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Abstract

Let P be a Markov kernel on a measurable state space $(\mathbb{X}, \mathcal{X})$ admitting some small-set $S \in \mathcal{X}$, that is: $P(x, A) \geq \nu(1_A)1_S(x)$ for any $x \in \mathbb{X}$, $A \in \mathcal{X}$ and for some positive measure ν. Let π be a $P-$invariant probability measure such that $\pi(1_S) > 0$. Using the non-negative residual kernel $R := P - \nu(\cdot)1_S$, we study the rate of convergence to π, in weighted or standard total variation norms, of normalized versions of the series $\sum_{n=1}^{+\infty} \nu \circ R^{n-1}$. Under drift-type conditions on R, we provide geometric/polynomial convergence bounds of the rate of convergence. Theses bounds are fully explicit and are as simple as possible. Their proofs do not require to introduce the split chain in the non-atomic case, the renewal theory, the coupling method, or the spectral theory.

1 Introduction

Let $(X_n)_{n\geq 0}$ be a Markov chain on a measurable state space $(\mathbb{X}, \mathcal{X})$ with transition kernel P. Let $\mathcal{M}^+$ (resp. $\mathcal{M}^+_*$) denote the set of finite non-negative (resp. positive) measures on $(\mathbb{X}, \mathcal{X})$. For any $\mu \in \mathcal{M}^+$ and any μ-integrable function $f : \mathbb{X} \to \mathbb{C}$, $\mu(f)$ denotes the integral $\int f d\mu$. Throughout the paper, the existence of a small-set S for P is assumed, that is

$$\exists S \in \mathcal{X},\ \exists \nu \in \mathcal{M}^+_*, \quad \forall x \in \mathbb{X},\ \forall A \in \mathcal{X}, \quad P(x, A) \geq \nu(1_A)\, 1_S(x). \tag{$\mathbf{S}$}$$

Under Condition ($\mathbf{S}$), we introduce the substochastic kernel R, called the residual kernel,

$$\forall x \in \mathbb{X},\ \forall A \in \mathcal{X}, \quad R(x, A) := P(x, A) - \nu(1_A)1_S(x) \tag{1}$$

and the following sequence $(\beta_k)_{k\geq 1} \in (\mathcal{M}^+)^{\mathbb{N}}$:

$$\beta_1 := \nu \quad \text{and} \quad \forall n \geq 2, \quad \beta_n := \nu \circ R^{n-1}. \tag{2}$$

Then the following statements are proved in Section 2 under the sole condition (S) (see Proposition 2.1). First the following equivalence holds:

$$P \text{ has an invariant probability measure } \pi \text{ such that } \pi(1_S) > 0 \iff \sum_{k=1}^{+\infty} \beta_k(1_{\mathbb{X}}) < \infty. \tag{3}$$

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Moreover, if we assume that $\sum_{k=1}^{+\infty}\beta_k(1_{\mathbb{X}}) < \infty$ and we set $\mu := \sum_{k=1}^{+\infty}\beta_k \in \mathcal{M}_*^+$, then $\mu(1_S) = 1$ and

$$\pi := \mu(1_{\mathbb{X}})^{-1}\mu \tag{4}$$

is a $P-$invariant probability measure on $(\mathbb{X}, \mathcal{X})$ such that $\pi(1_S) = \mu(1_{\mathbb{X}})^{-1} > 0$. Finally, for every $n \geq 1$, consider $\mu_n \in \mathcal{M}_*^+$ and the probability measure $\widetilde{\mu}_n$ on $(\mathbb{X}, \mathcal{X})$ defined by

$$\mu_n := \sum_{k=1}^{n}\beta_k \quad \text{and} \quad \widetilde{\mu}_n := \mu_n(1_{\mathbb{X}})^{-1}\,\mu_n. \tag{5}$$

Then, if $\|\cdot\|_{TV}$ is the total variation norm, we have $\lim_n \|\pi - \widetilde{\mu}_n\|_{TV} = 0$.

Hence a natural issue is: Can we specify the error approximation $\|\pi - \widetilde{\mu}_n\|_{TV}$? The same question is raised with respect to any weighted total variation norm (see (9)). First let us motivate such a study. Approximating π by $\widetilde{\mu}_n$ is less natural than that provided by the iterates P^n. In particular, the objective of the paper is not to present a new numerical method to approximate π. Actually the effective computation of $\widetilde{\mu}_n$ may not be necessary in problems only involving the error term $\|\pi - \widetilde{\mu}_n\|_{TV}$. In particular this may be an alternative theoretical tool in problems usually involving $\|\pi - P^n\|_{TV}$, provided that the control of $\|\pi - \widetilde{\mu}_n\|_{TV}$ is improved. For example, if P_θ is a perturbed Markov kernel of P_{θ_0}, then the quantities $\pi_\theta - \widetilde{\mu}_{n,\theta}$ defined from P_θ can be used as intermediate error terms to control $\pi_\theta - \pi_{\theta_0}$, where π_θ (resp. π_{θ_0}) is the invariant probability measure for P_θ (resp. P_{θ_0}). Note that only the error bounds for $\pi_\theta - \widetilde{\mu}_{n,\theta}$ are useful in this perturbation issue: neither $\widetilde{\mu}_{n,\theta}$ nor $\widetilde{\mu}_{n,\theta_0}$ need to be computed. The resulting error bounds for $\pi_\theta - \pi_{\theta_0}$ will be more accurate than those obtained with the intermediate term $\pi_\theta - P_\theta^n$, whenever the error bounds for $\pi_\theta - \widetilde{\mu}_{n,\theta}$ are better. Such a program is proposed in [HL22], generalizing in particular the results of [LL18, Sec. 2 and 3] for truncation approximations of atomic discrete Markov chains to general perturbed Markov kernels defined on a general state space.

Now let us return to the error approximation $\|\pi - \widetilde{\mu}_n\|_{TV}$, starting with the geometric case and the following contractive condition on the residual kernel R: $RV \leq \delta\, V$ for some $\delta \in (0,1)$ and some measurable function $V : \mathbb{X} \to [1, +\infty)$, called a Lyapunov function. Then it is easily deduced from (2) and (5) that $\|\pi - \widetilde{\mu}_n\|_{TV} = \mathrm{O}(\delta^n)$. More generally, if P satisfies the above contractive condition and PV is bounded on S, then it follows from [HL24, App. A] that there exists an explicit exponent $\alpha_0 \in (0,1]$ so that $RV^{\alpha_0} \leq \delta^{\alpha_0}V^{\alpha_0}$. The case $\alpha_0 = 1$ contains the atomic case but not only. Iterating $RV^{\alpha_0} \leq \delta^{\alpha_0}V^{\alpha_0}$ and using (2) (see (12)), it is easily checked that the following estimate holds

$$\forall n \geq 1, \quad \|\pi - \widetilde{\mu}_n\|_{TV} \leq \frac{2\,\nu(V^{\alpha_0})}{1-\delta^{\alpha_0}}\,\delta^{\alpha_0 n}.$$

This paper deals with the more difficult polynomial case, for which we use a similar approach which consists in introducing a basic drift condition on R (see (7) below) and then in finding appropriate procedures to return to this basic condition under more standard drift conditions. The main idea of this paper is to modify the Lyapunov function in order to fit the target case. In [HL24], under the standard geometric drift condition $PV \leq \delta V + K\,1_S$, new spectral properties of P are derived using such an approach. Here we show that this approach is specially fruitful to derive simple polynomial error bounds for $\|\pi - \widetilde{\mu}_n\|_{TV}$, or for some weighted total variation norms. The central point is that all the convergence bounds are fully explicit and are as simple as possible. Moreover the proofs can be thought of as self-contained in that we do not need to introduce the concepts of irreducibility, recurrence, or splitting technique for Markov chains. Of course, the drift conditions used here are directly inspired from that of the regeneration method (e.g. see [Num84, MT93, DMPS18, and references therein]). Finally, the residual kernel R has been used in the perturbation analysis of general Markov chains in [Kar81, Kar96]. We refer to [LL18, Sec. 3] for a recent contribution for atomic discrete Markov chains, where the condition $RV \leq \delta V$ for some $\delta \in (0,1)$ is used to get bounds on the truncation approximations of π in terms of the residual matrix R.

Under Condition (**S**), the following results are obtained in this paper. In Section 2 Equivalence (3) is specified in Proposition 2.1. Then, restricting the discussion here to the standard total variation norm, we prove in Theorem 2.2 that π given by Formula (4) is approximated in total variation norm by $(\widetilde{\mu}_n)_{n\geq 1}$ with the following error estimates

$$\|\pi - \widetilde{\mu}_n\|_{TV} \leq 2\,\mu(1_{\mathbb{X}})^{-1}\,\varepsilon_n \leq 2\,\varepsilon_n \quad \text{with} \quad \varepsilon_n := \sum_{k=n+1}^{+\infty} \beta_k(1_{\mathbb{X}}) \xrightarrow[n\to+\infty]{} 0. \tag{6}$$

In Section 3 the following polynomial drift-type conditions on R are introduced to study the rate of convergence of $(\varepsilon_n)_{n\geq 1}$: There exists a collection $\{V_i\}_{i=0}^m$ of Lyapunov functions with $m \geq 1$ such that

$$\forall i \in \{0,\dots,m-1\}, \quad RV_i \leq V_i - V_{i+1}. \tag{7}$$

Under Condition (7), we prove that $\lim_n n^{m-1}\varepsilon_n = 0$ in Theorem 3.1. The sequence $(\beta_n(V_m))_{n\geq 1}$ is analyzed in Theorem 3.2 to obtain computable rates of convergence for $(\varepsilon_n)_{n\geq 1}$. In particular the following property is stated in Corollary 3.4: if $m \geq 2$, then

$$\forall n \geq 1, \quad \varepsilon_n \leq \frac{C_m\,\nu(V_0)}{(m-1)}\,\frac{1}{n^{m-1}} \quad \text{with} \quad C_m := 2^{\frac{m(m+1)}{2}-1}. \tag{8}$$

It turns out that (7) is our target polynomial drift condition, as the condition $RV \leq \delta\,V$ was in the geometric case. In Section 4 appropriate procedures to fit Condition (7) are provided when starting with the following drift condition on R: $\exists\alpha \in [0,1)$, $\exists c > 0$, $RV \leq V - c\,V^\alpha$. This is adapted from a standard polynomial drift condition on P introduced in [JR02]. In the atomic case, using an iterative procedure, we prove that, under Conditions $RV \leq V - c\,V^\alpha$ and $\sup_S PV < \infty$, then the bound (8) holds with $m := \lfloor (1-\alpha)^{-1} \rfloor$, where $\lfloor\cdot\rfloor$ denotes the integer part function on $\mathbb{R}$ (see Corollary 4.1). The key point in the atomic case is that (7) always holds on S when $V_{i+1} \leq V_i$ since $R = 0$ on S. In the non atomic case, this property is no longer automatically satisfied. However the previous iterative procedure can be adapted under standard polynomial drift conditions, replacing the inequality $RV \leq V - c\,V^\alpha$ by $R\widehat{V} \leq \widehat{V} - \widehat{c}\,\widehat{V}^{\widehat{\alpha}}$ with $\widehat{V} = V^{\eta_0}$ for some explicit $\eta_0, \widehat{\alpha} \in (0,1]$. Then, if $\eta_0 \geq 1-\alpha$ and if V, PV are bounded on S, the bound (8) holds with $m := \lfloor \eta_0(1-\alpha)^{-1} \rfloor$ (Proposition 4.4).

The above error bounds actually hold in $W-$weighted total variation norm (see (9)) for any $W \geq 1$ such that $\mu(W) < \infty$ in Section 2, and for $W = V_i$ in Sections 3-4.

2 Basic material

For any Lyapounov function W, the W-weighted total variation norm $\|\lambda_1 - \lambda_2\|_W$ for any $(\lambda_1, \lambda_2) \in (\mathcal{M}^+)^2$ is defined by

$$\|\lambda_1 - \lambda_2\|_W := \sup_{|f|\leq W} \big|\lambda_1(f) - \lambda_2(f)\big|. \tag{9}$$

If $W := 1_{\mathbb{X}}$, then $\|\lambda_1 - \lambda_2\|_{1_{\mathbb{X}}} = \|\lambda_1 - \lambda_2\|_{TV}$ is the standard total variation norm. When λ_1 and λ_2 are probability measures, $\|\lambda_1 - \lambda_2\|_{TV}$ is their standard total variation distance.

Let P be a Markov kernel on $(\mathbb{X}, \mathcal{X})$ satisfying Condition (**S**). Consider the associated non-negative residual kernel $R := P - \nu(\cdot)1_S$ in (1) and the sequence $(\beta_k)_{k\geq 1} \in (\mathcal{M}^+)^{\mathbb{N}}$ defined in (2). First we prove in Proposition 2.1 that, under the sole Condition (**S**), P has an invariant probability measure π with $\pi(1_S) > 0$ if, and only if, $\sum_{k=1}^{+\infty} \beta_k(1_{\mathbb{X}}) < \infty$. In particular, the Nummelin-type representation (10) of π below is in force in this work. Such a result is well-known under various recurrence assumptions on the Markov chain. The reader can consult [Num84, Th. 5.2, Cor. 5.2]), [MT93, Chap. 10]) where comments on the story of such kind of results are provided. An analytic proof of Proposition 2.1 is provided in Appendix A and allows us to get general statements in a very efficient way. In particular, we do not need to introduce the concepts of irreducibility, recurrence, atom or splitted chain associated with $(X_n)_{n\in\mathbb{N}}$.

Proposition 2.1 *If P satisfies Condition* (**S**)*, then the following assertions are equivalent.*

(i) *There exists an $P-$invariant probability measure π on $(\mathbb{X}, \mathcal{X})$ such that $\pi(1_S) > 0$.*

(ii) $\displaystyle\sum_{k=1}^{+\infty} \beta_k(1_{\mathbb{X}}) < \infty.$

Under any of these two conditions, the sequence $(\beta_k(1_{\mathbb{X}}))_k$ is decreasing, and

$$\pi := \mu(1_{\mathbb{X}})^{-1}\,\mu \quad \textit{with} \quad \mu := \sum_{k=1}^{+\infty} \beta_k \in \mathcal{M}_*^+ \tag{10}$$

is an $P-$invariant probability measure on $(\mathbb{X}, \mathcal{X})$ with $\mu(1_S) = 1$ and $\pi(1_S) = \mu(1_{\mathbb{X}})^{-1} > 0$.

Under Assumption (**S**) and $\sum_{k=1}^{+\infty} \beta_k(1_{\mathbb{X}}) < \infty$, μ is the $P-$invariant positive measure given in (10) and for every $n \geq 1$ recall that $\mu_n := \sum_{k=1}^{n} \beta_k \in \mathcal{M}^+$, $\widetilde{\mu}_n := \mu_n(1_{\mathbb{X}})^{-1}\mu_n$. The next theorem gives a simple estimate of the error term $\pi - \widetilde{\mu}_n$ used throughout the Sections 3-4.

Theorem 2.2 *Assume that P satisfies Condition* (**S**) *and that W is a Lyapunov function satisfying $\mu(W) < \infty$. Let $\pi := \mu/\mu(1_{\mathbb{X}})$. Then*

$$\forall n \geq 1, \quad \|\pi - \mu(1_{\mathbb{X}})^{-1}\mu_n\|_W = \mu(1_{\mathbb{X}})^{-1}\,\varepsilon_{n,W} \leq \varepsilon_{n,W} \tag{11a}$$

$$\forall n \geq 1, \quad \|\pi - \widetilde{\mu}_n\|_W \leq \mu(1_{\mathbb{X}})^{-1}\big(\varepsilon_{n,W} + \mu_n(W)\,\mu_n(1_{\mathbb{X}})^{-1}\,\varepsilon_n\big) \tag{11b}$$

$$\textit{with} \quad \forall n \geq 1, \quad \varepsilon_{n,W} := \sum_{k=n+1}^{+\infty} \beta_k(W) \quad \textit{and} \quad \varepsilon_n := \varepsilon_{n,1_{\mathbb{X}}} = \sum_{k=n+1}^{+\infty} \beta_k(1_{\mathbb{X}}). \tag{11c}$$

In (11a)-(11b) we have $\mu(1_{\mathbb{X}})^{-1} = \pi(1_S) \leq 1$. Under the assumptions of Theorem 2.2, since $\mu(1_{\mathbb{X}}) \leq \mu(W) < \infty$, the Estimates (11a)-(11b) can be used with $W := 1_{\mathbb{X}}$ to get

$$\forall n \geq 1, \quad \|\pi - \widetilde{\mu}_n\|_{TV} \leq 2\mu(1_{\mathbb{X}})^{-1}\,\varepsilon_n \leq 2\,\varepsilon_n. \tag{12}$$

Proof. We have $\|\pi - \mu(1_{\mathbb{X}})^{-1}\mu_n\|_W = \mu(1_{\mathbb{X}})^{-1}(\mu - \mu_n)(W) = \mu(1_{\mathbb{X}})^{-1}\varepsilon_{n,W}$ since $\pi = \mu/\mu(1_{\mathbb{X}})$ and $\mu - \mu_n \in \mathcal{M}^+$. Thus $\|\mu - \mu_n\|_W = (\mu - \mu_n)(W) = \varepsilon_{n,W}$ from (11c). The last inequality in (11a) follows from $\mu(1_{\mathbb{X}}) \geq \mu(1_S) = 1$ (see the last assertion of Proposition 2.1). To prove (11b) let $f : \mathbb{X} \to \mathbb{R}$ measurable such that $|f| \leq W$. Then

$$\big|\pi(f) - \widetilde{\mu}_n(f)\big| = \left|\pi(f) - \frac{\mu_n(f)}{\mu_n(1_{\mathbb{X}})}\right| \leq \left|\pi(f) - \frac{\mu_n(f)}{\mu(1_{\mathbb{X}})}\right| + |\mu_n(f)| \times \left|\frac{1}{\mu(1_{\mathbb{X}})} - \frac{1}{\mu_n(1_{\mathbb{X}})}\right|$$
$$\leq \frac{\varepsilon_{n,W}}{\mu(1_{\mathbb{X}})} + \mu_n(W)\left|\frac{\mu_n(1_{\mathbb{X}}) - \mu(1_{\mathbb{X}})}{\mu(1_{\mathbb{X}})\mu_n(1_{\mathbb{X}})}\right|$$

by using the triangle inequality, (11a) and $|\mu_n(f)| \leq \mu_n(W)$. This gives Inequality (11b) using $|\mu_n(1_{\mathbb{X}}) - \mu(1_{\mathbb{X}})| = (\mu - \mu_n)(1_{\mathbb{X}}) = \varepsilon_n$ from (11c). □

It is clear from Estimates (11a)-(11b) or (12) and from Definition (11c) of $\varepsilon_{n,W}$ and ε_n that the rate of convergence to 0 of $\|\pi - \tilde{\mu}_n\|_W$ can be derived from good estimates of the convergence of the sequences $(\beta_n(W))_{n\geq 1}$ and $(\beta_n(1_{\mathbb{X}}))_{n\geq 1}$. This is the objective of Sections 3-4 for the polynomial case.

3 Error bounds under a polynomial drift condition on R

Let P be a Markov kernel satisfying Condition (**S**). Any Lyapunov function V is assumed to satisfy:

$$\forall x \in \mathbb{X},\ (PV)(x) < \infty.$$

Introduce the following polynomial drift conditions on R: There exists a familly $\{V_i\}_{i=0}^m$ of Lyapunov functions with $m \geq 1$ such that

$$\forall i \in \{0, \ldots, m-1\}, \quad RV_i \leq V_i - V_{i+1}. \tag{13}$$

Since $R \geq 0$, the sequence $\{V_i\}_{i=0}^m$ in (13) is decreasing. Moreover, since $(PV_0)(\cdot) < \infty$ by hypothesis, we have under Assumption (**S**): $\forall i \in \{0, \ldots, m\},\ \nu(V_i) \leq \nu(V_0) < \infty$.

Under Conditions (13), the convergence rate in Estimates (11a)-(11b) is shown to be polynomial in Theorem 3.1. Next, more explicit rates of convergence are provided in Corollary 3.4. Denote by $(\vartheta_j)_{j\geq 0}$ the following sequence of positive real numbers

$$\vartheta_0 := 1 \quad \text{and} \quad \forall \ell \geq 1,\ \vartheta_\ell := \sum_{j=0}^{\ell-1} C_\ell^j \vartheta_j \quad \text{with} \quad C_\ell^j := \frac{\ell\,!}{j\,!(\ell-j)\,!}. \tag{14}$$

Theorem 3.1 *Let P be a Markov kernel satisfying Condition* (**S**). *Assume that Conditions* (13) *hold for some collection $\{V_i\}_{i=0}^m$ of Lyapunov functions. Then we have*

$$\forall i \in \{1, \ldots, m\}, \quad \sum_{k=1}^{+\infty} k^{i-1}\, \beta_k(V_i) \leq \vartheta_{i-1}\, \nu(V_0) < \infty. \tag{15}$$

Moreover, for any $i = 1, \cdots, m$, we have $\pi(V_i) \leq \mu(V_i) = \sum_{k=1}^{+\infty} \beta_k(V_i) < \infty$, and Estimates (11a)-(11b) *hold with $W := V_i$ and with $(\varepsilon_{n,V_i})_{n\geq 1}$ and $(\varepsilon_n)_{n\geq 1}$ satisfying*

$$\lim_{n\to+\infty} n^{i-1}\varepsilon_{n,V_i} = 0 \qquad \textit{and} \qquad \lim_{n\to+\infty} n^{m-1}\varepsilon_n = 0. \tag{16}$$

Proof. Let us prove Inequalities (15) by an induction on m. Assume that (13) holds with $m = 1$, that is $RV_0 \leq V_0 - V_1$, or equivalently: $V_1 \leq V_0 - RV_0$. Then

$$\forall k \geq 0, \quad R^k V_1 \leq R^k V_0 - R^{k+1} V_0$$

where $R^0(x,\cdot) = \delta_x$ is the Dirac distribution at x. Then we obtain that

$$\forall n \geq 1, \quad \sum_{k=0}^{n} R^k V_1 \leq \sum_{k=0}^{n} \left[R^k V_0 - R^{k+1} V_0\right] \leq V_0$$

$$\text{and} \quad \forall n \geq 1, \quad \sum_{k=1}^{n+1} \beta_k(V_1) \leq \nu(V_0) \quad \text{(from (2))}.$$

This proves (15) when $m = 1$. Now suppose that Inequalities (15) hold for some $m \geq 1$ and that (13) holds at order $m+1$. Then using $V_{m+1} \leq V_m - RV_m$, we get

$$\forall k \geq 0, \quad R^k V_{m+1} \leq R^k V_m - R^{k+1} V_m$$

so that we have for every $n \geq 1$

$$\begin{aligned}\sum_{k=0}^{n} (k+1)^m \, R^k V_{m+1} &\leq \sum_{k=0}^{n} (k+1)^m \, R^k V_m - \sum_{k=0}^{n+1} k^m \, R^k V_m \leq \sum_{k=0}^{n} \left[(k+1)^m - k^m\right] R^k V_m \\ &\leq \sum_{j=0}^{m-1} C_m^j \sum_{k=0}^{n} k^j \, R^k V_m \leq \sum_{j=1}^{m} C_m^{j-1} \sum_{k=0}^{n} k^{j-1} \, R^k V_j\end{aligned}$$

using $\forall j \in \{1, \ldots, m\}$, $V_m \leq V_j$ for the last inequality. Inequalities (15) at order $m+1$ follows from (2) and from the induction hypothesis, that is we have

$$\begin{aligned}\sum_{k=1}^{+\infty} k^m \, \beta_k(V_{m+1}) \leq \sum_{j=1}^{m} C_m^{j-1} \sum_{k=0}^{+\infty} k^{j-1} \, \beta_{k+1}(V_j) &\leq \sum_{j=1}^{m} C_m^{j-1} \sum_{k=1}^{+\infty} k^{j-1} \, \beta_k(V_j) \\ &\leq \left(\sum_{j=1}^{m} C_m^{j-1} \vartheta_{j-1}\right) \nu(V_0) \quad = \vartheta_m \, \nu(V_0).\end{aligned}$$

Now let us prove the last assertion of Theorem 3.1. First note that for any $i = 1, \cdots, m$ we get $\pi(V_i) \leq \mu(V_i) = \sum_{k=1}^{+\infty} \beta_k(V_i) < \infty$ from (10) and (15). Next we have

$$\forall i \in \{1, \cdots, m\}, \quad \varepsilon_{n,V_i} = \sum_{k=n+1}^{+\infty} \beta_k(V_i) \leq \frac{1}{(n+1)^{i-1}} \sum_{k=n+1}^{+\infty} k^{i-1} \beta_k(V_i).$$

Then the first assertion in (16) follows from (15). In particular we have $\lim_n n^{m-1} \varepsilon_{n,V_m} = 0$, so that $\lim_{n\to+\infty} n^{m-1} \varepsilon_n = 0$ since $\varepsilon_n \leq \varepsilon_{n,V_m}$ from $1_{\mathbb{X}} \leq V_m$. □

Under the assumptions of Theorem 3.1, the following statement specifies the asymptotic behaviour of the sequence $(\beta_k(V_m))_{k\geq 1}$ assumed to be decreasing.

Theorem 3.2 *Let P be a Markov kernel satisfying Condition* (**S**)*. Assume Conditions (13) for some collection $\{V_i\}_{i=0}^m$ of Lyapunov functions. Then the following assertions hold.*

(i) $\forall i \in \{0, \ldots, m\}$, $\forall k \geq 1$, $\beta_k(V_i) < \infty$.

(ii) If the sequence $(\beta_k(V_m))_{k\geq 1}$ is decreasing, then

$$\forall n \geq 1, \quad \beta_n(V_m) \leq C_m \, \nu(V_0) \, \frac{1}{n^m} \quad \textit{with} \quad C_m := 2^{\frac{m(m+1)}{2} - 1}. \tag{17}$$

If moreover $\mu(V_0) := \sum_{k=1}^{+\infty} \beta_k(V_0) < \infty$, then

$$\forall n \geq 1, \quad \beta_n(V_m) \leq D_m \, \mu(V_0) \, \frac{1}{n^{m+1}} \quad \textit{with} \quad D_m := 2^{\frac{(m+1)(m+2)}{2} + 1}. \tag{18}$$

Lemma 3.3 *Assume that P satisfies Condition* $(\mathbf{S})$*. Let V and W be two Lyapunov functions such that*

$$RV \le V - W. \tag{19}$$

Then the following assertions hold.

(a) $\forall k \ge 1,\ \beta_k(V) < \infty$.

(b) The sequence $(\beta_k(V))_{k\ge 1}$ *is decreasing.*

(c) If the sequence $(\beta_k(W))_{k\ge1}$ *is decreasing, then we have for every $k \ge 1$ and $\varepsilon \in \{0,1\}$*

$$\beta_k(W) \le \nu(V)\,\frac{1}{k} \quad \text{and} \quad \beta_{2k-\varepsilon}(W) \le \beta_k(V)\,\frac{1}{k}.$$

(d) If the sequence $(\beta_k(W))_{k\ge1}$ *is decreasing and* $\mu(V) := \sum_{k=1}^{+\infty} \beta_k(V) < \infty$*, then*

$$\forall n \ge 1, \quad \beta_n(W) \le 16\,\mu(V)\frac{1}{n^2}.$$

Proof. Note that $W \le V$ from (19) and $R \ge 0$. Next we deduce from (19) that we have $\forall j \ge 1,\ R^j V \le R^{j-1}(V - W)$. Then (2) gives

$$\forall j \ge 1, \quad \beta_{j+1}(V) \le \beta_j(V) - \beta_j(W) \le \beta_j(V) \quad \text{in } [0, +\infty].$$

Using $\beta_1(V) = \nu(V) < \infty$, Assertion (a) is obtained by induction, and Assertion (b) is then obvious. Next rewrite the previous inequalities as

$$\forall j \ge 1, \quad 0 \le \beta_j(W) \le \beta_j(V) - \beta_{j+1}(V) \tag{20}$$

and suppose that $(\beta_j(W))_{j\ge1}$ is decreasing. Then it follows from (20) that

$$\forall k \ge 1, \quad k\,\beta_k(W) \le \sum_{j=1}^{k} \beta_j(W) \le \beta_1(V) - \beta_{k+1}(V) \le \nu(V),$$

from which we deduce the first inequality in Assertion (c). Moreover (20) also gives

$$\forall k \ge 1,\ \forall \varepsilon \in \{0,1\} \quad k\,\beta_{2k-\varepsilon}(W) \le \sum_{j=k}^{2k-\varepsilon} \beta_j(W) \le \beta_k(V) - \beta_{2k-\varepsilon+1}(V) \le \beta_k(V), \tag{21}$$

from which we deduce the second inequality in Assertion (c). Finally, to prove Assertion (d), note that for every $\ell \ge 1$ and every $\varepsilon \in \{0,1\}$

$$\ell\,\beta_{2\ell-\varepsilon}(V) \le \sum_{j=\ell}^{2\ell-\varepsilon} \beta_j(V) \le \mu(V) < \infty \tag{22}$$

since $(\beta_j(V))_{j\ge1}$ is decreasing from Assertion (b). Let $n \ge 1$ and write $n = 2(2\ell - \varepsilon_1) - \varepsilon_2$ with $\ell \ge 1$ and $(\varepsilon_1, \varepsilon_2) \in \{0,1\}^2$. Then it follows from (21) and (22) that

$$\beta_n(W) \le \frac{\beta_{2\ell-\varepsilon_1}(V)}{2\ell - \varepsilon_1} \le \frac{\mu(V)}{\ell(2\ell-1)} \le \frac{\mu(V)}{\ell^2} = \frac{16\,\mu(V)}{(n + 2\varepsilon_1 + \varepsilon_2)^2} \le \frac{16\,\mu(V)}{n^2}.$$

□

Proof of Theorem 3.2. Lemma 3.3-(a) applied with $V := V_0$ and $W := V_1$ proves that: $\forall k \geq 1,\ \beta_k(V_0) < \infty$. Then Theorem 3.2-($i$) holds since $V_i \leq V_0$. Now let us prove by induction on m that Property (17) holds. If $m = 1$, then the first inequality in Lemma 3.3-(c) applied with $V := V_0$ and $W := V_1$ provides

$$\forall n \geq 1, \quad \beta_n(V_1) \leq \frac{\nu(V_0)}{n}.$$

Hence (17) holds with $C_1 = 1$ when $m = 1$. Now suppose that (17) holds for some $m \geq 1$. Let $\{V_i\}_{i=0}^{m+1}$ be a collection of Lyapunov functions such that $\forall i \in \{0, \ldots, m\}$, $RV_i \leq V_i - V_{i+1}$ and such that the sequence $(\beta_k(V_{m+1}))_{k \geq 1}$ is decreasing. Note that Lemma 3.3-(b) applied with $V := V_m$ and $W := V_{m+1}$ ensures that the sequence $(\beta_k(V_m))_{k \geq 1}$ is decreasing. Hence we have from the induction hypothesis

$$\forall k \geq 1, \quad \beta_k(V_m) \leq \frac{C_m\, \nu(V_0)}{k^m} \quad \text{with} \quad C_m := 2^{\frac{m(m+1)}{2} - 1}. \tag{23}$$

Next let $n \geq 1$ and write $n = 2k - \varepsilon$ with $k \geq 1$ and $\varepsilon \in \{0, 1\}$. Then the second inequality in Lemma 3.3-(c) applied with $V := V_m$ and $W := V_{m+1}$ gives

$$\beta_n(V_{m+1}) \leq \frac{\beta_k(V_m)}{k} \tag{24}$$

so that $\beta_n(V_{m+1}) \leq C_m \nu(V_0)/k^{m+1}$ from (23). Hence

$$\beta_n(V_{m+1}) \leq \frac{2^{m+1}\, C_m\, \nu(V_0)}{(n+\varepsilon)^{m+1}} \leq \frac{C_{m+1}\, \nu(V_0)}{n^{m+1}} \quad \text{with} \quad C_{m+1} = 2^{m+1}\, C_m = 2^{\frac{(m+1)(m+2)}{2} - 1}.$$

The proof of Property (17) is complete.

Property (18) follows the same induction procedure. Indeed, if $m = 1$, then Lemma 3.3-(d) applied with $V := V_0$ and $W := V_1$ provides

$$\forall n \geq 1, \quad \beta_n(V_1) \leq \frac{16\, \mu(V_0)}{n^2}.$$

Hence (18) holds with $D_1 = 16$ when $m = 1$. Now, assume that (18) is true at some order $m \geq 1$, and consider a collection $\{V_i\}_{i=0}^{m+1}$ of Lyapunov functions as in the above induction proof. Then, writing $n \geq 1$ as $n = 2k - \varepsilon$ with $k \geq 1$ and $\varepsilon \in \{0, 1\}$, we deduce from (24) and from the induction hypothesis that

$$\beta_n(V_{m+1}) \leq \frac{\beta_k(V_m)}{k} \leq \frac{D_m\, \mu(V_0)}{k^{m+2}} \quad \text{with} \quad D_m := 2^{\frac{(m+1)(m+2)}{2} + 1}.$$

Hence

$$\beta_n(V_{m+1}) \leq \frac{2^{m+2}\, D_m\, \mu(V_0)}{(n+\varepsilon)^{m+2}} \leq \frac{D_{m+1}\, \mu(V_0)}{n^{m+2}} \quad \text{with} \quad D_{m+1} = 2^{m+2}\, D_m.$$

This proves (18). □

Under Conditions (13), since $V_m \geq 1_{\mathbb{X}}$, the last function V_m can be replaced by $1_{\mathbb{X}}$. Since the sequence $(\beta_k(1_{\mathbb{X}}))_k$ is decreasing from Proposition 2.1, the following computable bounds for ε_n and ε_{n,V_i} in (11a)-(11b) can be derived from Theorem 3.2.

Corollary 3.4 *Let P be a Markov kernel satisfying Condition* (**S**). *Assume that Conditions (13) hold for some collection $\{V_i\}_{i=0}^{m}$ of Lyapunov functions. Then the following assertions hold with the positive constants C_i and D_i defined in Theorem 3.2.*

(a) If $m \geq 2$, then $\mu(1_{\mathbb{X}}) < \infty$, and Estimate (12) *holds with*

$$\forall n \geq 1, \quad \varepsilon_n \leq \frac{C_m\, \nu(V_0)}{m-1}\, \frac{1}{n^{m-1}}. \tag{25}$$

Moreover, if $m \geq 3$, then for every $i \in \{2, \ldots, m-1\}$ we have $\pi(V_i) \leq \mu(V_i) < \infty$, and Estimates (11a)-(11b) *hold with $W := V_i$ and*

$$\forall n \geq 1, \quad \varepsilon_{n,V_i} \leq \frac{C_i\, \nu(V_0)}{i-1}\, \frac{1}{n^{i-1}}. \tag{26}$$

(b) If $m \geq 1$ and $\mu(V_0) < \infty$, then Estimate (12) *holds with*

$$\forall n \geq 1, \quad \varepsilon_n \leq \frac{D_m\, \mu(V_0)}{m}\, \frac{1}{n^m}. \tag{27}$$

If $m \geq 2$, then for any $i \in \{1, \ldots, m-1\}$ Estimates (11a)-(11b) *hold with $W := V_i$ and*

$$\forall n \geq 1, \quad \varepsilon_{n,V_i} \leq \frac{D_i\, \mu(V_0)}{i}\, \frac{1}{n^i}. \tag{28}$$

Proof. As previously mentioned, the function V_m in (13) can be replaced by $1_{\mathbb{X}}$, and the sequence $(\beta_k(1_{\mathbb{X}}))_{k\geq 1}$ is decreasing. Hence it follows from (17) that

$$\forall n \geq 1, \quad \beta_n(1_{\mathbb{X}}) \leq C_m\, \nu(V_0)\, \frac{1}{n^m}. \tag{29}$$

If $m \geq 2$, then Proposition 2.1-(ii) holds from (29). Then (25) is deduced from

$$\forall n \geq 1, \quad \varepsilon_n = \sum_{k=n+1}^{+\infty} \beta_k(1_{\mathbb{X}}) \leq C_m \nu(V_0) \sum_{k=n+1}^{+\infty} \frac{1}{k^m} \leq C_m \nu(V_0) \int_n^{+\infty} \frac{dt}{t^m} = \frac{C_m\, \nu(V_0)}{(m-1)\, n^{m-1}}.$$

Now assume that $\{V_i\}_{i=0}^m$ satisfies Conditions (13) with $m \geq 3$. Let $i \in \{2, \ldots, m-1\}$. The sequence $(\beta_k(V_i))_{k\geq 1}$ is decreasing from Lemma 3.3-(b), and obviously $\{V_j\}_{j=0}^i$ also satisfies Conditions (13). Then it follows from (17) that

$$\forall n \geq 1, \quad \beta_n(V_i) \leq C_i\, \nu(V_0)\, \frac{1}{n^i} \quad \text{with} \quad C_i := 2^{\frac{i(i+1)}{2}-1}. \tag{30}$$

Thus $\pi(V_i) \leq \mu(V_i) < \infty$ since $i \geq 2$, and (26) follows from comparison sums/integrals as above. Finally assume that $\mu(V_0) < \infty$ and $m \geq 1$. We deduce from (18) that

$$\forall n \geq 1, \quad \beta_n(1_{\mathbb{X}}) \leq D_m\, \mu(V_0)\, \frac{1}{n^{m+1}}. \tag{31}$$

Then (27) can be derived from comparison sums/integrals. Next assume that $m \geq 2$, and let $i \in \{1, \ldots, m-1\}$. Then Property (28) can be established by using as above the family $\{V_j\}_{j=0}^i$ and the fact that the sequence $(\beta_k(V_i))_{k\geq 1}$ is decreasing, then by applying (18) to V_i (in place of V_m), and finally by using again comparison sums/integrals. □

4 Picking Lyapunov functions to fit the target Conditions (13)

Here we consider assumptions under which the drift conditions (13) are satisfied so that Theorem 3.2 and Corollary 3.4 apply. Let V be a Lyapunov function and introduce the following drift condition on the residual kernel R: $\exists\alpha \in [0,1), \exists c > 0,\ RV \le V - c\,V^\alpha$, or separating the condition on S and S^c:

$$\exists\alpha \in [0,1), \exists c > 0, \quad \forall x \in S, \quad (RV)(x) \le V(x) - c\,V(x)^\alpha \qquad (\boldsymbol{Sub_{\alpha,S}})$$
$$\forall x \in S^c, \quad (PV)(x) \le V(x) - c\,V(x)^\alpha. \qquad (\boldsymbol{Sub_{\alpha,S^c}})$$

When PV is bounded on S, Condition ($\boldsymbol{Sub_{\alpha,S^c}}$) is equivalent to $\exists\alpha \in [0,1), \exists c > 0, \exists K > 0$, $PV \le V - c\,V^\alpha + K1_S$. Such a condition has been used to establish polynomial ergodicity of Markov chains (e.g. see [JR02, DFMS04, DMPS18]).

First consider the atomic case. When S is an atom and $\nu(\cdot) := P(a_0, \cdot)$ for $a_0 \in S$ in (**S**), we have: $\forall x \in S,\ (RV)(x) = 0$ and $\forall x \in S^c,\ (RV)(x) = (PV)(x)$. Then Conditions (13) rewrite as follows

$$\forall i \in \{0, \dots, m-1\}, \quad \begin{cases} \forall x \in S, & V_{i+1}(x) \le V_i(x) \\ \forall x \in S^c, & (PV_i)(x) \le V_i(x) - V_{i+1}(x). \end{cases} \tag{32}$$

Note that the second condition in (32) ensures that $V_{i+1} \le V_i$ on S^c too. For any $\alpha \in [0,1)$ define the integer $m \equiv m(\alpha) \ge 1$ by

$$m := \lfloor (1-\alpha)^{-1} \rfloor. \tag{33}$$

Corollary 4.1 (Atomic case) *Let P be a Markov kernel satisfying Conditions* (**S**) *with an atom S and $\nu(\cdot) := P(a_0, \cdot)$ for $a_0 \in S$. Assume that Condition* ($\boldsymbol{Sub_{\alpha,S^c}}$) *holds for some Lyapunov function V and $\alpha \in [0,1)$. Then all the assertions of Theorem 3.2 and Corollary 3.4 hold with the positive integer m in* (33) *and the functions $\{V_i\}_{i=0}^m$ specified in the proof.*

To prove Corollary 4.1 we use the following lemma which is based on [JR02, Lem. 3.5].

Lemma 4.2 *Let $S \in \mathcal{X}$, and W be a Lyapunov function such that PW is bounded on S. Let $0 < \theta_2 < \theta_1 < 1$ be such that*

$$\exists c > 0,\ \forall x \in S^c, \quad (PW^{\theta_1})(x) \le W(x)^{\theta_1} - c\,W(x)^{\theta_2}.$$

Then

$$\exists c' > 0,\ \forall x \in S^c, \quad (PW^{\theta_2})(x) \le W(x)^{\theta_2} - c'\,W(x)^{\theta_3} \qquad \textit{with} \quad \theta_3 := 2\theta_2 - \theta_1.$$

Note that the condition $c' > 0$ prevents to take $\theta_2 = 0$ in Lemma 4.2.

Proof. We have $\sup_{x\in S}(PW)(x) < \infty$ and $PW^{\theta_1} \le W^{\theta_1} - c\,(W^{\theta_1})^{\theta_2/\theta_1}$ on S^c. Thus

$$\forall\eta \in (0,1],\ \exists c' > 0, \quad PW^{\eta\theta_1} \le W^{\eta\theta_1} - c'\,(W^{\theta_1})^{\theta_2/\theta_1+\eta-1} \quad \text{on } S^c$$

from [JR02, Lem. 3.5]. The claimed inequality is obtained with $\eta := \theta_2/\theta_1 < 1$. □

Proof of Corollary 4.1. If the properties (**S**) and ($\boldsymbol{Sub_{\alpha,S^c}}$) hold for an atom S, $\nu(\cdot) := P(a_0, \cdot)$ with $a_0 \in S$ and some Lyapunov function V, we must prove that Conditions (32) hold for some decreasing family of Lyapunov functions $\{V_i\}_{i=0}^m$ with m given in (33).

Let $\alpha_1 := 1 - 1/m \in [0,1)$ with m given in (33). Note that $\alpha_1 \le \alpha$. Then we have

$$PV \le V - c_1\,V^{\alpha_1} \quad \text{on } S^c \tag{34}$$

from ($\boldsymbol{Sub_{\alpha,S^c}}$). Note that we can choose $c_1 < 1$.

- If $\alpha_1 = 0$, i.e. $m = 1$ or $\alpha \in [0, 1/2)$, then Conditions (32) hold with $V_0 := c_1^{-1} V \geq V_1 := 1_{\mathbb{X}}$.

- If $\alpha_1 = 1/2$, i.e. $m = 2$ or $\alpha \in [1/2, 2/3)$, then we deduce from (34) and Lemma 4.2 with $W := V, \theta_1 = 1, \theta_2 = \alpha_1$ that
$$\exists c_2 > 0, \quad PV^{\alpha_1} \leq V^{\alpha_1} - c_2\, V^{\alpha_2} \quad \text{on } S^c \tag{35}$$
with $\alpha_2 := 2\alpha_1 - 1 = 0$. Again note that we can choose $c_2 < 1$. Then the procedure stops, and Conditions (32) hold with $V_0 := c_1^{-1} c_2^{-1} V \geq V_1 := c_2^{-1} V^{\alpha_1} \geq V_2 := 1_{\mathbb{X}}$.

- If $\alpha_1 > 1/2$, then Lemma 4.2 can be used recursively to provide inequalities of the form $PV^{\alpha_{i-1}} \leq V^{\alpha_{i-1}} - c_i\, V^{\alpha_i}$ on S^c with $c_i < 1$ and $\alpha_i = 2\alpha_{i-1} - \alpha_{i-2} = (\alpha_1 - 1)\, i + 1$. Actually Lemma 4.2 can only be used until the value $i = m$ since $\alpha_m = 0$ and $\alpha_i < 0$ for $i > m$. Then Conditions (32) hold with
$$V_0 := \big[\prod_{k=1}^{m} c_k\big]^{-1} V, \quad 1 \leq i \leq m-1: \ V_i := \big[\prod_{k=i+1}^{m} c_k\big]^{-1} V^{\alpha_i}, \quad V_m := 1_{\mathbb{X}}. \tag{36}$$

The proof of Corollary 4.1 is complete. □

Now consider the non-atomic case. Let P satisfying Conditions (**S**) and ($\boldsymbol{Sub_{\alpha,S^c}}$), where S is not an atom, $\alpha \in [0, 1)$ and V is a Lyapunov function such that V and PV are bounded on S. Contrarily to the atomic case, Condition ($\boldsymbol{Sub_{\alpha,S}}$) does not hold automatically here. However, using ($\boldsymbol{Sub_{\alpha,S^c}}$) and combining Lemma 4.2 and the next Lemma 4.3, we can prove that $RV^{\eta_0} \leq V^{\eta_0} - \widehat{c}_1\, V^{\eta_0 \widehat{\alpha}_1}$ for some $\eta_0, \widehat{\alpha}_1 \in (0, 1]$ and $\widehat{c}_1 > 0$, from which the procedure of the atomic case (Corollary 4.1) can be extended.

Lemma 4.3 *Let P satisfying Condition* (**S**)*, and let V be a Lyapunov function such that PV is bounded on S. Then for any $\varepsilon \in (0, \nu(1_{\mathbb{X}}))$, there exists $\eta_0 \equiv \eta_0(\varepsilon) \in (0, 1]$ such that*
$$\forall \eta \in (0, \eta_0],\ \forall x \in S, \quad (RV^\eta)(x) \leq V(x)^\eta - \varepsilon. \tag{37}$$

Proof of Lemma 4.3. Set $M_S := \sup_{x \in S} (PV)(x) < \infty$. We have
$$\forall x \in S, \quad (RV^\eta)(x) - V(x)^\eta = (PV^\eta)(x) - \nu(V^\eta) - V(x)^\eta \leq {M_S}^\eta - \nu(1_{\mathbb{X}}) - 1$$
from Jensen's inequality and $1_{\mathbb{X}} \leq V^\eta$. Then (37) follows from the following property
$$\exists \eta_0 \in (0, 1],\ \forall \eta \in (0, \eta_0], \quad {M_S}^\eta - 1 \leq \nu(1_{\mathbb{X}}) - \varepsilon$$
which holds since ${M_S}^\eta \to 1$ when $\eta \to 0$. □

Now let $\varepsilon \in (0, \nu(1_{\mathbb{X}}))$ be fixed and $\eta_0 \equiv \eta_0(\varepsilon)$ provided by Lemma 4.3. If $\eta_0 \geq 1 - \alpha$ with $\alpha \in [0, 1)$ given in ($\boldsymbol{Sub_{\alpha,S^c}}$), define the positive integer $m \equiv m(\varepsilon, \alpha, \eta_0)$ as follows
$$m := \lfloor \eta_0\, (1 - \alpha)^{-1} \rfloor. \tag{38}$$

Proposition 4.4 *Assume that P satisfies Conditions* (**S**) *and* ($\boldsymbol{Sub_{\alpha,S^c}}$) *with V and PV bounded on S. Let $\varepsilon \in (0, \nu(1_{\mathbb{X}}))$. Assume that the real number $\eta_0(\varepsilon)$ given in* (37) *is such that $\eta_0 \geq 1 - \alpha$. Then all the assertions of Theorem 3.2 and Corollary 3.4 hold with the integer $m \equiv m(\varepsilon, \alpha, \eta_0) > 0$ in* (38) *and the functions $\{V_i\}_{i=0}^m$ specified in the proof.*

Proof. Let $M_1 := \sup_{x\in S} V(x) < \infty$ and $M_2 := \sup_{x\in S}(PV)(x) < \infty$. For every $\eta \in (0,1]$, we have $\sup_{\in S}(PV^\eta)(x) \leq {M_2}^\eta < \infty$ from Jensen's inequality. Note that Condition ($\boldsymbol{Sub_{\alpha,S}}$), that is $RV \leq V - c\,V^\alpha$ on S, may fail here. To initialize the procedure under ($\boldsymbol{Sub_{\alpha,S^c}}$) and $M_2 < \infty$, choose $c < 1$ in ($\boldsymbol{Sub_{\alpha,S^c}}$) and note that $PV \leq V - cV^\alpha + M_2 1_S$. Then it follows from [JR02, Lem. 3.5] that

$$\exists c_{\eta_0} > 0,\ \exists b' > 0, \quad PV^{\eta_0} \leq V^{\eta_0} - c_{\eta_0}\, V^{\alpha+\eta_0-1} + b' 1_S$$

with η_0 given in (37). This gives ($\boldsymbol{Sub_{\eta_0,S^c}}$), that is:

$$\forall x \in S^c, \quad (PV^{\eta_0})(x) \leq V(x)^{\eta_0} - c_{\eta_0}\, V(x)^{\alpha+\eta_0-1}. \tag{39}$$

If $\alpha+\eta_0-1 < 0$, then Inequality (39) cannot be used to apply Corollary 3.4 since the function V_1 in Conditions (13) must take its values in $[a, +\infty)$ for some $a > 0$. Now if $\alpha + \eta_0 - 1 \geq 0$ then prove that Condition ($\boldsymbol{Sub_{\eta_0,S}}$) holds. Up to the reduction of its value, c_{η_0} in (39) can be chosen such that $c_{\eta_0} {M_1}^{\alpha+\eta_0-1} \leq \varepsilon$, so that we have from (37)

$$\forall x \in S, \quad (RV^{\eta_0})(x) - V(x)^{\eta_0} + c_{\eta_0}\, V(x)^{\alpha+\eta_0-1} \leq -\varepsilon + c_{\eta_0}\, V(x)^{\alpha+\eta_0-1} \leq 0. \tag{40}$$

Next, let m defined in (38) and set

$$\widehat{V} := V^{\eta_0}, \quad \widehat{\alpha}_1 := 1 - \frac{1}{m} \quad \text{and} \quad \widehat{c}_1 := c_{\eta_0}.$$

Note that $m = \lfloor (1-\widehat{\alpha})^{-1} \rfloor$ with $\widehat{\alpha} = 1 - (1-\alpha)/\eta_0$, and that $\widehat{\alpha}_1 \leq \widehat{\alpha}$. We get from (39)-(40)

$$R\widehat{V} \leq \widehat{V} - \widehat{c}_1\, \widehat{V}^{\widehat{\alpha}_1}.$$

Then, starting from this inequality and iterating Lemma 4.2, we can proceed exactly as in the proof of Corollary 4.1, provided that Conditions (13) hold on S at each step. Namely, at each step, Lemma 4.2 provides an inequality of the form

$$R\widehat{V}^{\widehat{\alpha}_{i-1}} = P\widehat{V}^{\widehat{\alpha}_{i-1}} \leq \widehat{V}^{\widehat{\alpha}_{i-1}} - \widehat{c}_i\, \widehat{V}^{\widehat{\alpha}_i} \quad \text{on } S^c \tag{41}$$

with some $\widehat{c}_i > 0$ and with $\widehat{\alpha}_i = 2\widehat{\alpha}_{i-1} - \widehat{\alpha}_{i-2} = (\widehat{\alpha}_1 - 1)\, i + 1$. This procedure is repeated only until the value $i = m$ since $\widehat{\alpha}_m = 0$ and $\widehat{\alpha}_i < 0$ for $i > m$. Next we must check that the condition $R\widehat{V}^{\widehat{\alpha}_{i-1}} \leq \widehat{V}^{\widehat{\alpha}_{i-1}} - \widehat{c}_i\, \widehat{V}^{\widehat{\alpha}_i}$ also holds on S. Note that $\widehat{\alpha}_{i-1} \leq 1$ and that

$$R\widehat{V}^{\widehat{\alpha}_{i-1}} - \widehat{V}^{\widehat{\alpha}_{i-1}} = RV^{\eta_i} - V^{\eta_i} \quad \text{with} \quad \eta_i := \eta_0 \widehat{\alpha}_{i-1} \in (0, \eta_0].$$

Since $\widehat{c}_i$ in (41) can be chosen such that $\widehat{c}_i\, {M_1}^{\widehat{\alpha}_i} \leq \varepsilon$, it follows from (37) that

$$\forall x \in S, \quad (R\widehat{V}^{\widehat{\alpha}_{i-1}})(x) - \widehat{V}^{\widehat{\alpha}_{i-1}}(x) + \widehat{c}_i\, \widehat{V}^{\widehat{\alpha}_i}(x) \leq -\varepsilon + \widehat{c}_i\, \widehat{V}^{\widehat{\alpha}_i}(x) \leq 0. \tag{42}$$

Then Conditions (13) hold with V_i defined as in (36) replacing V by $\widehat{V}$, and α_i, c_i by $\widehat{\alpha}_i, \widehat{c}_i$. Note that $1_{\mathbb{X}} = V_m \leq \cdots \leq V_0$. Thus the proof of Corollary 4.4 is complete. □

The following proposition shows that a simpler condition than (37) in Lemma 4.3 can be used to choose η_0 for a large class of Markov chains.

Proposition 4.5 *Assume that any one of the two following conditions holds:*

1. *$\mathbb{X}$ is discrete and S is finite.*
2. *$\mathbb{X}$ is a metric space, S is compact and the functions V and PV^η for any $\eta \in (0,1]$ are continuous on S.*

Then there exists $\eta_0 \in (0,1]$ such that

$$\forall x \in S, \quad (RV^{\eta_0})(x) < V(x)^{\eta_0} \tag{43}$$

and such η_0 can be used in Proposition 4.4.

Proof. The existence of $\eta_0 \in (0,1]$ satisfying (43) is provided in the proof of Lemma 4.3. Now observe that the proof of Proposition 4.4 is still valid when Condition (42) holds with some $\varepsilon_i > 0$ for $i \in \{1,\ldots,m\}$ in place of $\varepsilon > 0$. Then $\widehat{c}_i$ in (41) has to be chosen such that $\widehat{c}_i\, {M_1}^{\eta_0} \le \varepsilon_i$, and the functions V_i are defined as in the previous proof from such $\widehat{c}_i$. Consequently, we have to prove in the case 1. or 2. that (43) implies that

$$\forall \eta \in (0,\eta_0],\ \forall x \in S, \quad (RV^{\eta})(x) < V(x)^{\eta}. \tag{44}$$

Recall that, for any $x \in S$, $R(x,\cdot) \in \mathcal{M}^+$ from (**S**), and note that $R(x,1_{\mathbb{X}}) = 1-\nu(1_{\mathbb{X}})$ does not depend on x. Set $r := 1-\nu(1_{\mathbb{X}})$. If $r = 0$, we have $(RV^{\eta}) = 0$ on S for any $\eta \in (0,1]$, so that (44) is obvious. Now assume that $r > 0$. Let us introduce $\widehat{V} := V^{\eta_0}$. Note that (43) reads as $(R\widehat{V})(x) < V(x)^{\eta_0}$ for any $x \in S$. Since $0 < \eta/\eta_0 \le 1$ for any $\eta \in (0,\eta_0]$, it easily follows from Jensen's Inequality applied to the probability measure $r^{-1}R(x,\cdot)$ that

$$\forall \eta \in (0,\eta_0],\ \forall x \in S, \quad (RV^{\eta})(x) = \big(R\widehat{V}^{\eta/\eta_0}\big)(x) \le \frac{r}{r^{\eta/\eta_0}}\,(R\widehat{V})(x)^{\eta/\eta_0} < \frac{r}{r^{\eta/\eta_0}} V(x)^{\eta}.$$

Since $0 < r < 1$, we obtain (44). Therefore the proof is complete. □

A Proof of Proposition 2.1

Let $\mathcal{B} := \{f : \mathbb{X} \to \mathbb{R} : \|f\| := \sup_{x\in\mathbb{X}} |f(x)| < \infty\}$. For bounded linear operators Q_1, Q_2 on $\mathcal{B}$, $Q_1 \le Q_2$ stands for: $\forall f \in \mathcal{B}$, $f \ge 0$, $Q_1 f \le Q_2 f$. Let P satisfying Condition (**S**) and T be the following operator on $\mathcal{B}$:

$$\forall f \in \mathcal{B}, \quad Tf := \nu(f)\, 1_S = \beta_1(f)\, 1_S.$$

Consider $(\beta_k)_{k\ge1} \in (\mathcal{M}^+)^{\mathbb{N}}$ in (2). Set $T_0 := 0$ and $T_n := P^n - R^n$ for $n \ge 1$. Then

$$\forall n \ge 1,\ 0 \le T_n \le P^n,\ T_n - T_{n-1}P = (P^{n-1} - T_{n-1})T \text{ and } T_n = \sum_{k=1}^{n} \beta_k(\cdot) P^{n-k} 1_S. \tag{45}$$

The first property follows from $0 \le R \le P$. The second one is deduced from $P^n - T_n = (P^{n-1} - T_{n-1})(P-T)$. Finally, the last one is clear for $n = 1$ and it holds for $n \ge 2$ by an easy induction based on $T_n = P^{n-1}T + T_{n-1}R$ and (2).

Now, let us prove Proposition 2.1. Assume that Assertion (i) holds. We deduce from (45) that $0 \le \pi\big((P^n - T_n)1_{\mathbb{X}}\big) = 1 - \pi(T_n 1_{\mathbb{X}}) = 1 - \pi(1_S)\sum_{k=1}^{n} \beta_k(1_{\mathbb{X}})$ from which it follows that $\sum_{k=1}^{+\infty} \beta_k(1_{\mathbb{X}}) \le \pi(1_S)^{-1} < \infty$ since $\pi(1_S) > 0$ by hypothesis. This gives Assertion (ii). Conversely if Assertion (ii) holds then $\mu := \sum_{k=1}^{+\infty} \beta_k \in \mathcal{M}_*^+$ since $\mu(1_{\mathbb{X}}) \ge \beta_1(1_{\mathbb{X}}) = \nu(1_{\mathbb{X}}) > 0$. Note that, for any $f \in \mathcal{B}$, the series $\sum_{k=1}^{+\infty} \beta_k(f)$ absolutely converges in $\mathbb{C}$ since $|\beta_k(f)| \le \|f\|\, \beta_k(1_{\mathbb{X}})$. We obtain that

$$\begin{aligned}
\forall f \in \mathcal{B},\ \mu(Pf) &= \sum_{k=1}^{+\infty} \nu\big(P^k f - T_{k-1}Pf\big) \qquad \text{from (2) and (45)} \\
&= \sum_{k=1}^{+\infty} \nu\big(P^k f - T_k f\big) + \sum_{k=1}^{+\infty} \nu\big(P^{k-1}Tf - T_{k-1}Tf\big) \qquad \text{from (45)} \\
&= \mu(f) + \mu(Tf) - \nu(f) \qquad \text{from (2) and } \beta_1(f) = \nu(f).
\end{aligned}$$

Thus $0 = \nu(1_{\mathbb{X}})\mu(1_S) - \nu(1_{\mathbb{X}})$, which gives $\mu(1_S) = 1$ since $\nu(1_{\mathbb{X}}) > 0$. Thus μ is $P-$invariant, so that $\pi := \mu(1_{\mathbb{X}})^{-1}\,\mu$ is an $P-$invariant distribution such that $\pi(1_S) = \mu(1_{\mathbb{X}})^{-1} > 0$.

Finally we prove that $(\beta_k(1_{\mathbb{X}}))_k$ is decreasing. Note that $R(1_{\mathbb{X}}) = 1_{\mathbb{X}} - \nu(1_{\mathbb{X}})1_S$, so that using (2) we get $\beta_{k+1}(1_{\mathbb{X}}) = \beta_k \circ R(1_{\mathbb{X}}) = \beta_k(1_{\mathbb{X}}) - \nu(1_{\mathbb{X}})\,\beta_k(1_S)$ for any $k \geq 1$. This gives the desired statement.

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