x)\big)\\ &= (\nabla\Psi)^{-1}\big(X(t,x)\big)\,\zeta\big(X(t,x)\big) = b\big(X(t,x)\big).\end{aligned}$$

Therefore, the mapping X defined by (2.12) is actually the flow associated with the vector field b defined by (2.8) through the ODE (1.1).

Finally, since $\Psi(x) = x - \Phi(x)$ for $x \in \mathbb{R}^d$, the desired expression (2.3) of the flow X directly follows from the composition of equality (2.12) by Ψ, and the fine asymptotic expansion (2.1) is an immediate consequence of the $\mathbb{Z}^d$-periodicity of the vector-valued Φ.

This concludes the proof of Proposition 2.3. $\square$

Finally, the following result provides sufficient conditions to obtain two vector-valued functions ζ and Φ satisfying the expression (2.3) of the flow X, and to also derive fine asymptotic expansion (2.2) in some sets of $\mathbb{R}^d$.

Theorem 2.1 *Let $b \in C^1_\sharp(Y_d)^d$ be a vector field in $\mathbb{R}^d$, $d \geq 2$.*

i) Assume that the vector field b satisfies the positivity condition

$$\exists\,\nabla u \in C^0_\sharp(Y_d)^d, \quad b \cdot \nabla u > 0 \;\text{ in } Y_d. \tag{2.15}$$

Also assume that there exists a vector-valued function ζ such that X satisfies the asymptotics

$$\forall\, x \in Y_d, \quad \lim_{t\to\infty} \frac{X(t,x)}{t} = \zeta(x). \tag{2.16}$$

Then, the vector field ζ is invariant for the flow X, and there exists $\Phi \in C^1(\mathbb{R}^d)^d$ such that the expression (2.3) of the flow X holds.

ii) Replace in part i) condition (2.15) by the stronger gradient invertibility condition

$$\exists\,\nabla u_1 \in C^0_\sharp(Y_d)^d, \quad b \cdot \nabla u_1 = 1 \;\text{ in } Y_d. \tag{2.17}$$

Then, the fine asymptotic expansion (2.2) holds in any strip of $\mathbb{R}^d$ orthogonal to the direction $\xi := \overline{\nabla u_1}$ of type

$$\big\{x \in \mathbb{R}^d : x \cdot \xi \in [a,b]\big\} \quad \text{for } -\infty < a < b < +\infty. \tag{2.18}$$

iii) Replace in part ii) condition (2.17) by the existence of a vector field $U = (u_1, \ldots, u_d)$ satisfying

$$\nabla U \in C^0_\sharp(Y_d)^{d\times d} \quad \text{with} \quad \begin{cases} b \cdot \nabla u_1 = 1,\\ b \cdot \nabla u_2 = \cdots = b \cdot \nabla u_d = 0, \quad \text{in } Y_d.\\ \det(\nabla U) \neq 0,\end{cases} \tag{2.19}$$

Then, the fine asymptotic expansion (2.1) is satisfied through the expression (2.3) obtained with the vector field

$$\Phi(x) := x - \big(\overline{\nabla U}\big)^{-1} U(x) \;\text{ for } x \in \mathbb{R}^d \quad \text{and} \quad \zeta := \big(\overline{\nabla U}\big)^{-1} e_1. \tag{2.20}$$

Remark 2.2 *In dimension two Peirone [22, Theorem 3.1] proved remarkably that the asymptotics* (2.16) *of the flow* X *is always satisfied when the vector field* b *does not vanish in* Y_2*, while this asymptotics is generally false in higher dimension [22, Section 4] and in dimension two with a vanishing vector field* b *[23].*

Proof of Theorem 2.1.

Proof of part i). First of all, due to the asymptotics (2.16) the invariance of the vector-valued function ζ for the flow X follows from the equalities (2.9).

Next, following [5, Remark 3.6] we can consider for each $x \in \mathbb{R}^d$ the unique time $\tau(x)$ for the orbit $X(\cdot, x)$ to meet the equipotential $\{u = 0\}$, *i.e.*

$$u\big(X(\tau(x), x)\big) = 0. \tag{2.21}$$

Using the positivity (2.15) and the C^1-regularity of the flow X, the implicit function theorem implies that the function τ belongs to $C^1(\mathbb{R}^d)$. By the uniqueness of τ combined with the semi-group property of X we also have

$$\forall\, t \geq 0, \quad \tau\big(X(t, x)\big) = \tau(x) - t. \tag{2.22}$$

Now, consider the vector-valued function Φ (neither necessarily bounded in $\mathbb{R}^d$ nor $\mathbb{Z}^d$-periodic) defined by

$$\Phi(x) = \int_0^{\tau(x)} \big(\zeta(x) - b\big(X(s, x)\big)\big)\, ds \quad \text{for } x \in \mathbb{R}^d. \tag{2.23}$$

Then, we have for any $t \geq 0$ and any $x \in \mathbb{R}^d$,

$$\Phi\big(X(t, x)\big) = \int_0^{\tau(x)-t} \big(\zeta(x) - b\big(X(s + t, x)\big)\big)\, ds = \int_t^{\tau(x)} \big(\zeta(x) - b\big(X(s, x)\big)\big)\, ds.$$

Hence, taking the t-derivative of $\Phi\big(X(t, x)\big)$ at point $t = 0$, we get that

$$\forall\, x \in \mathbb{R}^d, \quad \nabla\Phi(x)\, b(x) = b(x) - \zeta(x),$$

which is exactly the first equality of (2.4). This combined with the invariance of ζ for X yields (2.4). Therefore, by virtue of Proposition 2.2 we deduce the equivalent expression (2.3) of the flow X.

Proof of part ii). From equation (2.17) we deduce that

$$\forall\, (t, x) \in [0, \infty) \times \mathbb{R}^d, \quad u_1\big(X(t, x)\big) = t + u_1(x).$$

Then, the solution $\tau(x)$ to the equation (2.21) with the function u_1 is given by $\tau(x) = -\, u_1(x)$, and the vector-valued function Φ defined by (2.23) reads as for any $x \in \mathbb{R}^d$,

$$\Phi(x) = \int_0^{-\, u_1(x)} \left(\zeta(x) - \frac{\partial X}{\partial s}(s, x)\right) ds = -\, u_1(x)\, \zeta(x) - X(-u_1(x), x) + x.$$

Since ∇u_1 is in $C^0_\sharp(Y_d)^d$, the function u_1 can be written $u_1(x) = \xi \cdot x - v_1(x)$ where $\xi = \overline{\nabla u_1}$ and $v_1 \in C^1_\sharp(Y_d)$. Then, we have for any point x in the affine hyperplane $x \cdot \xi = c$,

$$\Phi(x) = \big(v_1(x) - c\big)\, \zeta(x) + x - X\big(v_1(x) - c, x\big) = \big(v_1(x) - c\big)\, \zeta(x) - \int_0^{v_1(x)-c} b\big(X(s, x)\big)\, ds, \tag{2.24}$$

and for any $t \geq 0$,

$$\Phi\big(X(t,x)\big) = \big(v_1\big(X(t,x)\big) - c\big)\ \zeta(x) - \int_0^{v_1(X(t,x))-c} b\big(X(s+t,x)\big)\, ds.$$

Hence, since the functions v_1 and ζ are $\mathbb{Z}^d$-periodic and continuous in Y_d, we get that for any $t \geq 0$ and any x in the affine hyperplane $x \cdot \xi = c$,

$$\big|\Phi\big(X(t,x)\big) - \Phi(x)\big| \leq 2\left(|c| + \|v_1\|_{L^\infty_\sharp(Y_d)}\right)\left(\|\zeta\|_{L^\infty_\sharp(Y_d)^d} + \|b\|_{L^\infty_\sharp(Y_d)^d}\right).$$

Therefore, taking into account the expression (2.3) of the flow given by the part i), we obtain the fine asymptotic expansion (2.2) in any strip defined by (2.18).

Proof of part iii). This result has been obtained in [3, Theorem 3.3] for obtaining a class of ODE's flows whose Herman's rotation sets are reduced to a unit set. In the present context, by (2.19) and (2.20) we get immediately the equality

$$(I_d - \nabla\Phi)\, b = \big(\overline{\nabla U}\big)^{-1} DU\, b = \big(\overline{\nabla U}\big)^{-1} e_1 = \zeta \quad \text{in } Y_d,$$

which by virtue of Proposition 2.2 implies the fine asymptotic expansion (2.1).

The proof of Theorem 2.1 is done. $\square$

3 The incommensurable two-dimensional case

We have the following result.

Theorem 3.1

I) *Let b be a non vanishing vector field at least in $C^2_\sharp(Y_2)^2$ admitting an invariant probability measure $\sigma(x)dx$ where σ is a positive function at least in $C^5_\sharp(Y_2)$, such that*

$$\overline{\sigma b} \ \text{ is incommensurable in } \mathbb{R}^2 \quad \text{and} \quad \text{the ratio } \frac{\overline{\sigma b_1}}{\overline{\sigma b_2}} \text{ is a Diophantine number.} \tag{3.1}$$

Then, provided that b and σ are regular enough, the flow X defined by (1.1) satisfies the fine asymptotic expansion

$$\forall\, t \geq 0,\ \forall\, x \in \mathbb{R}^d, \quad X(t,x) = x + t\,\overline{\sigma b} + O(1), \tag{3.2}$$

where $O(1)$ is a vector-valued function which is bounded uniformly with respect to t and x.

II) *Let ξ be a unit vector of $\mathbb{R}^2$ such that ξ_1/ξ_2 is a Liouville number. Then, there exists a positive function $a \in C^\infty_\sharp(Y_2)$ such that the Stepanoff flow X associated with the vector field $b = a\,\xi$ does not satisfies the fine asymptotic expansion (2.1).*

Remark 3.1 *In view of the two cases of Theorem 3.1 – restricting ourselves to the class of smooth two-dimensional vector fields b such that each associated flow has an invariant probability measure $\sigma(x)dx$ with a smooth Lebesgue's density $\sigma > 0$ and an incommensurable rotation vector ξ $(= \overline{\sigma b}$ in (3.1)$)$ – we obtain that a necessary and sufficient condition to derive systematically the fine asymptotic assumption (2.1) in $\mathbb{R}^2$ with $\zeta(x) = \xi$, is that the ratio ξ_1/ξ_2 is a Diophantine number.*

On the one hand, by virtue of the Kolmogorov theorem [18] (see also [26, Lecture 11]) the Diophantine property of some rotation number permits to prove that the two-dimensional ODE (1.1) *can be mapped to a linear ODE through a suitable diffeomorphism on Y_2, provided that the vector field b is smooth and non vanishing in Y_2 and that the associated flow X has an invariant probability measure with a smooth Lebesgue's density. On the other hand, the conjugacy Denjoy theorem (see [15, Section 12.1]) claims that any smooth diffeomorphism on the circle $\mathbb{S}_1$ with an irrational rotation number ρ is topologically equivalent to the rotation of angle ρ. It turns out that the Arnold theorem [1] (see [15, Sections 12.3 and 12.5] and [9, Chapter 3, §5]) shows that the Diophantine property of the rotation number is essential to show that the conjugating map involved in Denjoy's theorem is smooth (at last differentiable). The construction of the Peirone two-dimensional counterexample [23] (recall Remark 2.2) is also based on some Diophantine rotation number for the ODE's flow. Alternatively, Theorem 3.1 seems to be, up to our best knowledge, a new example of the essential role played by the Diophantine property of the rotation number.*

Proof of Theorem 3.1.

PROOF OF PART I).

First step: Reduction to a Stepanoff flow.
By the Kolmogorov theorem [18] combined with enough regularity for the vector field b (at least C^2) and the invariant probability measure $\sigma(x)dx$ (at least C^5) ([2]), there exists a diffeomorphism Ψ on the torus Y_2 (see, *e.g.*, [5, Remark 2.1]) of class C^2 (at least) satisfying

$$\forall\, x \in \mathbb{R}^d, \quad \Psi(x) = Mx + \Psi_\sharp(x), \tag{3.3}$$

where $M \in \mathrm{SL}_2^\pm(\mathbb{Z})$ (*i.e.* M is a unimodular matrix) and $\Psi_\sharp \in C^2_\sharp(Y_2)^2$, such that the flow $\widehat{X}$ obtained from the flow X through the diffeomorphism Ψ by

$$\widehat{X}(t,y) := \Psi\big(X(t,\Psi^{-1}(y))\big) \quad \text{for } (t,y) \in \mathbb{R}\times Y_2, \tag{3.4}$$

is actually the flow associated with the vector field $\hat{b} \in C^1_\sharp(Y_2)^2$ defined by

$$\hat{b}(y) = \big((\nabla\Psi\, b)\circ\Psi^{-1}\big)(y) = a(y)\,\xi \quad \text{for } y \in Y_2, \tag{3.5}$$

where a is a non vanishing function in $C^1_\sharp(Y_2)$ (at least) and ξ a non null vector of $\mathbb{R}^2$. Moreover, we easily check that

$$\forall\, y \in Y_2, \quad \lim_{t\to\infty} \frac{\widehat{X}(t,y)}{t} = M\left(\lim_{t\to\infty}\frac{X(t,\Psi^{-1}(y))}{t}\right), \tag{3.6}$$

if one of the two limits does exist. However, by virtue of Liouville theorem (see, *e.g.*, [26, Lecture 11]) the vector field $\sigma\, b$ is divergence free in Y_2, so that there exists $u \in C^2_\sharp(Y_2)$ satisfying

$$\sigma\, b = R_\perp \nabla u \quad \text{or equivalently} \quad b = \sigma^{-1} R_\perp \nabla u \quad \text{in } Y_2.$$

By hypothesis the mean value of $\sigma\, b$ is incommensurable, so is the mean value of ∇u. Then, by virtue of [5, Corollary 3.4] the Herman rotation set associated with the vector field b is the unit set

$$\mathsf{C}_b = \big\{\overline{\sigma b}\big\}.$$

[2] See the remark of [12, p. 8-9] in connection with the Denjoy counterexample (see [14].

By [5, Proposition 2.1] this combined with (3.6) implies that

$$\forall\, y \in Y_d, \quad \lim_{t\to\infty} \frac{\widehat{X}(t,y)}{t} = M\left(\lim_{t\to\infty} \frac{X(t,\Psi^{-1}(y))}{t}\right) = M\,\overline{\sigma b}$$

which is also an incommensurable vector due to $M \in \mathrm{SL}_2^{\pm}(\mathbb{Z})$. Hence, again applying [5, Proposition 2.1] but with the Stepanoff flow $\widehat{X}$, using the results [6, Section 2.4] on the asymptotics of Stepanoff's flows, and recalling (3.5) we get that

$$\mathsf{C}_{\hat b} = \{\underline{a}\,\xi\} = \big\{M\,\overline{\sigma b}\big\}. \tag{3.7}$$

Hence, due to $M \in \mathrm{SL}_2^{\pm}(\mathbb{Z})$ it follows that ξ is an incommensurable vector of $\mathbb{R}^2$ as $\overline{\sigma b}$, and ξ_1/ξ_2 is a Diophantine number as the equivalent number $\overline{\sigma b_1}/\overline{\sigma b_2}$. Therefore, we are led to a Stepanoff's flow satisfying the same assumption (3.1) as the original flow X.

Now, it remains to derive the asymptotic (3.2) for any Stepanoff's flow satisfying condition (3.1) with $\sigma = \underline{a}/a$ and a regular enough. This is the aim of the following step.

Second step: The Stepanoff flow in the incommensurable case.
Assume that $\hat b = a\,\xi$ where a is a positive function in $C^1_\sharp(Y_2)$ and ξ is an incommensurable vector of $\mathbb{R}^2$ such that ξ_1/ξ_2 is a Diophantine number.

First of all, following [6, Section 2.4] recall some general results about the Stepanoff flow [25] in the incommensurable case, namely associated with the vector field $\hat b = a\,\xi$ where a is a positive function in $C^1_\sharp(Y_d)$ and ξ is an incommensurable unit vector of $\mathbb{R}^d$ for $d \geq 2$. Let θ be the function defined by

$$\begin{aligned} \theta(y) \;&:= \int_0^{y\cdot\xi} \left(\frac{\underline{a}}{a\big(t\,\xi + (y\cdot\xi^i)\,\xi^i\big)} - 1\right) dt \\ (s = t - y\cdot\xi) \;&= \int_{-y\cdot\xi}^{0} \left(\frac{\underline{a}}{a\big(s\,\xi + y\big)} - 1\right) ds \qquad\qquad \text{for } y \in \mathbb{R}^d, \end{aligned} \tag{3.8}$$

where $(\xi^2,\ldots,\xi^d)$ is an orthonormal basis of $(\mathbb{R}\,\xi)^\perp$ so that for any $y \in \mathbb{R}^d$,

$$y = (y\cdot\xi)\,\xi + (y\cdot\xi^i)\,\xi^i \quad \text{with} \quad (y\cdot\xi^i)\,\xi^i = (\xi^2\cdot y)\,\xi^2 + \cdots + (\xi^d\cdot y)\,\xi^d,$$

according to Einstein's convention. The function θ is in $C^1(\mathbb{R}^d)$ and satisfies for any $y \in \mathbb{R}^d$,

$$\begin{aligned} \nabla\theta(y)\cdot\xi \;&= \left(\frac{\underline{a}}{a(y)} - 1\right)\xi\cdot\xi + \int_0^{y\cdot\xi} \left[(\xi^i\otimes\xi^i)\,\nabla\Big(\frac{\underline{a}}{a}\Big)\big(t\,\xi + (y\cdot\xi^i)\,\xi^i\big)\right]\cdot\xi\,dt \\ &= \frac{\underline{a}}{a(y)} - 1 + \int_0^{y\cdot\xi} \left[\xi^i\cdot\nabla\Big(\frac{\underline{a}}{a}\Big)\,\big(t\,\xi + (y\cdot\xi^i)\,\xi^i\big)\right]\underbrace{(\xi^i\cdot\xi)}_{=0}\,dt \\ &= \frac{\underline{a}}{a(y)} - 1. \end{aligned} \tag{3.9}$$

On the other hand, the two-dimensional flow $\widehat{X}$ associated with the vector field $\hat b$ explicitly reads as

$$\widehat{X}(t,y) = F_y^{-1}(t)\,\xi + y \quad \text{where} \quad F_y(t) := \int_0^t \frac{ds}{a(s\,\xi + y)}, \tag{3.10}$$

and F_y^{-1} denotes the reciprocal function of F_y. By (3.9) we have

$$\underline{a}\,F_y(t) = t + \int_0^t \frac{\partial}{\partial s}\big(\theta(s\,\xi + y)\big)\,ds = t + \theta(t\,\xi + y) - \theta(y).$$

Therefore, replacing t by $F_y^{-1}(t)$ in the previous equality and using the expression (3.10) of the flow, we get that

$$\forall\, y \in \mathbb{R}^d, \quad \begin{cases} \forall\, t \geq 0, \quad \widehat{X}(t,y) = \underline{a}\, t\, \xi + y + \theta(y)\, \xi - \theta\big(\widehat{X}(t,y)\big)\, \xi \\ \displaystyle \lim_{t\to\infty} \frac{\widehat{X}(t,y)}{t} = \underline{a}\, \xi. \end{cases} \tag{3.11}$$

Now, assume that $d = 2$ and that ξ_1/ξ_2 is a Diophantine number. Consider the function $\alpha \in C^1_\sharp(Y_2)$ and its Fourier expansion defined by

$$\alpha(y) := \frac{\underline{a}}{a(y)} - 1 = \sum_{n\in\mathbb{Z}^2\setminus\{0_{\mathbb{R}^2}\}} \hat{\alpha}(n)\, e^{2i\pi\,(y\cdot n)} \quad \text{for } y \in Y_2, \tag{3.12}$$

where $\hat{\alpha}(n)$ denote the Fourier coefficients of α. Then, putting the Fourier expansion (3.12) in the second integral of (3.8), we may permute the integral and the series due to $\hat{\alpha} \in \ell^1(\mathbb{Z}^2)$, which implies that for any $x \in Y_2$,

$$\theta(y) = \sum_{n\in\mathbb{Z}^2\setminus\{0_{\mathbb{R}^2}\}} \frac{\hat{\alpha}(n)}{2i\pi\,(\xi\cdot n)} \left(e^{2i\pi\,(y\cdot n)} - e^{2i\pi\,(y-(y\cdot\xi)\,\xi)\cdot n}\right). \tag{3.13}$$

Next, since ξ_1/ξ_2 is a Diophantine number, by (1.10) there exists a non negative integer m_ξ such that

$$\#\left(\left\{(p,q) \in \mathbb{Z}\times\mathbb{N} : \left|\frac{\xi_1}{\xi_2} - \frac{p}{q}\right| \leq \frac{1}{q^{m_\xi+1}}\right\}\right) < \infty. \tag{3.14}$$

Also assume that $a \in C^{m_\xi+2}_\sharp(Y_2)$. Then, by the Cauchy-Schwarz inequality we get that

$$\left(n \in \mathbb{Z}^2\setminus\{0_{\mathbb{R}^2}\} \longmapsto |n|^{m_\xi}\, |\hat{\alpha}(n)| = \frac{|\hat{\alpha}(n)|\, |n|^{m_\xi+2}}{|n|^2}\right) \in \ell^1(\mathbb{Z}^2\setminus\{0_{\mathbb{R}^2}\}), \tag{3.15}$$

since by the Parseval identity applied with the tensor-valued function $\nabla^{(m_\xi+2)}\alpha \in C^0_\sharp(Y_2)^{2^{(m_\xi+2)}}$ we have

$$\sum_{n\in\mathbb{Z}^2\setminus\{0_{\mathbb{R}^2}\}} \frac{1}{|n|^4} < \infty \quad \text{and} \quad \sum_{n\in\mathbb{Z}^2\setminus\{0_{\mathbb{R}^2}\}} |n|^{2(m_\xi+2)}\, |\hat{\alpha}(n)|^2 \leq c\, \|\nabla^{(m_\xi+2)}\alpha\|^2_{\ell^2(\mathbb{Z})^{2^{(m_\xi+2)}}}.$$

Moreover, by (3.14) we have for any $n = (n_1, n_2) \in \mathbb{Z}^2\setminus\{0_{\mathbb{R}^2}\}$ with $|n| \geq N$ large enough,

$$|\xi\cdot n| = \begin{cases} |\xi_2\, n_2| \geq |\xi_2| & \text{if } n_1 = 0 \\ |\xi_2|\, |n_1|\, |\xi_1/\xi_2 + n_2/n_1| \geq |\xi_2|/|n_1|^{m_\xi} & \text{if } n_1 \neq 0, \end{cases}$$

which implies that

$$\exists\, c > 0,\ \forall\, n \in \mathbb{Z}^2\setminus\{0_{\mathbb{R}^2}\}, \quad |\xi\cdot n| \geq \frac{c}{|n|^{m_\xi}}. \tag{3.16}$$

This combined with (3.15) thus yields

$$\forall\, n \in \mathbb{Z}^2\setminus\{0_{\mathbb{R}^2}\} \text{ with } |n| \geq N, \quad \frac{|\hat{\alpha}(n)|}{|\xi\cdot n|} \leq C\, |\hat{\alpha}(n)|\, |n|^{m_\xi} = \frac{|\hat{\alpha}(n)|\, |n|^{m_\xi+2}}{|n|^2} \in \ell^1(\mathbb{Z}^2\setminus\{0_{\mathbb{R}^2}\}).$$

Therefore, we deduce that the asymptotic expansion of (3.11) satisfies the uniform estimate

$$\forall\, t \geq 0,\ \forall\, y \in \mathbb{R}^2, \quad \left|\widehat{X}(t,y) - t\, \underline{a}\, \xi - y\right| \leq c \sum_{n\in\mathbb{Z}^2\setminus\{0_{\mathbb{R}^2}\}} \frac{|\hat{\alpha}(n)|}{|\xi\cdot n|} < \infty, \tag{3.17}$$

which establishes the asymptotic expansion (3.2) for the Stepanoff flow in the Diophantine case.

Let us conclude the proof of part I). Starting from formula (3.4), multiplying formula (3.3) by the matrix M^{-1}, and using the estimate (3.17) of $\widehat{X}$ combined with the equality (3.7) and the boundedness of $\Psi_\sharp$, we get that for any $t \geq 0$ and any $x \in Y_2$,

$$\begin{aligned} X(t,x) = \Psi^{-1}\big(\widehat{X}(t,\Psi(x))\big) &= M^{-1}\big(\widehat{X}(t,\Psi(x))\big) - M^{-1}\big(\Psi_\sharp \circ \Psi^{-1}\big)\big(\widehat{X}(t,\Psi(x))\big) \\ &= M^{-1}\big(t\,\overline{a}\,\xi + Mx + \Psi_\sharp(x) + O(1)\big) - O(1) \\ &= t\,\overline{\sigma b} + x + O(1), \end{aligned}$$

which finally yields the desired fine asymptotic expansion (3.2).

Proof of part II).

Since ξ_1/ξ_2 is a Liouville number, by (1.11) there exist two sequences of integers $(p_n)_{n\in\mathbb{N}}$ in $\mathbb{Z}^\mathbb{N}$ and $(q_n)_{n\in\mathbb{N}}$ in $\mathbb{N}^\mathbb{N}$ satisfying

$$\forall\, n \in \mathbb{N}, \qquad \left|\frac{\xi_1}{\xi_2} - \frac{p_n}{q_n}\right| < \frac{1}{(q_n)^n}\,, \tag{3.18}$$

or equivalently,

$$\forall\, n \in \mathbb{N}, \quad |\xi \cdot k_n| < \frac{|\xi_2|}{(q_n)^{n-1}} \quad \text{where} \quad k_n := q_n\, e_1 - p_n\, e_2 \in \mathbb{Z}^2. \tag{3.19}$$

Up to extract a subsequence of the sequence $(q_n)_{n\in\mathbb{N}}$ (which converges to ∞) still denoted by $(q_n)_{n\in\mathbb{N}}$, we can assume in addition that

$$\forall\, n \geq 3, \quad q_n \geq |\xi \cdot k_{n-1}|^{\frac{1}{3-n}} + n + \sum_{i=1}^{n-1} q_i \qquad \text{and} \qquad \sum_{n=3}^{\infty} \frac{2\pi\,|\xi_2|}{(q_n)^{n-2}} < 1, \tag{3.20}$$

which implies in particular that $(q_n)_{n\in\mathbb{N}}$ is increasing. Then, define the positive function a in $C^\infty_\sharp(Y_2)$ by its inverse

$$\frac{1}{a(x)} := 1 + \sum_{n=3}^{\infty} \alpha_n \cos\left(2\pi\, k_n \cdot x\right) \quad \text{for } x \in Y_2, \quad \text{where} \quad \alpha_n := 2\pi\, q_n\, \xi \cdot k_n. \tag{3.21}$$

The function a is well defined and positive due to the second inequality of (3.20) combined with inequality (3.19). Moreover, since by (3.19) and (3.21) we have for any $m \in \mathbb{N}$,

$$\sum_{n=m+2}^{\infty} \alpha_n\,|k_n|^m \leq \sum_{n=m+2}^{\infty} 2\pi\,|\xi_2|\,\frac{q_n\,(|p_n| + q_n)^m}{(q_n)^{n-1}} \leq c \sum_{n=m+2}^{\infty} \frac{1}{(q_n)^{n-m-2}} < \infty,$$

the function a belongs to $C^\infty_\sharp(Y_2)$.

On the other hand, define the sequence $(\tau_n)_{n\in\mathbb{N}}$ by

$$\tau_n := \frac{1}{4\,\xi \cdot k_n} \quad \text{for } n \in \mathbb{N}. \tag{3.22}$$

Then, the function θ defined by the first integral of (3.8) with $1/a$ defined by the series expansion (3.21), satisfies for any integer $m \geq 4$ (note that $\underline{a} = 1$)

$$\begin{aligned}
\theta(\tau_m\,\xi) &= \int_0^{\tau_m} \left(\sum_{n=3}^{\infty} \alpha_n \, \cos\big(2\pi\,(\xi\cdot k_n)\,t\big) \right) dt \\
&= \sum_{n=3}^{\infty} \alpha_n \, \frac{\sin\big(2\pi\,(\xi\cdot k_n)\,\tau_m\big)}{2\pi\,(\xi\cdot k_n)} \\
&= q_m + \sum_{n=3}^{m-1} \alpha_n \, \frac{\sin\big(2\pi\,(\xi\cdot k_n)\,\tau_m\big)}{2\pi\,(\xi\cdot k_n)} + \sum_{n=m+1}^{\infty} \alpha_n \, \frac{\sin\big(2\pi\,(\xi\cdot k_n)\,\tau_m\big)}{2\pi\,(\xi\cdot k_n)},
\end{aligned}$$

which by the first inequalities of (3.20) and (3.19) implies that

$$\begin{aligned}
\theta(\tau_m\,\xi) &\geq q_m - \sum_{n=3}^{m-1} \frac{|\alpha_n|}{2\pi\,|\xi\cdot k_n|} - \sum_{n=m+1}^{\infty} |\tau_m|\,|\alpha_n| \\
&\geq q_m - \sum_{n=3}^{m-1} q_n - \frac{\pi}{2} \sum_{n=m+1}^{\infty} q_n \, \frac{|\xi\cdot k_n|}{|\xi\cdot k_m|} \\
&\geq m - \frac{\pi\,|\xi_2|}{2} \sum_{n=m+1}^{\infty} \frac{1}{(q_n)^{n-2}} \, \frac{1}{|\xi\cdot k_m|}.
\end{aligned} \tag{3.23}$$

Moreover, applying the first inequality of (3.20) with $n = m+1$, we get that for any $n \geq m+1$,

$$q_n \geq q_{m+1} \geq |\xi\cdot k_m|^{\frac{1}{2-m}} \quad \text{so that} \quad \frac{1}{(q_n)^{n-m}} \geq \frac{1}{(q_n)^{n-2}} \, \frac{1}{|\xi\cdot k_m|}.$$

This combined with (3.23) and the increase of $(q_n)_{n\in\mathbb{N}}$ thus yields

$$\theta(\tau_m\,\xi) \geq m - \frac{\pi\,|\xi_2|}{2} \sum_{n=m+1}^{\infty} \frac{1}{(q_n)^{n-m}} = m - \frac{\pi\,|\xi_2|}{2} \sum_{n=1}^{\infty} \frac{1}{(q_{n+m})^{n}} \geq m - \frac{\pi\,|\xi_2|}{2} \underbrace{\sum_{n=1}^{\infty} \frac{1}{(q_n)^{n}}}_{<\infty}.$$

Hence, we deduce that

$$\lim_{m\to\infty} \theta(\tau_m\,\xi) = \infty. \tag{3.24}$$

Finally, by the expression (3.10) of the Stepanoff flow for $y = 0_{\mathbb{R}^2}$, we have for any $m \in \mathbb{N}$,

$$\widehat{X}(t_m, 0_{\mathbb{R}^2}) = \tau_m\,\xi \quad \text{where} \quad t_m := F_{0_{\mathbb{R}^2}}(\tau_m).$$

Therefore, using the expression (3.11) of the flow $\widehat{X}$ for $y = 0_{\mathbb{R}^2}$ and limit (3.24), we obtain that

$$\big|\widehat{X}(t_m, 0_{\mathbb{R}^2}) - t_m\,\xi\big| = \big|\theta(0_{\mathbb{R}^2}) - \theta\big(\tau_m\,\xi\big)\big| \underset{m\to\infty}{\longrightarrow} \infty,$$

which shows that the fine asymptotic expansion (2.1) does not hold for the Stepanoff flow $\widehat{X}$.

The proof of part II) is done, which also concludes the proof of Theorem 3.1. □

Remark 3.2 *In higher dimension and in spirit of the case iii) of Theorem 2.1, assume that there exists a vector-valued function* $U := (u_1, \ldots, u_d)$ *satisfying besides condition* (2.19) *the following one*

$$\nabla U \in C^1_\sharp(Y_d)^{d\times d} \quad with \quad \begin{cases} b\cdot\nabla u_1 > 0, \\ b\cdot\nabla u_2 = \cdots = b\cdot\nabla u_d = 0, \quad in\ Y_d. \\ \det(\nabla U) \neq 0, \end{cases} \tag{3.25}$$

Then, following [3, Theorem 3.3] the matrix $\overline{\nabla U}$ *is invertible and the diffeomorphism on the torus* $\Psi := MU$ *with* $M := \left(\overline{\nabla U}\right)^{-1}$ (3), *satisfies*

$$\nabla\Psi \in C^1(Y_d)^{d\times d}, \quad \overline{\nabla\Psi} = I_d, \quad \nabla\Psi\, b = (b\cdot\nabla u_1)\,\xi \ \ in\ Y_d, \quad with\ \xi := M e_1. \tag{3.26}$$

Hence, Ψ *is a* C^2*-diffeomorphism on the torus* Y_d *(recall* (3.3)*) which maps the flow* X *associated with* b *to the Stepanoff flow* $\widehat{X}$ (3.4) *associated with the vector field*

$$\hat{b} := a\,\xi \quad where \quad a(y) := \big((b\cdot\nabla u_1)\circ\Psi^{-1}\big)(y) > 0 \quad for\ y\in Y_d. \tag{3.27}$$

When the vector ξ *satisfies the extension of* (3.16)

$$\exists\, c>0,\ \exists\, m_\xi\in\mathbb{N},\ \forall\, n\in\mathbb{Z}^d\setminus\{0_{\mathbb{R}^d}\}, \quad |\xi\cdot n| \geq \frac{c}{|n|^{m_\xi}}, \tag{3.28}$$

and $a \in C^{m_\xi+p}_\sharp(Y_d)$ *for some integer* $p > d/2$*, we get similarly to the proof of the second part of Theorem 3.1, that the flow* X *satisfies the fine asymptotic expansion* (2.1)*.*

In the part *iii)* of Theorem 4.1 below we will again use the previous diffeomorphism Ψ on Y_d with $d > 2$, in the case where the vector ξ is commensurable in $\mathbb{R}^d$.

4 The commensurable case in any dimension

We have the following result.

Theorem 4.1 *Let* $b \in C^1_\sharp(Y_d)^d$ *be a vector field in* $\mathbb{R}^d$*.*

i) Let A *be a non-empty subset of* $\mathbb{R}^d$*. Assume that there exist* $T_A, k_A \in (0,\infty)$ *such that the flow* X *satisfies the periodicity property*

$$\begin{aligned} &\forall\, x\in A,\ \exists\, T(x)\in(0,T_A],\ \exists\, k(x)\in\mathbb{Z}^d \text{ with } |k(x)|\leq k_A,\ \forall\, t\geq 0, \\ &\qquad\qquad X\big(t+T(x),x\big) = X(t,x) + k(x). \end{aligned} \tag{4.1}$$

Then, the flow X *associated with* b *satisfies the fine asymptotic expansion* (2.2) *in* A *with* $\zeta(x) := k(x)/T(x)$ *for* $x \in A$*.*

(3) Actually, the authors have recently discovered that the mapping Ψ used in [3] was previously introduced by Kozlov in [20, Theorems 1,2] to extend in some way the two-dimensional Kolmogorov theorem [18] to higher dimension.

ii) *Assume that b is a non vanishing vector field in $C^2_\sharp(Y_2)^2$ admitting an invariant probability measure $\sigma(x)\,dx$, where σ is a positive function in $C^5_\sharp(Y_2)$ with mean value 1, such that*

$$\overline{\sigma b} \ \textit{ is commensurable in } \mathbb{R}^2. \tag{4.2}$$

Then, there exists $T>0$ such that the flow X satisfies the fine asymptotic expansion (2.1) *with*

$$\zeta\big(\Psi(x)\big) := \left(\frac{1}{T}\int_0^T \frac{dt}{a\big(t\,\xi+\Psi(x)\big)}\right)^{-1}\xi \quad \textit{for } x\in Y_2, \tag{4.3}$$

where the C^2-diffeomorphism Ψ on Y_2 maps the flow X on the Stepanoff flow $\widehat{X}$ associated with the vector field $\hat b$ through equalities (3.3), (3.4), (3.5).

iii) *Assume that for $d>2$, the vector field b satisfies* (3.25) *with $DU\in C^1_\sharp(Y_d)^{d\times d}$, and that the vector $\xi := \big(\overline{\nabla U}\big)^{-1}e_1$ in* (3.26) *is commensurable,* i.e. *there exists $T>0$ such that $T\,\xi\in\mathbb{Z}^d$.*

Then, the flow X still satisfies the fine asymptotic expansion (2.1) *with the vector-valued function ζ defined by* (4.3) *in Y_d, where the C^2-diffeomorphism $\Psi = MU$ on Y_d maps the flow X on the Stepanoff flow associated with the vector field $\hat b$ through equalities* (3.25), (3.26), (3.27).

Remark 4.1 *By virtue of [11, Theorem 1.2] it is known that the rotation set C_b of the ODE's flow* (1.1) *associated with a vector field $b\in C^1_\sharp(Y_2)$ is always a closed line segment of $\mathbb{R}^2$ carried by a line passing through $0_{\mathbb{R}^2}$. This combined with [10, Theorem B] implies that if C_b contains a non null commensurable vector ζ, then the flow X satisfies a fine asymptotic expansion in the direction $\zeta^\perp$,* i.e. *there exists a constant $C\geq 0$ such that*

$$\forall\,t\geq 0,\ \forall\,x\in\mathbb{R}^2,\quad \big|\big(X(t,x)-x\big)\cdot\zeta^\perp\big|\leq C, \tag{4.4}$$

where the first-order term $t\,\zeta(x)$ does not appear due to $\zeta(x)\in\mathsf{C}_b\subset\mathbb{R}\,\zeta$. Estimate (4.4) *extends the one obtained in the first case of the proof of [22, Theorem 3.1] where the constant does depend on x* a priori.

Proof of Theorem 4.1.

Proof of part i). First of all, for $t\geq 0$ and $x\in A$, let $n_{t,x}$ be the integer satisfying

$$n_{t,x}\,T(x)\leq t<(n_{t,x}+1)\,T(x). \tag{4.5}$$

Reiterating equality (4.1) we get that

$$\begin{aligned} X(t,x) &= X\big(t-n_{t,x}\,T(x),x\big)+n_{t,x}\,k(x)\\ &= x+t\,\frac{k(x)}{T(x)}+\left(n_{t,x}-\frac{t}{T(x)}\right)k(x)+X\big(t-n_{t,x}\,T(x),x\big)-x, \end{aligned}$$

and by (4.5) we have

$$\begin{aligned} \left|\left(n_{t,x}-\frac{t}{T(x)}\right)k(x)+X\big(t-n_{t,x}\,T(x),x\big)-x\right| &\leq |k(x)|+\left|\int_0^{t-n_{t,x}\,T(x)} b\big(X(s,x)\big)\,ds\right|\\ &\leq k_A+T_A\,\|b\|_{L^\infty(Y_d)^d}. \end{aligned}$$

Therefore, we obtain the fine asymptotic expansion (2.2) for the flow X in the subset A with $\zeta(x) := k(x)/T(x)$ for $x \in A$.

Proof of part ii). Proceeding as the first step of Theorem 3.1 thanks to Kolmogorov's theorem we are led to Stepanoff flow associated with the vector field $\hat{b} = a\,\xi$, where a is a positive function in $C^1_\sharp(Y_2)$ and ξ is a vector of $\mathbb{R}^2$ such that $T\,\xi = k \in \mathbb{Z}^2$ for some $T \in (0,\infty)$. Indeed, due to (3.7) with $M \in \mathrm{SL}_2^{\pm}(\mathbb{Z})$ and to condition (4.2), the vector

$$\xi := \frac{1}{\underline{a}}\,M\,\overline{\sigma b} \ \text{ is commensurable in } \mathbb{R}^2. \tag{4.6}$$

Moreover, by the expression (3.10) of the Stepanoff flow $\widehat{X}$ combined with the $\mathbb{Z}^d$-periodicity of a, we have for any $t \geq 0$ and any $y \in \mathbb{R}^d$,

$$F_y(t+T) = F_y(t) + \int_0^T \frac{ds}{a(s\,\xi + y)} = F_y(t) + \frac{T}{m(y)} \quad \text{where} \quad m(y) := \left(\frac{1}{T}\int_0^T \frac{ds}{a(s\,\xi + y)}\right)^{-1}$$

Hence, replacing t by $F_y^{-1}(t)$ in the previous equality we obtain that

$$\widehat{X}\left(t + \frac{T}{m(y)}, y\right) = F_y^{-1}\left(t + \frac{T}{m(y)}, y\right)\xi + y = F_y^{-1}(t)\,\xi + T\,\xi + y = \widehat{X}(t,y) + k,$$

which implies condition (4.1) with $A := \mathbb{R}^d$, $T(y) := T/m(y)$ bounded by $T_A := T\,\|a^{-1}\|_{L^\infty(Y_2)}$, and $k(x) := k$. Therefore, the fine asymptotic expansion (2.1) holds with the vector-valued function ζ defined by (4.3), *i.e.*

$$\zeta(y) = m(y)\,\xi \quad \text{and} \quad \widehat{X}(t,y) = y + t\,\zeta(y) + O(1).$$

Hence, since the vector-valued functions $\big(y \mapsto \Psi^{-1}(y) - y\big)$ and $\big(x \mapsto \Psi(x) - x\big)$ are $\mathbb{Z}^2$-periodic and continuous thus bounded in $\mathbb{R}^2$, mapping the previous equality by Ψ^{-1} and using the relation (3.4) between the two flows X and $\widehat{X}$, we deduce that for any $t \geq 0$ and any $x := \Psi^{-1}(y) \in \mathbb{R}^2$,

$$X(t,x) = \Psi^{-1}\big(y + t\,\zeta(y) + O(1)\big) = \Psi(x) + t\,\zeta\big(\Psi(x)\big) + O(1) = x + t\,\zeta\big(\Psi(x)\big) + O(1),$$

which is the desired fine asymptotic expansion (2.1) satisfied by X.

Proof of part iii). The proof is quite similar to the one of case *ii*), which concludes the proof of Theorem 4.1. □

5 Examples

5.1 Cases with a non vanishing vector field

Let us start by a very simple example illustrating explicitly Theorem 3.1.

Example 5.1 Let ξ be an incommensurable vector of $\mathbb{R}^2$, and let b be the vector field

$$b(x) := \frac{\xi}{2 + \cos(2\pi x_1)} \quad \text{for } x \in Y_2.$$

Then, an explicit computation of formulas (3.8), (3.10) and (3.11) leads us to

$$\left\{\begin{aligned} & X(t,x) = x + \left[\frac{1}{2}\,t + \frac{\sin(2\pi x_1)}{4\pi\xi_1} - \frac{\sin\big(2\pi(x_1 + F_x^{-1}(t)\,\xi_1)\big)}{4\pi\xi_1}\right]\xi \\ & F_x(t) := 2\,t + \frac{\sin\big(2\pi(x_1 + t\,\xi_1)\big) - \sin(2\pi x_1)}{2\pi\xi_1}, \end{aligned}\right. \qquad \text{for } t \geq 0,\ x \in Y_2.$$

Therefore, the flow X associated with the vector field b satisfies Theorem 3.1, and consequently the fine asymptotic expansion (2.1) with the vector-valued function $\zeta(x) \equiv \frac{1}{2}\,\xi$.

The following example revisits the two-dimensional flow of [7, Example 2.7] in the light of the fine asymptotic expansion (2.1).

Example 5.2 Consider the non vanishing two-dimensional vector field b defined by

$$b(x) := e_1 + 2\pi\,\sin(2\pi x_2)\,e_2 = \nabla u(x) \quad \text{where} \quad u(x) := x_1 - \cos(2\pi x_2) \qquad \text{for } x \in \mathbb{R}^2. \tag{5.1}$$

By [7, Example 2.12] a tedious but easy computation shows that the flow X associated with the vector field (5.1) is given explicitly by

$$X(t,x) = \left\{\begin{array}{ll} (t+x_1)\,e_1 + \left[n + \frac{1}{\pi}\arctan\!\left(e^{4\pi^2 t}\tan(\pi x_2)\right)\right] e_2, & |x_2 - n| < \frac{1}{2} \\ (t+x_1)\,e_1 + \left(n+\frac{1}{2}\right) e_2, & x_2 = n + \frac{1}{2}, \end{array}\right. \quad \text{for } n \in \mathbb{Z}. \tag{5.2}$$

Condition (2.15) is clearly satisfied with $u(x) = x_1$.

Moreover, we have

$$\forall\,x \in Y_2, \quad \lim_{t\to\infty} \frac{X(t,x)}{t} = e_1, \tag{5.3}$$

so that by [5, Proposition 2.1] Herman's rotation set is the unit set $\mathsf{C}_b = \{e_1\}$. By the analysis of [7, Example 2.12] it is surprising to observe that the flow X (5.2) has no invariant measure of type $\sigma(x)dx$ where σ is a positive function in $C^0_\sharp(Y_2)$. However, the Radon measure $dx_1 \otimes \delta_{x_2=0}$ on Y_2 is invariant for the flow X. Indeed, we have

$$\forall\,\varphi \in C^1_\sharp(Y_2), \quad \int_{Y_2} b(x)\cdot\nabla\varphi(x)\,(dx_1 \otimes \delta_{x_2=0}) = \int_0^1 \frac{\partial\varphi}{\partial x_1}(x_1,0)\,dx_1 = 0,$$

which owing to Liouville theorem (see, *e.g.*, [5, Proposition 2.2]) yields the invariance.

Finally, the expression (5.2) of the flow shows directly that for any $t \geq 0$ and any $x \in \mathbb{R}^2$ such that $x_2 \in \left[n - \frac{1}{2}, n - \frac{1}{2}\right]$ with $n \in \mathbb{Z}$,

$$\left|X(t,x) - x - t\,e_1\right| \leq |n - x_2| + \frac{1}{2} \leq 1. \tag{5.4}$$

Therefore, the flow X satisfies the fine asymptotic expansion (2.1) with $\zeta = e_1$ and a uniformly bounded term.

However, following Proposition 2.2 it is interesting to recover the fine asymptotic expansion (2.3) from a suitable bounded vector-valued function Φ. To this end, the general definition (2.23) with asymptotics (5.3) leads us to the vector field Φ defined for $x \in \mathbb{R}^2$, by

$$\Phi(x) := \int_0^{\tau(x)} \big(e_1 - b(X(t,x))\big)\,dt \quad \text{where } \tau(x) \text{ is solution to} \quad u\big(X(\tau(x),x)\big) = 0, \tag{5.5}$$

which similarly to (2.3) yields the expression of the flow

$$\forall\, t \geq 0,\ \forall\, x \in \mathbb{R}^d, \quad X(t,x) = x + t\, e_1 + \Phi\big(X(t,x)\big) - \Phi(x). \tag{5.6}$$

Then, due to (5.2) we have

$$0 = u\big(X(\tau(x),x)\big) = X_1(\tau(x),x) - \cos\big(2\pi X_2(\tau(x),x)\big) = \tau(x) + x_1 - \cos\big(2\pi X_2(\tau(x),x)\big),$$

which implies that

$$\Phi(x) = -\,2\pi\, e_2 \int_0^{-x_1+\cos(2\pi X_2(\tau(x),x))} \sin\big(2\pi X_2(t,x)\big)\, dt. \tag{5.7}$$

By (5.2) we have for any $t \geq 0$ and any $x \in \mathbb{R}^2$ such that $x_2 \in \big(n - \frac{1}{2}, n + \frac{1}{2}\big)$ with $n \in \mathbb{Z}$,

$$\sin\big(2\pi X_2(t,x)\big) = \sin\Big[2\, \arctan\big(e^{4\pi^2 t} \tan(\pi x_2)\big)\Big] = \frac{2\, e^{4\pi^2 t} \tan(\pi x_2)}{1 + e^{8\pi^2 t} \tan^2(\pi x_2)}. \tag{5.8}$$

Therefore, we deduce the inequality

$$\forall\, x \in \mathbb{R}^2, \quad |\Phi(x)| \leq \int_{-\infty}^{\infty} \frac{4\pi\, e^{4\pi^2 t} \tan(\pi x_2)}{1 + e^{8\pi^2 t} \tan^2(\pi x_2)}\, dt = \frac{1}{\pi}\Big[\arctan\big(e^{4\pi^2 t} \tan(\pi x_2)\big)\Big]_{-\infty}^{\infty} = 1.$$

which yields the uniform boundedness of $\Phi\big(X(t,x)\big) - \Phi(x)$ with respect to t and x in (5.6).

5.2 Cases with a vanishing vector field

In the first example a vector field with separate variables is investigated.

Example 5.3 Let be the vector field $b(x) := \big(b_1(x_1), \ldots, b_d(x_d)\big) \in C^1_\sharp(Y_d)^d$ such that 0 is the unique root of the functions $b_1, \ldots, b_d$ in Y_1, which implies that b has 0_{Y_d} as unique root in Y_d.

First of all, it is clear that property (2.15) does not hold, since the vector field b does vanish. Then, the flow $X = (X_1, \ldots, X_d)$ associated with b is given for $i = 1, \ldots, d$ and $x \in Y_d$, by (see, *e.g.*, [6, Section 2.4])

$$\begin{cases} X_i(t,x) = F_{i,x}^{-1}(t) + x_i & \text{for } t \geq 0 \\ F_{i,x}(t) := \displaystyle\int_0^t \frac{ds}{b_i(s + x_i)} & \text{for } t \in \big([x_i] - x_i, 1 + [x_i] - x_i\big), \end{cases} \tag{5.9}$$

where $F_{i,x}^{-1}$ is the reciprocal function of $F_{i,x}$, and $[x_i]$ is the integer satisfying $[x_i] \leq x_i < [x_i] + 1$. Since the zero set of each function b_i is $\mathbb{Z}$, b_i has a constant sign in the interval $\big([x_i], 1 + [x_i]\big)$, and for any $\big([x_i], 1 + [x_i]\big)$,

$$\int_0^{[x_i]-x_i} \frac{ds}{b_i(s + x_i)} = -\int_0^{1+[x_i]-x_i} \frac{ds}{b_i(s + x_i)} \in \{-\infty, \infty\}.$$

Hence, the function $F_{i,x}^{-1}$ is a bijection from $\mathbb{R}$ on the interval $\big([x_i] - x_i, 1 + [x_i] - x_i\big) \subset [-1, 1]$. Therefore, the range of the flow X is contained in $[-1, 1]^d$, so that X satisfies the fine asymptotic expansion (2.1) with the vector-valued function $\zeta(x) \equiv 0$.

The following example deals with a two-dimensional Stepanoff flow associated with a vector field which has isolated roots in Y_2.

Example 5.4 Let $b \in C^\infty_\sharp(Y_2)^2$ be the vector field defined by

$$b(x) := \cos^2(\pi x_1)\,(e_1 + \gamma\, e_2) \quad \text{for } x \in Y_2, \quad \text{with } \gamma \in \mathbb{R}.$$

The flow X associated with b is given by the explicit formula

$$X(t,x) = \begin{cases} x + \left[\frac{1}{\pi}\arctan\big(\pi\, t + \tan(\pi(x_1 - n))\big) + n - x_1\right](e_1 + \gamma\, e_2) & \text{if } |x_1 - n| < \frac{1}{2} \\ x & \text{if } x_1 = n + \frac{1}{2}, \end{cases} \quad n \in \mathbb{Z}.$$

Therefore, the flow X satisfies the inequality

$$\forall\, t \geq 0,\ \forall\, x \in \mathbb{R}^2, \quad |X(t,x) - x| \leq \sqrt{1+\gamma^2},$$

which provides the fine asymptotic expansion (2.1) with the vector-valued function $\zeta(x) \equiv 0$.

The following general result shows that any two-dimensional Stepanoff flow associated with a vector field having one root in Y_2 and an incommensurable direction ξ in $\mathbb{R}^2$, does not satisfy the fine asymptotic expansion (2.2) in the set $A := \mathbb{R}\,\xi + \mathbb{Z}^d$.

Proposition 5.1 *Let $b = a\,\xi$ be a two-dimensional vector field such that $a \in C^1_\sharp(Y_2)$ has 0_{Y_2} as unique root in Y_2, and ξ is any incommensurable unit vector of $\mathbb{R}^2$.*
Then, the flow X satisfies the asymptotics

$$\forall\, x \in \mathbb{R}^2, \quad \zeta_\pm(x) := \lim_{t\to\pm\infty} \frac{X(t,x)}{t} = \begin{cases} \underline{a}\,\xi & \text{if } x \in \mathbb{R}^2 \backslash (\mathbb{R}\,\xi + \mathbb{Z}^2) \\ \underline{a}\,\xi & \text{if } x \in \mathbb{R}\,\xi + \mathbb{Z}^2,\ \pm\tau_x < 0 \\ 0_{\mathbb{R}^2} & \text{if } x \in \mathbb{R}\,\xi + \mathbb{Z}^2,\ \pm\tau_x \geq 0, \end{cases} \tag{5.10}$$

where τ_x is the unique real number satisfying

$$x + \tau_x\,\xi = k_x \in \mathbb{Z}^2. \tag{5.11}$$

Moreover, the fine asymptotic expansion (2.2) is not fulfilled in the set $A := \mathbb{R}\,\xi + \mathbb{Z}^2$, and the following large deviation holds

$$\forall\, v \in \mathbb{S}_1 \text{ s.t. } \xi\cdot v \neq 0, \quad \sup_{t\in\mathbb{R},\, x\in A} \big(X(t,x) - x - t\,\zeta_\pm(x)\big)\cdot v = \infty. \tag{5.12}$$

Remark 5.1 *Taking into account the asymptotics of the flow (5.10), by virtue of [21, Theorem 2.4, Remark 2.5, Corollary 2.6] the Herman rotation set is given by the non degenerate closed line segment*

$$\mathsf{C}_b = \mathrm{conv}\big(\zeta(\mathbb{R}^2)\big) = [0, \underline{a}]\,\xi.$$

Therefore, in the present case of a Stepanoff flow associated with a vanishing vector field and an incommensurable vector, we recover directly from the asymptotics of the flow the result of [6, Section 2.4] obtained by a perturbation result.

Contrary to the hypothesis of Proposition 5.1, the function a of the Stepanoff vector field $b = a\,\xi$, has non isolated roots in Example 5.4. It turns out that the fine asymptotic expansion (2.1) holds in Example 5.4 for any vector ξ in $\mathbb{R}^2$, while it fails in Proposition 5.1 for any incommensurable vector ξ in $\mathbb{R}^2$.

Proof of Proposition 5.1. First of all, make some considerations on the set $\mathbb{R}\,\xi+\mathbb{Z}^2$. By the incommensurability of ξ, for any $x \in \mathbb{R}\,\xi + \mathbb{Z}^2$ there exists a unique $\tau_x \in \mathbb{R}$ satisfying (5.11). Let y be a point in $\mathbb{R}^2 \setminus (\mathbb{R}\,\xi + \mathbb{Z}^2)$. Since ξ is incommensurable, the set $\mathbb{R}\,\xi + \mathbb{Z}^2$ is dense into $\mathbb{R}^2$. Then, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $(\mathbb{R}\,\xi+\mathbb{Z}^2)^{\mathbb{N}}$ which converges to y. We have

$$x_n = -\,\tau_{x_n}\,\xi + k_{x_n} \quad \text{with} \quad \tau_{x_n} \in \mathbb{R} \;\text{ and }\; k_{x_n} \in \mathbb{Z}^2, \tag{5.13}$$

where

$$\lim_{n\to\infty} |k_{x_n}| = \infty \quad \text{and consequently} \quad \lim_{n\to\infty} |\tau_{x_n}| = \infty. \tag{5.14}$$

Indeed, assume that the first limit of (5.14) does not hold. Then, there exists a subsequence of the integer vectors sequence $(k_{x_n})_{n\in\mathbb{N}}$ which is stationary, so that by (5.13) the corresponding subsequence of $(\tau_{x_n})_{n\in\mathbb{N}}$ converges, which implies that $y \in \mathbb{R}\,\xi+\mathbb{Z}^2$, a contradiction. Up to consider $-\,y$ with $\tau_{-y} = -\,\tau_y$, and to extract a subsequence we can assume that $\tau_{x_n} > 0$ for any $n \in \mathbb{N}$. We have just established the existence of a sequence $(x_n)_{n\in\mathbb{N}}$ in $(\mathbb{R}\,\xi+\mathbb{Z}^2)^{\mathbb{N}}$ satisfying

$$\forall\, n \in \mathbb{N}, \;\; x_n + \tau_{x_n}\,\xi \in \mathbb{Z}^2, \quad \lim_{n\to\infty} x_n = y \quad \text{and} \quad \lim_{n\to\infty} \tau_{x_n} = \infty. \tag{5.15}$$

On the other hand, due the uniqueness of the representation (5.11) τ_x is the unique root of the function $\big(t \mapsto a(t\,\xi + x)\big)$ in $\mathbb{R}$. Moreover, since the continuous function a does not vanish in the connected set $\mathbb{R}^2 \setminus \mathbb{Z}^2$, it has a constant sign in $\mathbb{R}^2 \setminus \mathbb{Z}^2$. Without loss of generality we can assume that a is positive in $\mathbb{R}^2 \setminus \mathbb{Z}^2$. Then, defining for each $x \in \mathbb{R}^2$ the function F_x by

$$F_x(t) := \begin{cases} \displaystyle\int_0^t \frac{ds}{a(s\,\xi + x)} & \text{for } t \in \mathbb{R}, \qquad\qquad \text{if } x \in \mathbb{R}^2\backslash(\mathbb{R}\,\xi+\mathbb{Z}^2) \\ \displaystyle\int_0^t \frac{ds}{a(s\,\xi + x)} & \text{for } t \in (-\infty,\tau_x), \;\; \text{if } x \in \mathbb{R}\,\xi+\mathbb{Z}^2,\;\; \tau_x > 0 \\ \displaystyle\int_0^t \frac{ds}{a(s\,\xi + x)} & \text{for } t \in (\tau_x,\infty), \qquad \text{if } x \in \mathbb{R}\,\xi+\mathbb{Z}^2,\;\; \tau_x < 0 \\ \qquad 0 & \text{for } t \in \mathbb{R}, \qquad\qquad \text{if } x \in \mathbb{Z}^2 \;(i.e.\;\; \tau_x = 0), \end{cases} \tag{5.16}$$

the function F_x is increasing in the first cases of (5.16) due to the positivity of a. Then, the reciprocal application F_x^{-1} is an increasing bijection from $\mathbb{R}$ onto $(-\infty,\tau_x)$ if $\tau_x > 0$, and from $\mathbb{R}$ onto (τ_x,∞) if $\tau_x < 0$. Hence, by formula (3.10) the flow X associated with the vector field $b = a\,\xi$ satisfies

$$\forall\, t \in \mathbb{R}, \quad X(t,x) = \begin{cases} F_x^{-1}(t)\,\xi + x & \text{if } x \in \mathbb{R}^2\backslash\,\mathbb{Z}^2 \\ \qquad x & \text{if } x \in \mathbb{Z}^2 \;(i.e.\;\; \tau_x = 0), \end{cases} \tag{5.17}$$

which combined with (5.16) and $\tau_{x_n} > 0$, implies in particular that

$$\forall\, n \in \mathbb{N}, \quad \lim_{t\to\infty} X(t,x_n) = \tau_{x_n}\,\xi + x_n. \tag{5.18}$$

Therefore, the formula (5.17) of the flow X together with the formula (5.16) of the function F_x (see also the positive case of [6, Section 2.4]) yield the desired asymptotics (5.10), which in return implies that

$$\forall\, n \in \mathbb{N}, \quad \lim_{t\to-\infty} \left(\frac{X(t,x_n)}{t}\right) = \underline{a}\,\xi. \tag{5.19}$$

Finally, applying (5.10) and (5.18) with the sequences $(\pm x_n)_{n\in\mathbb{N}}$ satisfying (5.15), we get that for any vector $v \in \mathbb{S}_1$ such that $\xi \cdot v \neq 0$,

$$\forall\, n \in \mathbb{N}, \qquad \zeta_\pm(\pm x_n) = 0_{\mathbb{R}^2} \quad \text{and} \quad \lim_{t\to\pm\infty} \big(X(t, \pm x_n) - \pm x_n\big) \cdot v = \tau_{\pm x_n}\, \xi \cdot v.$$

Hence, it follows that the fine asymptotic expansion (2.2) is not fulfilled in the set $A := \mathbb{R}\,\xi + \mathbb{Z}^2$, and that the following large deviation in any direction $v \in \mathbb{S}_1$ such that $\xi \cdot v \neq 0$, holds

$$\sup_{t\in\mathbb{R},\, x\in A} \big(X(t,x) - x - t\,\zeta_\pm(x)\big) \cdot v \geq \lim_{n\to\infty} (\tau_{\pm x_n}\, \xi \cdot v) = \infty \quad \text{if } \pm\,\xi \cdot v > 0, \tag{5.20}$$

which yields equality (5.12).

This concludes the proof of Proposition 5.1. $\square$

5.3 Euler flows on the torus

5.3.1 A two-dimensional Euler flow

Consider the vector field b defined on the torus $Y_2 := \mathbb{R}^2/2\pi\mathbb{Z}^2$, by

$$b(x) := -\,(A\cos x_1 + B\sin x_2)\, e_1 + (A\sin x_1 + B\cos x_2)\, e_2, \quad x \in Y_2, \quad \text{for } A, B \in \mathbb{R}. \tag{5.21}$$

The field b represents the velocity solution to the steady Euler equation (see, *e.g.*, [28])

$$\partial_t\omega + \psi_1\, \omega_2 - \psi_2\, \omega_1 = 0, \tag{5.22}$$

where the function ω denotes the (scalar) fluid vorticity, and ψ solution to $-\,\Delta\psi = \omega$, denotes the stream function. Here, by definition (5.21) we have

$$\omega(x) = (\operatorname{curl} b)(x) = (\partial_{x_1} b_2 - \partial_{x_2} b_1)(x) = A\cos x_1 + B\cos x_2 \quad \text{and} \quad \psi(x) = \omega(x).$$

According to [28] and the recent paper [8] the study of the stability of the flow (1.1) associated with the vector field (5.21) is relevant for the atmospheric flows

- of the outer planets of the solar system (Jupiter, Saturn, Uranus, Neptune), considering as in [8] the non stationary Euler equation on a rotating sphere,
- of Saturn's E Ring which may be regarded as a two-dimensional torus, considering as in [28] (see also [8, & 1.4.]) the non stationary Euler equation (5.22).

For the moment, we have not succeeded to deduce from our results the complete asymptotic picture for the flow associated with the vector field (5.21). Actually, there is generally no positive regular Y_2-periodic invariant function σ for the associated flow. Moreover, even if such an invariant function σ does exist, it must be enough regular to apply Kolmogorov's procedure of [18]. Indeed, Denjoy has built a counter-example when σ is only of class $C^{2-\varepsilon}$ (see [14]).

Let us now prove that there does not exist any positive regular Y_2-periodic invariant function when $A \neq 0$ and $B = 0$. To this end, assume by contradiction the existence of a positive invariant function $\sigma \in C^1_\sharp(Y_2)$ for the vector field b (5.21) with $A = 1$ and $B = 0$, *i.e.* σ is solution to the equation $\operatorname{div}(\sigma b) = 0$ in Y_2 by virtue of Liouville theorem (see, *e.g.*, [26, Lecture 11]). Then, we have

$$-\,\partial_{x_1}\big(\sigma(x)\cos x_1\big) + \partial_{x_2}\big(\sigma(x)\sin x_1\big) = 0 \quad \text{in } Y_2,$$

which is equivalent to the existence of a stream function ψ with $\nabla\psi \in C^0_\sharp(Y_2)^2$ – *i.e.* $\psi(x)$ can be written as the sum of an affine function $\lambda\cdot x = \lambda_1 x_1 + \lambda_2 x_2$, plus a periodic function – such that

$$\sigma(x)\cos x_1 = \partial_{x_2}\psi \quad \text{and} \quad \sigma(x)\sin x_1 = \partial_{x_1}\psi \quad \text{in } Y_2.$$

Integrating the a.e. positive periodic function $\big(x \mapsto \sigma(x)\cos^2 x_1\big)$ on the torus, we get that

$$\begin{aligned} 0 < 4\pi^2 \int_{Y_2} \sigma(x)\cos^2 x_1\,dx &= \int_0^{2\pi}\left(\int_0^{2\pi} \partial_{x_2}\psi(x_1,x_2)\,dx_2\right)\cos x_1\,dx_1 \\ &= \int_0^{2\pi}\big(\psi(x_1,2\pi) - \psi(x_1,0)\big)\cos(x_1)\,dx_1 = \int_0^{2\pi} 2\pi\lambda_2\cos x_1\,dx_1 = 0, \end{aligned}$$

which leads us to a contradiction.

As a consequence, we can apply neither Theorem 3.1 I) nor Theorem 4.1 *ii*) in the particular case $A \neq 0$ and $B = 0$. However, the flow $X(t,x)$ associated with the vector field (5.21) in this case is given explicitly (through a lengthy but easy computation) for $A > 0$ and $n \in \mathbb{Z}$, by:

$$\begin{aligned} X_1(t,x) &= \begin{cases} -\dfrac{\pi}{2} + n\pi + 2\arctan\left[\tan\left(\dfrac{x_1 - n\pi}{2} + \dfrac{\pi}{4}\right)e^{-A\,t}\right] & \text{if } |x_1 - n\pi| < \dfrac{\pi}{2} \\ x_1 & \text{if } x_1 = \dfrac{\pi}{2} + n\pi, \end{cases} \\ X_2(t,x) &= \begin{cases} x_2 - A\,t + \ln\left[\dfrac{1 + \tan^2\left(\frac{x_1 - n\pi}{2} + \frac{\pi}{4}\right)}{1 + \tan^2\left(\frac{x_1 - n\pi}{2} + \frac{\pi}{4}\right)e^{-2A\,t}}\right] & \text{if } |x_1 - n\pi| < \dfrac{\pi}{2} \\ x_2 + A\,t & \text{if } x_1 = \dfrac{\pi}{2} + n\pi. \end{cases} \end{aligned} \tag{5.23}$$

Therefore, we deduce directly the following asymptotics for any $A > 0$,

$$X(t,x) = x \mp A\,t\,e_2 + O_K(1), \quad \forall \pm t \in \mathbb{R}_+,\ \forall x \in K \times \mathbb{R},$$

where $O_K(1)$ is bounded uniformly with respect to $\pm\,t \in \mathbb{R}_+$, to $x_2 \in \mathbb{R}$, and to x_1 in any fixed compact set K of $(-\frac{\pi}{2}, \frac{\pi}{2})$ modulo π.

In conclusion, the general case with $AB \neq 0$ is thus far to be evident, and would deserve a further analysis.

5.3.2 A three-dimensional Euler flow

Consider the Arnold-Beltrami-Childress (ABC) flow [2], for $A, B, C \in \mathbb{R}$, associated with the vector field b defined on the torus $Y_3 := \mathbb{R}^3/2\pi\mathbb{Z}^3$, by

$$b(x) := (A\sin x_3 + C\cos x_2)\,e_1 + (B\sin x_1 + A\cos x_3)\,e_2 + (C\sin x_2 + B\cos x_1)\,e_3, \quad x \in Y_3, \tag{5.24}$$

and let $\vartheta := \alpha\,e_1 + \beta\,e_2 + \gamma\,e_3$ be a fixed vector in $\mathbb{R}^3$. Due to the well known identity $\operatorname{curl} b := \nabla \times b = b$, the vector field $b^\vartheta := b + \vartheta$ represents the velocity v solution to the steady incompressible Euler equation (also called Bernoulli equation in [2])

$$\partial_t v = v \times (\nabla \times v) - \nabla\big(p + \tfrac{1}{2}\,|v|^2\big) + f \quad \text{and} \quad \operatorname{div} v = 0, \tag{5.25}$$

where the function $f := -\,\vartheta \times v$ denotes the additional Coriolis force with the constant angular velocity $-\,\vartheta$.

For our purpose, assume that $C = 0$ and $|\alpha\,\gamma| > |AB|$ which implies that $|\alpha| > |A|$ or $|\gamma| > |B|$. Without loss of generality we can thus assume that $\alpha > |A|$ (indeed, the negative case $\alpha < -|A|$ is quite similar). Then, the ABC flow is reduced to

$$\partial_t X(t,x) = \begin{cases} A\sin\big(X_3(t,x)\big) + \alpha \\ B\sin\big(X_1(t,x)\big) + A\cos\big(X_3(t,x)\big) + \beta & \text{for } t\in\mathbb{R}_+,\ x\in\mathbb{R}^3. \\ B\cos\big(X_1(t,x)\big) + \gamma \end{cases} \tag{5.26}$$

Now, define the Hamiltonian associated with the two-dimensional flow composed by the first and the third differential equations of (5.26), by

$$\varphi(x_1,x_3) := \gamma\,x_1 - \alpha\,x_3 + B\sin x_1 + A\cos x_3 \quad \text{for } (x_1,x_3)\in\mathbb{R}^2, \tag{5.27}$$

so that the dynamical system (5.26) can be rewritten

$$\begin{cases} \big(\partial_t X_1(t,x), \partial_t X_3(t,x)\big) = (\nabla^\perp\varphi)\big(X_1(t,x), X_3(t,x)\big) \\ X_2(t,x) = x_2 + \big(\beta + \varphi(x_1,x_3)\big)\,t - \displaystyle\int_0^t \big(\gamma\,X_1(s,x) - \alpha\,X_3(s,x)\big)\,ds, \end{cases} \tag{5.28}$$

where $\nabla^\perp := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\nabla$ denotes the orthogonal gradient.

In the sequel we denote $x' := x_1\,e_1 + x_3\,e_3$. Then, we distinguish the two following regimes depending on whether the vector $\xi := \alpha\,e_1 + \gamma\,e_3$ is incommensurable or not in $\mathbb{R}^2$:

- On the one hand, assume that ξ is incommensurable in $\mathbb{R}^2$, and that γ/α is a Diophantine number. Since $\nabla^\perp\varphi$ (recall (5.27)) is divergence free with mean value ξ, we obtain, by virtue of Theorem 3.1 I) applied to the dynamical system of (5.28), the following fine asymptotics
$$X_1(t,x)\,e_1 + X_3(t,x)\,e_3 = x' + t\,\xi + O(1), \tag{5.29}$$
where $O(1)$ is bounded uniformly with respect to $t\in\mathbb{R}$ and $x'\in\mathbb{R}^2$.

- On the other hand, assume that vector ξ is commensurable, *i.e.* there exists $T > 0$ such that
$$T\,\xi = T\,(\alpha\,e_1 + \gamma\,e_3) = \kappa \in 2\pi\,\mathbb{Z}^2. \tag{5.30}$$
Without loss of generality we can also assume that $\gamma\,(B\cos x_1 + \gamma) > 0$ in $\mathbb{R}$, *i.e.* $|\gamma| > |B|$.

  – Otherwise, let $k\in\mathbb{N}$ be such that (recall that $\alpha > |A|$)
$$k > \frac{|\gamma| + |B|}{\alpha - |A|}\,.$$
Following [27, Section 2] the change of variables defined by
$$y' := \begin{pmatrix} y_1 \\ y_3 \end{pmatrix} = K\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = K\,x' \quad \text{where } K := \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \in \mathrm{SL}_2^+(\mathbb{Z}),$$
then allows us to replace the dynamical system of (5.28) with the vector field $\nabla^\perp\varphi(x')$ by the one with the vector field
$$K\,\nabla^\perp\varphi(K^{-1}y') = \begin{pmatrix} A\sin(y_3 - k\,y_1) + \alpha & > 0 \\ B\cos y_1 + \gamma + k\,\big(A\sin(y_3 - k\,y_1) + \alpha\big) & > 0 \end{pmatrix}$$

which remains Y_2-periodic. –

Now, the mapping Ψ defined by

$$y' := \Psi(x') = \Psi(x_1, x_3) := (x_1 + B/\gamma \sin x_1)\, e_1 + (x_3 - A/\alpha \cos x_3)\, e_3 \quad \text{for } x' \in \mathbb{R}^2, \tag{5.31}$$

is a C^∞-diffeomorphism on the torus $Y_2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$, which satisfies (recall (3.5))

$$\nabla\Psi(x')\, \nabla^\perp\varphi(x') = a\big(\Psi(x')\big)\, \xi \quad \text{where} \quad a(y') := \frac{1}{\alpha\,\gamma}\, (A \sin x_3 + \alpha)\, (B \cos x_1 + \gamma) > 0.$$

Note that the function $a(y')$ which is defined implicitly *via* the reciprocal mapping Ψ^{-1}, is also Y_2-periodic in view of (5.31). Therefore, by virtue of Theorem 4.1 *ii)* we obtain the following fine asymptotics

$$\begin{aligned} & X_1(t,x)\, e_1 + X_3(t,x)\, e_3 = x' + t\, \zeta\big(\Psi(x')\big) + O(1) \\ \text{where} \qquad & \zeta\big(\Psi(x')\big) = \zeta(y') := \left(\frac{1}{T}\int_0^T \frac{dt}{a\big(t\,\xi + y'\big)}\right)^{-1} \xi\,, \end{aligned} \tag{5.32}$$

and $T > 0$ is given by (5.30).

Note that we have no relevant information on the asymptotics of $X_2(t,x)$ given by (5.28).

Acknowledgment. The authors wish to thank the unknown referee for having suggested the stimulating application to Euler flows of section 5.3, which would deserve a further analysis.

References

[1] V.I. Arnold: “Small denominators I. Mapping the circle onto itself” (Russian), *Izv. Akad. Nauk SSSR Ser. Mat.*, **25** (1961), 21-86. English translation: *Trans. Amer. Math. Soc. (Series 2)*, **46** (1965), 213-284.

[2] V.I. Arnold: “Sur la topologie des écoulements stationnaires des fluides parfaits” (French), *C. R. Acad. Sci. Paris*, **261** (1965), 17-20. *Vladimir I. Arnold – Collected Works Vol. II. Hydrodynamics, Bifurcation Theory, and Algebraic Geometry* 1965-1972, 17-20. Ed. by A.B. Givental, B.A. Khesin, A.N. Varchenko, V.A. Vassiliev, O.Ya. Viro, Springer-Verlag, Berlin, 2014, 465 pp.

[3] M. Briane: “Isotropic realizability of fields and reconstruction of invariant measures under positivity properties. Asymptotics of the flow by a non ergodic approach”, *SIAM J. App. Dyn. Sys.*, **18** (4) (2019), 1846-1866.

[4] M. Briane & L. Hervé: “A picture of the ODE’s flow in the torus: from everywhere or almost-everywhere asymptotics to homogenization of transport equations”, *J. Differential Equations*, **304** (2021), 165-190.

[5] M. Briane & L. Hervé: “Asymptotics of ODE’s flow on the torus through a singleton condition and a perturbation result. Applications”, *Dis. Con. Dyn. Sys.*, **42** (7) (2022), 3431-3463.

[6] M. Briane & L. Hervé: “Specific properties of the ODE’s flow in dimension two *versus* dimension three”, accepted and to appear in *J. Dyn. Diff. Equa.*, DOI: https://doi.org/10.1007/s10884-022-10154-1, (2021), pp. 42. arXiv:2111.02090.

[7] M. Briane, G.W. Milton & A. Treibergs : "Which electric fields are realizable in conducting materials?", *ESAIM: Math. Model. Numer. Anal.*, **48** (2) (2014), 307-323.

[8] A. Constantin & P. Germain : "Stratospheric planetary flows from the perspective of the Euler equation on a rotating sphere", *Arch. Ration. Mech. Anal.*, **245** (2022), 587-644.

[9] I.P. Cornfeld, S.V. Fomin & Ya.G. Sinaĭ: *Ergodic Theory*, translated from the Russian by A.B. Sosinskii, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] **245**, Springer-Verlag, New York, 1982, 486 pp.

[10] P. Dàvalos: "On annular maps of the torus and sublinear diffusion", *J. Inst. Math. Jussieu*, **17** (4) (2018), 913-978.

[11] J. Franks & M. Misiurewicz: "Rotation sets of toral flows", *Proc. Amer. Math. Soc.*, **109** (1) (1990), 243-249.

[12] F. Golse: "Moyennisation des champs de vecteurs et EDP" (French), [The averaging of vector fields and PDEs], *Journées Équations aux Dérivées Partielles*, Saint Jean de Monts 1990, Exp. no. **XVI**, École Polytech. Palaiseau, 1990, 17 pp.

[13] M.R. Herman: "Existence et non existence de tores invariants par des difféomorphismes symplectiques" (French), [Existence and nonexistence of tori invariant under symplectic diffeomorphisms], *Séminaire sur les Équations aux Dérivées Partielles 1987-1988*, **XIV**, École Polytech. Palaiseau, 1988, 24 pp.

[14] M.R. Herman: "Sur les courbes invariantes par les difféomorphismes de l'anneau" (French), *Astérisque*, **103-104** (1983), 1-221.

[15] Katok & B. Hasselblatt: *Introduction to the modern theory of dynamical systems. With a supplementary chapter by Katok and Leonardo Mendoza*, Encyclopedia of Mathematics and its Applications, **54**, Cambridge University Press, Cambridge, 1995, pp. 802.

[16] A. Kocsard: "On the dynamics of minimal homeomorphisms of $\mathbb{T}_2$ which are not pseudo-rotations", *Ann. Scient. Éc. Norm. Sup., 4e série* **54** (4) (2021), 991-1034.

[17] A. Kocsard & A. Koropecki: "A mixing-like property and inexistence of invariant foliations for minimal diffeomorphisms of the 2-torus", *Proc. Amer. Math. Soc.*, **137** (10) (2009), 3379-3386.

[18] A.N. Kolmogorov: "On dynamical systems with an integral invariant on the torus" (Russian), *Doklady Akad. Nauk SSSR (N.S.)*, **93** (1953), 763-766.

[19] A. Koropecki & F. A. Tal: "Area-preserving irrotational diffeomorphisms of the torus with sublinear diffusion", *Proc. Amer. Math. Soc.*, **142** (10) (2014), 3483-3490.

[20] V. V. Kozlov: "Dynamical systems on a torus with multivalued integrals", (Russian) *Tr. Mat. Inst. Steklova* **256** (2007), *Din. Sist. i Optim.*, 201-218, translation in *Proc. Steklov Inst. Math.*, **256** (1) (2007), 188-205.

[21] M. Misiurewicz & K. Ziemian: "Rotation sets for maps of tori", *J. London Math. Soc.(2)*, **40** (3) (1989), 490-506.

[22] R. Peirone: "Convergence of solutions of linear transport equations", *Ergodic Theory Dynam. Systems*, **23** (3) (2003), 919-933.

[23] R. Peirone: "A nonhomogenizable linear transport equation in $\mathbb{R}^2$", *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **8** (1) (2009), 175-206.

[24] R. Peirone: "Homogenization of ODE's in $\mathbb{R}^N$", *Ann. Mat. Pura Appl.* (4) **198** (3) (2019), 869-879.

[25] W. Stepanoff: "Sur une extension du théorème ergodique", *Compositio Math.*, **3** (1936), pp. 239-253.

[26] Ya.G. Sinai: *Introduction to Ergodic Theory*, Translated by V. Scheffer, Mathematical Notes **18**, Princeton University Press, Princeton, N.J., 1976, 144 pp.

[27] T. Tassa: "Homogenization of two-dimensional linear flows with integral invariance", *SIAM J. Appl. Math.*, **57** (5) (1997), 1390-1405.

[28] D. Wirosoetisno & T.G. Shepherd : "Nonlinear stability of Euler flows in two-dimensional periodic domains", *Geophys. Astrophys. Fluid Dynam.*, **90** (3-4) (1999), 229-246.