Explicit bounds for spectral theory of geometrically ergodic Markov kernels and applications

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Abstract

In this paper, we deal with a Markov chain on a measurable state space $(\mathbb{X}, \mathcal{X})$ which has a transition kernel P admitting an aperiodic small-set S and satisfying the standard geometric-drift condition. Under these assumptions, there exists $\alpha_0 \in (0, 1]$ such that $PV^{\alpha_0} \leq \delta^{\alpha_0} V^{\alpha_0} + \nu(V^{\alpha_0})1_S$. Hence P is $V^{\alpha_0}-$geometrically ergodic and its "second eigenvalue" ϱ_{α_0} provides the best rate of convergence. Setting $R := P - \nu(\cdot)1_S$ and $\Gamma := \{\lambda \in \mathbb{C},\ \delta^{\alpha_0} < |\lambda| < 1\}$, ϱ_{α_0} is shown to satisfy, either $\varrho_{\alpha_0} = \max\big\{|\lambda| \,:\, \lambda \in \Gamma,\ \sum_{k=1}^{+\infty} \lambda^{-k}\, \nu(R^{k-1}1_S) = 1\big\}$ if this set is not empty, or $\varrho_{\alpha_0} \leq \delta^{\alpha_0}$. Actually the set is finite in the first case and is composed by the spectral values of P in Γ. The second case occurs when P has no spectral value in Γ. Moreover, a bound of the operator-norm of $(zI - P)^{-1}$ allows us to derive an explicit formula for the multiplicative constant in the rate of convergence, which can be evaluated provided that any information of the "second eigenvalue" is available. Such numerical computation is carried out for a classical family of reflected random walks. Moreover we obtain a simple and explicit bound of the operator-norm of $(I - P + \pi(\cdot)1_{\mathbb{X}})^{-1}$ involved in the definition of the so-called fundamental solution to Poisson's equation. This allows us to specify the location of the eigenvalues of P and, then, to obtain a general bound on ϱ_{α_0}. The reversible case is also discussed. In particular, the bound of ϱ_{α_0} obtained for positive reversible Markov kernels is the expected one, and numerical illustrations are proposed for the Metropolis-Hastings algorithm and for the Gaussian autoregressive Markov chain. The bound for the operator-norm of $(I - P + \pi(\cdot)1_{\mathbb{X}})^{-1}$ is derived from an estimate, only depending on δ^{α_0}, of the operator-norm of $(I - R)^{-1}$ which provides another way to get a solution to Poisson's equation. This estimate is also shown to be of greatest interest to generalize the error bounds obtained for perturbed discrete and atomic Markov chains in [LL18] to the case of general geometrically ergodic Markov chains. These error estimates are the simplest that can be expected in this context. All the estimates in this work are expressed in the standard $V^{\alpha_0}-$weighted operator norm.

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1 Introduction

Let $(\mathbb{X},\mathcal{X})$ be a measurable space, and let $\mathcal{M}^+$ denote the set of finite non-negative measures on $(\mathbb{X},\mathcal{X})$. For any $\mu\in\mathcal{M}^+$ and any μ-integrable function $f:\mathbb{X}\to\mathbb{C}$, $\mu(f)$ denotes the integral $\int f d\mu$. For any measurable function $W\geq 1$ we denote by $(\mathcal{B}_W,\|\cdot\|_W)$ the Banach space of measurable functions $f:\mathbb{X}\to\mathbb{C}$ such that $\|f\|_W:=\sup_{x\in\mathbb{X}}|f(x)|/W(x)<\infty$. The identity map on $\mathcal{B}_W$ is denoted by I, and $(\mathcal{B}'_W,\|\cdot\|'_W)$ stands for the topological dual space of $\mathcal{B}_W$ (i.e. the Banach space of $\mathbb{C}$-valued bounded linear maps on $\mathcal{B}_W$). For any $\mu\in\mathcal{M}^+$ satisfying $\mu(W)<\infty$, the map $f\mapsto\mu(f)$ belongs to $\mathcal{B}'_W$, and for any such $(\mu_1,\mu_2)\in(\mathcal{M}^+)^2$, the norm $\|\mu_1-\mu_2\|'_W$ coincides with the standard W-weighted total variation norm, that is:

$$\|\mu_1-\mu_2\|'_W:=\sup_{|f|\leq W}\big|\mu_1(f)-\mu_2(f)\big|. \tag{1}$$

Throughout the paper P is a Markov kernel on $(\mathbb{X},\mathcal{X})$, and the existence of a small-set S for P is assumed, that is: there exist $S\in\mathcal{X}$ and $\nu\in\mathcal{M}^+$ such that

$$\nu(1_S)>0 \quad\text{and}\quad \forall x\in\mathbb{X},\ \forall A\in\mathcal{X},\quad P(x,A)\geq\nu(1_A)\,1_S(x). \tag{$\boldsymbol{S}$}$$

We also assume that there exists a measurable function $V:\mathbb{X}\to[1,+\infty)$ (called a Lyapunov function) satisfying the following geometric drift condition with $S^c:=\mathbb{X}\setminus S$:

$$\exists\delta\equiv\delta(P)\in(0,1),\ \forall x\in S^c,\quad (PV)(x)\leq\delta\,V(x) \tag{$\boldsymbol{D_{S^c}}$}$$

$$\text{and}\quad K:=\sup_{x\in S}(PV)(x)<\infty. \tag{$\boldsymbol{K}$}$$

Throughout the paper, Assumptions $(\boldsymbol{A})$ will stand for the set of the three assumptions $(\boldsymbol{S})$-$(\boldsymbol{D_{S^c}})$-$(\boldsymbol{K})$. Under Assumptions $(\boldsymbol{A})$ we know that there exists a unique $P-$invariant probability measure denoted by π on $(\mathbb{X},\mathcal{X})$ and that $\pi(V)<\infty$, e.g. see [MT93, RR04, Bax05, DMPS18]. In this paper, replacing the Lyapunov function V with V^{α_0} for some suitable constant $\alpha_0\in(0,1]$ derived from the data in $(\boldsymbol{A})$, we present new results concerning the spectral properties of P on the space $\mathcal{B}_{V^{\alpha_0}}$ in relation with the so-called $V^{\alpha_0}-$geometric ergodicity of P. These spectral results are applied to the study of the sensitivity with respect to the parameter θ of the invariant probability measure of transition kernels P_θ satisfying Assumptions $(\boldsymbol{A})$ in a uniform way in θ.

Let us recall some facts before specifying the main results of the paper. Under Assumptions $(\boldsymbol{A})$, we know from [HL22, Cor. 4.2] that there exists $\alpha_0\equiv\alpha_0(P)\in(0,1]$ such that

$$PV^{\alpha_0}\leq\delta^{\alpha_0}\,V^{\alpha_0}+\nu(V^{\alpha_0})1_S. \tag{$\boldsymbol{D^{\alpha_0}}$}$$

A first consequence of $(\boldsymbol{S})$ and $(\boldsymbol{D^{\alpha_0}})$ is that the non-negative kernel

$$R(x,dy):=P(x,dy)-1_S(x)\nu(dy)$$

defines a contraction on $\mathcal{B}_{V^{\alpha_0}}$, that is $\|R\|_{V^{\alpha_0}}\leq\delta^{\alpha_0}$ (see (13)). This is one of the key points of this work. A second consequence is that P is $V^{\alpha_0}-$geometrically ergodic, e.g. see [MT93, RR04, Bax05, DMPS18]: There exist $\rho\in(0,1)$ and $C_\rho\in(0,+\infty)$ such that

$$\forall f\in\mathcal{B}_{V^{\alpha_0}},\ \forall n\geq 1,\quad \|P^nf-\pi(f)1_{\mathbb{X}}\|_{V^{\alpha_0}}\leq C_\rho\,\rho^n\,\|f\|_{V^{\alpha_0}}. \tag{2}$$

We denote by ϱ_{α_0} the infimum bound of the positive real numbers ρ satisfying (2). The real number ϱ_{α_0} is sometimes called the "second eigenvalue" of P on $\mathcal{B}_{V^{\alpha_0}}$ (even though ϱ_{α_0} is not necessarily an eigenvalue of P), while $1-\varrho_{\alpha_0}$ is called the spectral gap of P on $\mathcal{B}_{V^{\alpha_0}}$. When P satisfies (2) and is reversible with respect to π, it follows from [Bax05, Th. 6.1] that

$$\forall f \in \mathbb{L}^2(\pi),\ \forall n \geq 1, \quad \|P^n f - \pi(f)1_{\mathbb{X}}\|_{\mathbb{L}^2(\pi)} \leq 2\, \varrho_{\alpha_0}^{\ n}\, \|f\|_{\mathbb{L}^2(\pi)} \tag{3}$$

where $\mathbb{L}^2(\pi)$ is the standard Lebesgue space equipped with the norm $\|f\|_{\mathbb{L}^2(\pi)} = \pi(|f|^2)^{1/2}$. Thus, in this case, ϱ_{α_0} is an upper bound of the second eigenvalue of P on $\mathbb{L}^2(\pi)$. Finally recall that $\lambda \in \mathbb{C}$ is a spectral value of P on $\mathcal{B}_{V^{\alpha_0}}$ if $\lambda I - P$ is not invertible on $\mathcal{B}_{V^{\alpha_0}}$. The spectral value $\lambda \in \mathbb{C}$ is an eigenvalue of P on $\mathcal{B}_{V^{\alpha_0}}$ if $\lambda I - P$ is not injective on $\mathcal{B}_{V^{\alpha_0}}$.

Under Assumptions ($\boldsymbol{A}$) the following statements with $\alpha_0 \in (0,1]$ in ($\boldsymbol{D^{\alpha_0}}$) hold true.

- (Section 2) The transition kernel P is quasi-compact on $\mathcal{B}_{V^{\alpha_0}}$ with essential spectral radius less than δ^{α_0}. Then we obtain spectral results in Theorems 2.1 and 2.2 which can be summarized as follows. Let $a \in (\delta^{\alpha_0}, 1)$. The set $\mathcal{S}_a$ of spectral values λ of P on $\mathcal{B}_{V^{\alpha_0}}$ such that $a \leq |\lambda| \leq 1$ is finite and composed of eigenvalues of P. Note that $\lambda = 1 \in \mathcal{S}_a$. If $\mathcal{S}_a = \{1\}$, then $\varrho_{\alpha_0} \leq a$; Otherwise $\varrho_{\alpha_0} = \max\big\{|\lambda|,\, \lambda \in \mathcal{S}_a,\, \lambda \neq 1\big\}$. Moreover
$$\lambda \in \mathcal{S}_a \iff \mu_\lambda(1_S) = 1 \tag{4}$$
where $\mu_\lambda(1_S) := \sum_{k=1}^{+\infty} \lambda^{-k}\, \beta_k(1_S)$, with $\beta_k := \nu \circ R^{k-1} \in \mathcal{B}'_{V^{\alpha_0}}$. In other words this ensures that the following alternative holds for the second eigenvalue: Either ϱ_{α_0} equals to the largest solution (in modulus) to the equation $\mu_z(1_S) = 1$ in $\{z \in \mathbb{C} : \delta^{\alpha_0} < |z| < 1\}$ if such a solution exists; or $\varrho_{\alpha_0} \leq \delta^{\alpha_0}$ (see Corollary 2.1). This algebraic issue is difficult to address in practice since it involves the power series $\sum_{k=1}^{+\infty} z^{-k}\, \beta_k(1_S)$, but the equivalence (4) plays an important theoretical role in the proofs of the properties below.

- (Section 3) Recall that finding effective and computable rate of convergence and multiplicative constants in geometric ergodicity is a difficult issue, see [MT94, Bax05, and the references therein]. In Theorem 3.1, we propose the following bound for the resolvent of P: For every $z \in \mathbb{C} \setminus \{1\}$ such that $|z| \in \big(\max(\delta^{\alpha_0}, \varrho_{\alpha_0}), 1\big]$, the operator $zI - P$ is invertible on $\mathcal{B}_{V^{\alpha_0}}$ and we have
$$\forall f \in \mathcal{B}_{V^{\alpha_0}}, \quad \|(zI-P)^{-1}f\|_{V^{\alpha_0}} \leq \frac{1}{|z|-\delta^{\alpha_0}}\left(1 + \frac{\nu(V^{\alpha_0})\|1_S\|_{V^{\alpha_0}}}{\big|1-\mu_z(1_S)\big|\big(|z|-\delta^{\alpha_0}\big)}\right)\|f\|_{V^{\alpha_0}}. \tag{5}$$
Then it allows us to derive the following formula for the constant C_ρ in Inequality (2) for any $\rho \in \big(\max(\delta^{\alpha_0}, \varrho_{\alpha_0}), 1\big)$:
$$C_\rho = \frac{\rho}{2\pi(\rho - \delta^{\alpha_0})}\left(1 + \frac{\nu(V^{\alpha_0})\|1_S\|_{V^{\alpha_0}}}{m_\rho\big(\rho-\delta^{\alpha_0}\big)}\right) \quad \text{with } m_\rho := \min_{z\in\mathbb{C}:|z|=\rho} |1-\mu_z(1_S)| > 0. \tag{6}$$
Note that the equivalence (4) is crucial to prove the positivity of m_ρ. When the second eigenvalue of P is known (or at least bounded), Formula (6) is relevant provided that the numerical computation of m_ρ is tractable. Such computations are carried out for a classical family of reflected random walks (see Example 3.1).

- (Section 4) In this section we focus on the solutions in $\mathcal{B}_{V^{\alpha_0}}$ to the so-called Poisson equation. In Theorem 4.1, we prove that, for any $f \in \mathcal{B}_{V^{\alpha_0}}$ such that $\pi(f) = 0$, the function $\widetilde{f} := \sum_{n=0}^{+\infty} R^n f$ is a solution in $\mathcal{B}_{V^{\alpha_0}}$ to Poisson's equation $(I - P)g = f$. Moreover $\widetilde{f}$ and the classical solution $\widehat{f} := \sum_{n=0}^{+\infty} P^n f$ satisfy

$$\|\widetilde{f}\|_{V^{\alpha_0}} \leq \frac{1}{1-\delta^{\alpha_0}} \|f\|_{V^{\alpha_0}} \quad \text{and} \quad \|\widehat{f}\|_{V^{\alpha_0}} \leq \frac{1+\pi(V^{\alpha_0})\|1_{\mathbb{X}}\|_{V^{\alpha_0}}}{1-\delta^{\alpha_0}} \|f\|_{V^{\alpha_0}}. \tag{7}$$

The bound on $\widehat{f}$ is deduced from that on $\widetilde{f} = \sum_{n=0}^{+\infty} R^n f$. This control on the norm of these two solutions to Poisson's equation is central to Sections 5 and 6. The inequalities in (7) give true computable bounds for the $V^{\alpha_0}-$norm of $\widetilde{f}$ and $\widehat{f}$. In particular note that the inequality $\pi(V^{\alpha_0}) \leq \nu(V^{\alpha_0})/(1-\delta^{\alpha_0})$ easily derived from $(\boldsymbol{D^{\alpha_0}})$ may be used in (7) when π is unknown. The second bound in (7) concerning the classical solution $\widehat{f} = \sum_{n=0}^{+\infty} P^n f$ to Poisson's equation when $\pi(f) = 0$ is deduced from the first bound in (7) for the solution $\widetilde{f} = \sum_{n=0}^{+\infty} R^n f$.

- (Section 5) Using Inequality (5) and the bound on $\|\widehat{f}\|_{V^{\alpha_0}}$ in (7), we present results concerning the location of the eigenvalues of P on $B_{V^{\alpha_0}}$, from which we deduce a upper bound of the second eigenvalue ϱ_{α_0} in Corollary 5.1. This general bound of ϱ_{α_0} requires to prove that the real number $m_0 := \min_{\vartheta\in[\vartheta_0,2\pi-\vartheta_0]} |1-\mu_{e^{i\vartheta}}(1_S)|$ for some $\vartheta_0 \in (0, \pi/2)$ is positive: Again this is deduced from (4). When P is reversible with respect to π and satisfies a slight additional condition (see (49)), we obtain in Corollary 5.2 that

$$\varrho_{\alpha_0} \leq \psi(\eta_\infty) \leq \psi(\eta_n) \quad \text{with} \quad \psi(t) := \frac{\delta^{\alpha_0}(1-\delta^{\alpha_0})t + \nu(V^{\alpha_0})\|1_S\|_{V^{\alpha_0}}}{(1-\delta^{\alpha_0})t + \nu(V^{\alpha_0})\|1_S\|_{V^{\alpha_0}}} \tag{8}$$

where $\forall n \geq 1,\ \eta_n := 2\sum_{k=1}^{n} \beta_{2k-1}(1_S)$ and $\eta_\infty := 2\sum_{k=1}^{+\infty} \beta_{2k-1}(1_S)$. The interest in the bounds of ϱ_{α_0} obtained in Corollary 5.1 and in (8) is essentially theoretical. By contrast, if P is reversible and positive (i.e. $\forall f \in \mathbb{L}^2(\pi),\ \langle f, Pf\rangle_{\mathbb{L}^2(\pi)} \geq 0$), then we prove in Corollary 5.3 that the simple and explicit bound $\varrho_{\alpha_0} \leq \delta^{\alpha_0}$ holds (again under the slight condition (49)). Numerical applications of Corollary 5.3 to the Metropolis-Hastings algorithm for the Gaussian distribution and to Gaussian autoregressive Markov chain are carried out in Examples 5.1-5.2. Recall that, in the reversible case, any rate of convergence in (2) provides a rate of convergence in the $\mathbb{L}^2(\pi)-$geometric ergodicity (3), whatever the value of α_0.

- (Section 6) In this section the results of Section 4 are applied to get an explicit control on the invariant probability measure of a perturbed Markov chain. Specifically, let Θ be an open subset of some metric space, and let $\{P_\theta\}_{\theta\in\Theta}$ be a family of Markov kernels on $(\mathbb{X}, \mathcal{X})$ satisfying Assumptions $(\boldsymbol{A})$ in a uniform way in $\theta \in \Theta$, as well as the following condition:

$$\forall x \in \mathbb{X}, \quad \lim_{\theta\to\theta_0} \Delta_{\theta,\alpha_0}(x) = 0 \quad \text{with} \quad \Delta_{\theta,\alpha_0}(x) := \|P_\theta(x,\cdot) - P_{\theta_0}(x,\cdot)\|'_{V^{\alpha_0}} \tag{9}$$

where θ_0 is fixed in Θ. Let π_θ denote the $P_\theta-$invariant probability measure. Then we obtain in Theorem 6.1 that $\lim_{\theta\to\theta_0} \|\pi_\theta - \pi_{\theta_0}\|'_{V^{\alpha_0}} = 0$ and that for every $\theta \in \Theta$

$$\big\|\pi_\theta - \pi_{\theta_0}\big\|'_{V^{\alpha_0}} \leq \frac{1+\pi_{\theta_0}(V^{\alpha_0})\|1_{\mathbb{X}}\|_{V^{\alpha_0}}}{1-\delta^{\alpha_0}} \times \pi_\theta\big(\Delta_{\theta,\alpha_0}\big) \quad \text{with} \quad \lim_{\theta\to\theta_0} \pi_\theta\big(\Delta_{\theta,\alpha_0}\big) = 0. \tag{10}$$

The bound on $\|\widetilde{f}\|_{V^{\alpha_0}}$ in (7) plays a crucial role to prove (10). Note that the multiplicative constant in (10) is truly computable using again $\pi_{\theta_0}(V^{\alpha_0}) \leq \nu(V^{\alpha_0})/(1-\delta^{\alpha_0})$ when π is unknown. Moreover observe that the real number $\pi_\theta\big(\Delta_{\theta,\alpha_0}\big)$ in (10) is available when the function Δ_{θ,α_0} in (9) is known (or can be bounded) and π_θ is computable for $\theta \neq \theta_0$. This holds for instance when $\mathbb{X}$ is a discrete set and the perturbed Markov kernels are truncated stochastic matrices on a finite state space (see Remark 6.1). Finally note that $\|\pi_\theta - \pi_{\theta_0}\|'_{TV} \leq \|\pi_\theta - \pi_{\theta_0}\|'_{V^{\alpha_0}}$ where $\|\pi_\theta - \pi_{\theta_0}\|'_{TV}$ denotes the total variation distance between the two probability measures π_θ and π_{θ_0} (use (1) with $W = 1_{\mathbb{X}}$).

Under Assumptions $(\boldsymbol{A})$, it is proved in [Bax05, Th. 1.1] that P is $V-$geometrically ergodic. However it is worth noticing that our results only focus on the $V^{\alpha_0}-$weighted operator norm in Sections 2–5 and on $V^{\alpha_0}-$weighted total variation norm in Section 6, where $\alpha_0 \in (0,1]$ is given in $(\boldsymbol{D^{\alpha_0}})$. Hence, when $\alpha_0 < 1$, our results involve the smaller space $\mathcal{B}_{V^{\alpha_0}}$ in place of the expected one $\mathcal{B}_V$. This is the price to pay when working with the drift condition $(\boldsymbol{D^{\alpha_0}})$. The benefit is that the results obtained on $\mathcal{B}_{V^{\alpha_0}}$ from $(\boldsymbol{D^{\alpha_0}})$ have a fairly simple form.

The constant $\alpha_0 \in (0,1]$ in $(\boldsymbol{D^{\alpha_0}})$ can be easily computed from the data in Assumptions $(\boldsymbol{A})$ (see [HL22, (28)]). For convenience the proof that $(\boldsymbol{D^{\alpha_0}})$ holds true and the explicit computation of α_0 are recalled in Appendix A. The real number K in Condition $(\boldsymbol{K})$ plays an important role in the computation of α_0: roughly speaking, the larger K is compared to $\nu(V)$, the smaller α_0 is. If the small-set S in $(\boldsymbol{S})$ is an atom with ν given by $\nu = P(s,\cdot)$ for some $s \in S$, then $(\boldsymbol{D^{\alpha_0}})$ holds with $\alpha_0 = 1$ (see Appendix A). Note that the case $\alpha_0 = 1$ is not equivalent to the atomic case, in other words Property $(\boldsymbol{D^{\alpha_0}})$ may hold with $\alpha_0 = 1$ for non-atomic small set S (even in the continuous state space case, see Example 5.1). Of course there are probably instances of Markov chains satisfying Assumptions $(\boldsymbol{A})$, for which the use of Property $(\boldsymbol{D^{\alpha_0}})$ is not relevant because α_0 is too close to zero, so that δ^{α_0} is too close to one for the bounds (6), (7), (8) or (10) to be of interest. We believe that these unfavourable cases correspond to instances for which the minorization/drift conditions are not well suited for finding interesting rates of convergence in geometric ergodicity context, whatever the method used (see [QH21]).

The spectral properties for geometrically ergodic Markov chains have been investigated in many papers, e.g. see [KM03, KM05, Hen06, HL14a, HL14b, Del17]. The novelty of this work is that we obtain more simple and explicit results due to Condition $(\boldsymbol{D^{\alpha_0}})$. To the best of our knowledge, the results in this work are new. The numerical values of the multiplicative constant C_ρ in (2) derived from (6) for discrete random walks are quite realistic and consistent (see Table 1 in Example 3.1). When P is positive reversible and satisfies Assumptions $(\boldsymbol{A})$ with an atom S (thus $\alpha_0 = 1$), the bound $\varrho_1 \leq \delta$ was obtained in [Bax05, Sec. 2.3]. Thus the bound $\varrho_{\alpha_0} \leq \delta^{\alpha_0}$ in Corollary 5.3 extends this result to the non-atomic case. The numerical computations for the Metropolis-Hastings algorithm of the Gaussian distribution and the Gaussian autoregressive Markov chain (see Examples 5.1 and 5.2) show that the bound $\varrho_{\alpha_0} \leq \delta^{\alpha_0}$ is relevant in comparison with those provided in [Bax05, Sec. 2.3 (non-atomic case)]. That $\widetilde{f} := \sum_{n=0}^{+\infty} R^n f$ is solution to Poisson's equation when $\pi(f) = 0$ seems to be a new result which extends to our framework the statement [Kem81, Th. 2] involving generalized fundamental finite matrix. The bounds (7) and (10) have been proved for discrete state space Markov chains with a finite atom in [LL18, Prop. 1, Th. 2] thanks to renewal theory. Theorems 4.1 and 6.1 extend these results to the non-atomic case and to general state spaces. The bound (10) improves all the error bounds obtained under Condition (9) in the literature for the stationary distribution of perturbed geometrically ergodic Markov chains, provided we use the Lyapunov function V^{α_0} in place of V. Indeed, the bound (10) involves neither the iterates of the unperturbed Markov kernels, nor those of the perturbed Markov kernels, exactly as in the case of discrete Markov chains with an atom investigated in [LL18]. It is worth noticing that Condition (9) is much weaker than the standard operator norm continuity assumption introduced in [Kar86] (see Remark 6.2). The control of the perturbed invariant probability measure performed in [SS00, FHL13, HL14a, RS18, MARS20] requires that the rate and the multiplicative constant in geometric ergodicity are known. If this is not the case, then the bound in (10) is a relevant alternative. Finally mention that the operator $R = P - \nu(\cdot)1_S$ and its iterates have been considered in [KM03] to investigate the eigenvectors belonging to the dominated eigenvalue of the Laplace kernels associated with the Markov kernel P. This issue called "multiplicative Poisson equation" in [KM03] is used to prove limit theorems for the underlying Markov chain. This question is not addressed in our work.

2 Quasi-compactness of P

For any measurable function $W \geq 1$, if L is a bounded linear operator on $(\mathcal{B}_W, \|\cdot\|_W)$, we also denote by $\|L\|_W := \sup\{\|Lf\|_W, f \in \mathcal{B}_W, \|f\|_W \leq 1\}$ the operator norm of L on $\mathcal{B}_W$. First, recall the definition of the quasi-compactness and of the essential spectral radius of L, assuming for the sake of simplicity that its spectral radius $r(L) := \lim_n \|L^n\|_W{}^{1/n}$ is one. Then L is quasi-compact on $(\mathcal{B}_W, \|\cdot\|_W)$ if there exist $a \in (0,1)$, $m \in \mathbb{N}^*$, $(\lambda_i, p_i) \in \mathbb{C} \times \mathbb{N}^*$ for $i = 1, \ldots, m$, and finally a closed L-invariant subspace H of $\mathcal{B}_W$ such that

$$|\lambda_i| \geq a, \quad 1 \leq \dim \operatorname{Ker}(L - \lambda_i I)^{p_i} < \infty, \quad \inf_{n\geq 1} \Big(\sup_{h\in H,\, \|h\|_W \leq 1} \|L^n h\|_W \Big)^{1/n} < a \tag{11a}$$

$$\text{and} \quad \mathcal{B}_W = \overset{m}{\underset{i=1}{\oplus}} \operatorname{Ker}(L - \lambda_i I)^{p_i} \oplus H. \tag{11b}$$

Moreover the essential spectral radius of L, denoted by $r_{ess}(L)$, is given by

$$r_{ess}(L) = \inf \big\{ \; a \in (0,1) \text{ such that (11a)-(11b) are satisfied } \big\}. \tag{12}$$

The previous definition [Hen93, p. 628] may be compared with the reduction of matrices which is known to be relevant to derive convergence rates for finite Markov chains. Note that the infinum bound in the last condition of (11a) is nothing else but the spectral radius of the restriction of L to H. Various equivalent definitions of $r_{ess}(L)$ (see (16) below) can be found in the literature in link with, either the essential spectrum, or the quasi-compactness property, e.g. see [Hen93, Hen07] and [HH01, Chapter XIV] for a general context and [Wu04, Hen06, AP07, HL14b, HL14a, Del17] in the framework of V-geometrically ergodic Markov kernels. For the link between geometric ergodicity and spectral theory, see also [MT09, Chap. 20].

The quasi-compactness of P on $\mathcal{B}_V$ is investigated in [HL14a, Th. 5.2] under Assumptions $(\boldsymbol{A})$. However the bound obtained in [HL14a, Th. 5.2] for the essential spectral radius of P on $\mathcal{B}_V$ is not accurate. Moreover note that we can directly deduce from (2) that P is quasi-compact on $\mathcal{B}_{V^{\alpha_0}}$ and that its essential spectral radius satisfies $r_{ess}(P) \leq \varrho_{\alpha_0}$. Indeed, for any $\rho \in (0,1)$ such that (2) holds, we have $r_{ess}(P) \leq \rho$ from the above definition with $m = 1$, $\lambda_1 = 1$, $\operatorname{Ker}(P - I) = \mathbb{C} \cdot 1_{\mathbb{X}}$ and $H := \{f \in \mathcal{B}_{V^{\alpha_0}}, \pi(f) = 0\}$. Thus $r_{ess}(P) \leq \varrho_{\alpha_0}$ from the definition of ϱ_{α_0}. However, this bound of $r_{ess}(P)$ is not interesting in practice since

ϱ_{α_0} is unknown. In this section we prove that the expected bound $r_{ess}(P) \le \delta^{\alpha_0}$ holds under Condition ($\boldsymbol{D^{\alpha_0}}$) and that this bound provides as a by-product some interesting informations on the second eigenvalue ϱ_{α_0}.

Assume that P satisfies Assumptions ($\boldsymbol{A}$). Note that $\nu(V) < \infty$ due to ($\boldsymbol{S}$). Let $\alpha_0 \in (0,1]$ be given in ($\boldsymbol{D^{\alpha_0}}$). Then P and $T := \nu(\cdot)1_S$ are bounded linear operators on $\mathcal{B}_{V^{\alpha_0}}$. Define

$$R := P - T = P - \nu(\cdot)1_S.$$

We deduce from ($\boldsymbol{S}$) that R is a non-negative operator on $\mathcal{B}_{V^{\alpha_0}}$ (i.e. $\forall f \in \mathcal{B}_{V^{\alpha_0}} : f \ge 0 \Rightarrow Rf \ge 0$). Moreover ($\boldsymbol{D^{\alpha_0}}$) reads as $RV^{\alpha_0} \le \delta^{\alpha_0}\, V^{\alpha_0}$ due to the definition of T. Iterating this inequality gives: $\forall k \ge 1,\ R^k V^{\alpha_0} \le \delta^{\alpha_0 k}\, V^{\alpha_0}$. Then we deduce from the non-negativity of R^k that

$$\forall k \ge 1, \quad \|R^k\|_{V^{\alpha_0}} \le \delta^{\alpha_0 k} \tag{13}$$

since for every $f \in \mathcal{B}_{V^{\alpha_0}}$ we have $|R^k f| \le R^k|f| \le \|f\|_{V^{\alpha_0}} R^k V^{\alpha_0}$. Moreover let us define

$$\forall k \ge 1, \quad \beta_k := \nu \circ R^{k-1} \in \mathcal{B}'_{V^{\alpha_0}} \tag{14}$$

with the convention $R^0 = I$ so that $\beta_1 = \nu$. Recall that for every $n \ge 1$ the operator T_n on $\mathcal{B}_{V^{\alpha_0}}$ defined by $T_n := P^n - R^n$ satisfies (see [HL20, Prop. 2.1])

$$T_n = \sum_{k=1}^{n} \beta_k(\cdot) P^{n-k} 1_S. \tag{15}$$

Hence T_n is finite-rank. This fact and (13) are the key points to prove the next Theorem 2.1 using the notion of essential spectral radius and quasi-compactness. The adjoint operator of P acting on $\mathcal{B}'_{V^{\alpha_0}}$ is denoted by P^*.

Theorem 2.1 *Suppose that P satisfies Assumptions ($\boldsymbol{A}$), and let $\alpha_0 \in (0,1]$ be given in ($\boldsymbol{D^{\alpha_0}}$). Then, for any $a \in (\delta^{\alpha_0}, 1)$, the set $\mathcal{S}_a$ of spectral values λ of P on $\mathcal{B}_{V^{\alpha_0}}$ (or of P^* on $\mathcal{B}'_{V^{\alpha_0}}$) such that $a \le |\lambda| \le 1$ is finite and composed of eigenvalues of both P and P^*. Moreover the second eigenvalue ϱ_{α_0} of P on $\mathcal{B}_{V^{\alpha_0}}$ (see (2)) is such that:*

(a) Either $\mathcal{S}_a = \{1\}$ and $\varrho_{\alpha_0} \le a$.

(b) Or $\mathcal{S}_a \ne \{1\}$ and $\varrho_{\alpha_0} = \max\big\{|\lambda|,\, \lambda \in \mathcal{S}_a,\, \lambda \ne 1\big\}$.

First we prove the following simple lemma.

Lemma 2.1 *$\lambda = 1$ is the only eigenvalue of P on $\mathcal{B}_{V^{\alpha_0}}$ such that $\varrho_{\alpha_0} < |\lambda| \le 1$.*

Proof. Let $\lambda \in \mathbb{C} \setminus \{1\}$ be any eigenvalue of P on $\mathcal{B}_{V^{\alpha_0}}$. Let $f \in \mathcal{B}_{V^{\alpha_0}}$, $f \ne 0$, be such that $Pf = \lambda f$. Then $\pi(f) = 0$, so that (2) gives $|\lambda|^n = \mathrm{O}(\rho^n)$, thus $|\lambda| \le \rho$. Hence $|\lambda| \le \varrho_{\alpha_0}$ since ρ in (2) can be chosen arbitrarily close to ϱ_{α_0}.

□

Proof of Theorem 2.1. Recall that the essential spectral radius $r_{ess}(P)$ of P on $\mathcal{B}_{V^{\alpha_0}}$ is also given by

$$r_{ess}(P) := \lim_n \big(\inf_{U \in \mathcal{K}} \|P^n - U\|_{V^{\alpha_0}}\big)^{1/n} \tag{16}$$

where $\mathcal{K}$ denotes the space of all compact operators on $\mathcal{B}_{V^{\alpha_0}}$. Then we have

$$r_{ess}(P) \leq \delta^{\alpha_0} < 1 \tag{17}$$

from (16) and (13) since $P^n - T_n = R^n$ where T_n in (15) is a finite-rank operator so is compact on $\mathcal{B}_{V^{\alpha_0}}$. Hence P is quasi-compact on $\mathcal{B}_{V^{\alpha_0}}$ since the spectral radius of P on $\mathcal{B}_{V^{\alpha_0}}$ is one. It follows from quasi-compactness that the set $\mathcal{S}_a$ is composed of finitely many spectral values which are in fact eigenvalues, e.g. see [Hen93]. The alternative (a)-(b) then follows from the definition of ϱ_{α_0} (see (2)) and from classical arguments of spectral theory. For the sake of completeness, let us present the main arguments. First assume that $\mathcal{S}_a \neq \{1\}$ and define $\gamma_a = \max\{|\lambda|,\, \lambda \in \mathcal{S}_a,\, \lambda \neq 1\}$. From Lemma 2.1 we have $\gamma_a \leq \varrho_{\alpha_0}$. Moreover, it follows from the standard spectral theory that, for any $\gamma \in (\gamma_a, 1)$, we have the following equality

$$\forall n \geq 1, \quad P^n = \pi(\cdot)1_{\mathbb{X}} + \frac{1}{2i\pi} \oint_{|z|=\gamma} z^n (zI - P)^{-1}\, dz, \tag{18}$$

from which we deduce that the value $\rho = \gamma$ is allowed in (2). Thus $\varrho_{\alpha_0} \leq \gamma$, so that $\varrho_{\alpha_0} \leq \gamma_a$ since γ is arbitrarily close to γ_a. We have proved that $\varrho_{\alpha_0} = \gamma_a$ in Case (b). Finally assume that $\mathcal{S}_a = \{1\}$. Then property (18) applies to $\gamma = a$, so that the value $\rho = a$ is allowed in (2). Thus $\varrho_{\alpha_0} \leq a$.

□

Recall that $\beta_k \in \mathcal{B}'_{V^{\alpha_0}}$ is defined in (14). It follows from (13) that, for every $z \in \mathbb{C}$ such that $|z| > \delta^{\alpha_0}$, the following series

$$\mu_z := \sum_{k=1}^{+\infty} z^{-k}\, \beta_k \tag{19}$$

is absolutely convergent in $\mathcal{B}'_{V^{\alpha_0}}$, so that $\sum_{k=1}^{+\infty} z^{-k}\, \beta_k(1_S)$ is absolutely convergent in $\mathbb{C}$.

Theorem 2.2 *Assume that P satisfies $(\boldsymbol{A})$, and let $\alpha_0 \in (0,1]$ be given in $(\boldsymbol{D^{\alpha_0}})$. Let $\lambda \in \mathbb{C}$ be such that $\delta^{\alpha_0} < |\lambda| \leq 1$. Then the two following assertions are equivalent:*

(i) λ is an eigenvalue of P on $\mathcal{B}_{V^{\alpha_0}}$.

(ii) $\mu_\lambda(1_S) = \sum_{k=1}^{+\infty} \lambda^{-k}\, \beta_k(1_S) = 1$.

Moreover, under Condition (i) or (ii), the subspace $E_\lambda := \{\psi \in \mathcal{B}'_{V^{\alpha_0}} : \psi \circ P = \lambda\, \psi\}$ is spanned by μ_λ.

For the proof of Theorem 2.2 we may assume that $\alpha_0 = 1$ in $(\boldsymbol{D^{\alpha_0}})$, that is

$$PV \leq \delta\, V + \nu(V) 1_S. \tag{20}$$

If $\alpha_0 < 1$, then replace V, δ with V^{α_0}, δ^{α_0} respectively in the proof below. First we prove the following lemma.

Lemma 2.2 *For any $z \in \mathbb{C}$ such that $|z| > \delta$ we have*

$$\mu_z \circ P = z\mu_z - \nu + \mu_z \circ T. \tag{21}$$

Proof. Let $z \in \mathbb{C}$ be such that $|z| > \delta$. Then

$$\begin{aligned}
\mu_z \circ P &= \sum_{k=1}^{+\infty} z^{-k}\, \nu \circ R^{k-1} \circ P \qquad \text{(from (19) and (14))} \\
&= \sum_{k=1}^{+\infty} z^{-k}\, \nu \circ R^{k} + \sum_{k=1}^{+\infty} z^{-k}\, \nu \circ \big(R^{k-1} \circ T\big) \qquad \text{(since } P = R + T\text{)} \\
&= \sum_{k=1}^{+\infty} z^{-k}\, \beta_{k+1} + \sum_{k=1}^{+\infty} z^{-k}\, \beta_k \circ T \qquad \text{(from (14))} \\
&= z\mu_z - \nu + \mu_z \circ T \qquad \text{(from (19) and } \beta_1 = \nu\text{)}.
\end{aligned}$$

□

Proof of Theorem 2.2. Let λ be an eigenvalue of P (thus of P^*) such that $\delta < |\lambda| \leq 1$. Using $R^n = P^n - T_n$ and (15) we deduce from (13) (with $\alpha_0 = 1$ here) that

$$\forall n \geq 1,\ \forall f \in \mathcal{B}_V, \qquad \Big\| P^n f - \sum_{k=1}^{n} \beta_k(f) P^{n-k} 1_S \Big\|_V \leq \delta^n \, \|f\|_V. \tag{22}$$

Let $\psi \in E_\lambda$, $\psi \neq 0$. Composing on the left by ψ in (22) gives the following equality in $\mathcal{B}'_V$

$$\lambda^n \psi = \psi(1_S) \sum_{k=1}^{n} \lambda^{n-k}\, \beta_k + O(\delta^n),$$

so that $\psi = \psi(1_S) \sum_{k=1}^{n} \lambda^{-k}\, \beta_k + O((\delta/\lambda)^n)$. Hence $\psi = \psi(1_S)\, \mu_\lambda$. Note that $\psi(1_S) \neq 0$ since $\psi \neq 0$, so that $\mu_\lambda(1_S) = 1$. We have proved the last assertion of Theorem 2.2, as well as the implication $(i) \Rightarrow (ii)$. Now let us prove that $(ii) \Rightarrow (i)$. Let $\lambda \in \mathbb{C}$ be such that $\delta < |\lambda| < 1$ and assume that $\mu_\lambda(1_S) = 1$, so that $\mu_\lambda \neq 0$. Lemma 2.2 applied to $z = \lambda$ gives

$$\mu_\lambda \circ P = \lambda\, \mu_\lambda - \nu + \mu_\lambda \circ T.$$

Moreover, since $T = \nu(\cdot) 1_S$ we obtain that

$$\mu_\lambda \circ T = \sum_{k=1}^{+\infty} \lambda^{-k}\, \beta_k \circ T = \Big(\sum_{k=1}^{+\infty} \lambda^{-k}\, \beta_k(1_S) \Big) \nu = \mu_\lambda(1_S)\, \nu = \nu$$

from which it follows that $\mu_\lambda \circ P = \lambda\, \mu_\lambda$. Hence λ is an eigenvalue of P^*, thus of P. □

We deduce the following statement from Theorems 2.1-2.2.

Corollary 2.1 *Assume that P satisfies ($\boldsymbol{A}$) and let $\alpha_0 \in (0, 1]$ be given in ($\boldsymbol{D}^{\boldsymbol{\alpha_0}}$). Then the second eigenvalue ϱ_{α_0} of P on $\mathcal{B}_{V^{\alpha_0}}$ (see (2)) satisfies the following alternative.*

- *Either for every $\lambda \in \mathbb{C}$ such that $\delta^{\alpha_0} < |\lambda| < 1$ we have $\mu_\lambda(1_S) \neq 1$, so that $\varrho_{\alpha_0} \leq \delta^{\alpha_0}$.*
- *Or $\varrho_{\alpha_0} = \max\big\{ |\lambda| \,:\, \lambda \in \mathbb{C},\ \delta^{\alpha_0} < |\lambda| < 1,\ \mu_\lambda(1_S) = 1 \big\}$.*

As a complement to Theorem 2.2, we prove in Appendix B that any eigenvalue λ of P on $\mathcal{B}_{V^{\alpha_0}}$ such that $\delta^{\alpha_0} < |\lambda| \leq 1$ is of order one (i.e $\mathrm{Ker}(P - \lambda I)^2 = \mathrm{Ker}(P - \lambda I)$) if, and only if, $\mu'_\lambda(1_S) \neq 0$ where $\mu'_\lambda(1_S)$ is the derivative at $z = \lambda$ of $z \mapsto \mu_z(1_S)$.

3 A bound for the constant C_ρ in (2)

Let P satisfying Assumptions $(\boldsymbol{A})$ and let $\alpha_0 \in (0,1]$ be given in $(\boldsymbol{D^{\alpha_0}})$. Recall that ϱ_{α_0} denotes the infimum bound of the positive real numbers ρ such that the $V^{\alpha_0}-$geometric ergodicity property (2) holds true. Property (24b) below provides an explicit constant C_ρ in (2) when $\rho \in \big(\max(\delta^{\alpha_0}, \varrho_{\alpha_0}), 1\big)$. Recall that for every $z \in \mathbb{C}$ such that $|z| > \delta^{\alpha_0}$, the series $\mu_z(1_S) = \sum_{k=1}^{+\infty} z^{-k}\, \beta_k(1_S)$ is absolutely convergent (see (19)).

Theorem 3.1 *Let P satisfying Assumptions $(\boldsymbol{A})$. Let $\alpha_0 \in (0,1]$ be given in $(\boldsymbol{D^{\alpha_0}})$. Then, for every $z \in \mathbb{C}\setminus\{1\}$ such that $|z| \in \big(\max(\delta^{\alpha_0}, \varrho_{\alpha_0}), 1\big]$, the operator $zI - P$ is invertible on $\mathcal{B}_{V^{\alpha_0}}$, and*

$$\|(zI-P)^{-1}\|_{V^{\alpha_0}} \leq \frac{1}{|z|-\delta^{\alpha_0}}\left(1 + \frac{\nu(V^{\alpha_0})\|1_S\|_{V^{\alpha_0}}}{\big|1-\mu_z(1_S)\big|\big(|z|-\delta^{\alpha_0}\big)}\right). \tag{23}$$

Moreover, for every $\rho \in \big(\max(\delta^{\alpha_0}, \varrho_{\alpha_0}), 1\big)$, we have

$$\forall n \geq 1, \quad \|P^n - \pi(\cdot)1_{\mathbb{X}}\|_{V^{\alpha_0}} \ \leq\ C_\rho\, \rho^n \tag{24a}$$

$$\textit{with } C_\rho = \frac{\rho}{2\pi(\rho-\delta^{\alpha_0})}\left(1 + \frac{\nu(V^{\alpha_0})\|1_S\|_{V^{\alpha_0}}}{m_\rho\big(\rho-\delta^{\alpha_0}\big)}\right) \quad \textit{and} \quad m_\rho := \min_{z\in\mathbb{C}:|z|=\rho} |1-\mu_z(1_S)| > 0. \tag{24b}$$

The explicit $V^{\alpha_0}-$geometric ergodicity property (24a)-(24b) is interesting, on the one hand when ϱ_{α_0} is known or can be at least bounded from above, and on the other hand when m_ρ can be numerically computed or at least bounded from below by a positive real number. This is illustrated in Example 3.1.

Again, we may assume that $\alpha_0 = 1$ in $(\boldsymbol{D^{\alpha_0}})$, that is (20), for the following proofs. Moreover ϱ stands for ϱ_1 to simplify. If $\alpha_0 < 1$, then replace V, δ and ϱ with V^{α_0}, δ^{α_0} and ϱ_{α_0} respectively in the proof below. The following lemmas are used to prove Theorem 3.1.

Recall that $R = P - T = P - \nu(\cdot)1_S$ satisfies: $\forall k \geq 0,\ \|R^k\|_V \leq \delta^k$ (see (13)).

Lemma 3.1 *Let $z \in \mathbb{C}$ be such that $|z| > \delta$. Then $zI - R$ is invertible on $\mathcal{B}_V$ with*

$$(zI-R)^{-1} = \sum_{k=0}^{+\infty} z^{-(k+1)} R^k. \tag{25}$$

Moreover, with $\mu_z \in \mathcal{B}'_V$ defined in (19), we have

$$\forall f \in \mathcal{B}_V, \quad \nu\big((zI-R)^{-1}f\big) = \mu_z(f). \tag{26}$$

Lemma 3.2 *Let $z \in \mathbb{C}\setminus\{1\}$ be such that $|z| \in \big(\max(\delta, \varrho), 1\big]$. Then $zI - P$ is invertible on $\mathcal{B}_V$, and*

$$\forall f \in \mathcal{B}_V, \quad (zI-P)^{-1}f = (zI-R)^{-1}f + \frac{\mu_z(f)}{1-\mu_z(1_S)}(zI-R)^{-1}1_S. \tag{27}$$

Moreover we have

$$\|(zI-P)^{-1}\|_V \leq \frac{1}{|z|-\delta}\left(1 + \frac{\nu(V)\|1_S\|_V}{\big|1-\mu_z(1_S)\big|\big(|z|-\delta\big)}\right). \tag{28}$$

Proof of Theorem 3.1. Property (23) is established in Lemma 3.2. Now let $\rho \in \big(\max(\delta, \varrho), 1\big)$. We deduce from Lemma 2.1 and from Theorem 2.2 that

$$\forall z \in \mathbb{C},\ |z| = \rho, \quad \mu_z(1_S) := \sum_{k=1}^{+\infty} z^{-k}\beta_k(1_S) \neq 1.$$

Since $z \mapsto \mu_z(1_S) := \sum_{k=1}^{+\infty} z^{-k}\beta_k(1_S)$ is continuous on the compact set $\{z \in \mathbb{C} : |z| = \rho\}$, it follows that the constant m_ρ introduced in (24b) is well-defined and is positive. Finally let us prove (24a) with C_ρ defined in (24b). It follows from the spectral decomposition (18) applied here to $\gamma = \rho$ that

$$\forall n \geq 1, \quad \|P^n - \pi(\cdot)1_{\mathbb{X}}\|_V \leq \frac{\rho^{n+1}}{2\pi} \max_{z \in \mathbb{C}:|z|=\rho} \big\|(zI - P)^{-1}\big\|_V.$$

Consequently the constant C_ρ in (2) can be chosen as follows

$$\frac{\rho}{2\pi} \max_{z \in \mathbb{C}:|z|=\rho} \big\|(zI - P)^{-1}\big\|_V \quad \leq \quad C_\rho := \frac{\rho}{2\pi} \times \frac{1}{\rho - \delta}\bigg(1 + \frac{\nu(V)\|1_S\|_V}{m_\rho(\rho - \delta)}\bigg)$$

from (28) and m_ρ in (24b). This provides (24a) with C_ρ as in (24b).

□

Proof of Lemma 3.1. Let $z \in \mathbb{C}$ be such that $|z| > \delta$. Then $zI - R$ is invertible on $\mathcal{B}_V$ since the spectral radius of R is less than δ from $\|R^k\|_V \leq \delta^k$. Then Formula (25) is classical. Moreover note that for every $f \in \mathcal{B}_V$

$$\sum_{k=0}^{+\infty} \int_{\mathbb{X}} |z|^{-(k+1)} |R^k f|\, d\nu \leq |z|^{-1}\nu(V)\, \|f\|_V \sum_{k=0}^{+\infty} \big(|z|^{-1}\delta\big)^k < \infty$$

from $\|R^k\|_V \leq \delta^k$ and from $\delta < |z|$. Therefore the permutation of the integral and the series in the following equality is allowed:

$$\nu((zI - R)^{-1}f) = \nu(\sum_{k=0}^{+\infty} z^{-(k+1)} R^k f) = \sum_{k=0}^{+\infty} z^{-(k+1)} \nu(R^k f).$$

This gives (26) due to (14) and (19).

□

Proof of Lemma 3.2. Let $z \in \mathbb{C} \setminus \{1\}$ such that $|z| \in \big(\max(\delta, \varrho), 1\big]$. If $zI - P$ is not invertible on $\mathcal{B}_V$, then z is an eigenvalue of P from Theorem 2.1, which is impossible from Lemma 2.1. Thus $zI - P$ is invertible on $\mathcal{B}_V$. Next we have

$$zI - P = zI - R - T = U_z \circ (zI - R) \quad \text{with} \quad U_z := I - T \circ (zI - R)^{-1}. \tag{29}$$

We deduce from $T = \nu(\cdot)1_S$ and from (26) that

$$\forall f \in \mathcal{B}_V, \quad U_z f = f - \mu_z(f)1_S \quad \text{or} \quad f = U_z f + \mu_z(f)1_S.$$

Next

$$U_z f = \big(U_z \circ (zI - R)\big) \circ (zI - R)^{-1} f = (zI - P) \circ (zI - R)^{-1} f$$

using (29), so that

$$f = (zI - P) \circ (zI - R)^{-1} f + \mu_z(f) 1_S$$

and

$$(zI - P)^{-1} f = (zI - R)^{-1} f + \mu_z(f)(zI - P)^{-1} 1_S.$$

The last equality applied to $f = 1_S$ gives

$$(zI - P)^{-1} 1_S = \frac{1}{1 - \mu_z(1_S)} (zI - R)^{-1} 1_S$$

where $\mu_z(1_S) \neq 1$ from Corollary 2.1. This provides (27).

Next we have

$$\|(zI - R)^{-1}\|_V \leq \frac{1}{|z| - \delta}, \quad \text{in particular} \quad \|(zI - R)^{-1} 1_S\|_V \leq \frac{\|1_S\|_V}{|z| - \delta}$$

from (25) and $\|R^k\|_V \leq \delta^k$. Moreover we have

$$\forall f \in \mathcal{B}_V, \quad |\mu_z(f)| \leq \frac{\nu(V)}{|z| - \delta} \|f\|_V$$

from (14) and $\|R^k\|_V \leq \delta^k$. Then (28) follows from (27) and the previous inequalities.

□

In the atomic case, that is when ($\boldsymbol{S}$) holds with $S \in \mathcal{X}$ such that $\forall (a, a') \in S^2,\ P(a, \cdot) = P(a', \cdot)$ and with $\nu(\cdot) := P(s_0, \cdot)$ for some (any) $s_0 \in S$, then

$$\forall n \geq 1, \quad \beta_n(1_S) = \mathbb{P}_{s_0}\big(R_S = n\big) \tag{30}$$

where $R_S := \inf\{n \geq 1 : X_n \in S\}$ is the first return time in S (see [HL22, Sec. 2]). When the power series $\sum_{k=1}^{+\infty} \mathbb{P}_{s_0}\big(R_S = n\big)\, z^k$ can be computed, the positive constant m_ρ in (24b) is easily tractable. This is illustrated in the following example.

Example 3.1 (Reflected random walk) *Let $P = (P(i,j))_{(i,j)\in\mathbb{N}^2}$ be the reversible Markov kernel defined on $\mathbb{X} = \mathbb{N}$ by*

$$P(0,0) = \varepsilon,\ P(0,1) = 1 - \varepsilon \quad \textit{and} \quad \forall n \geq 1,\ P(n, n-1) := p,\ P(n, n+1) := q = 1 - p \tag{31}$$

with $\varepsilon \in (0,1)$ and $p > 1/2$. Define $\forall n \in \mathbb{N},\ V(n) = (p/q)^{n/2}$. It is well-known that P satisfies Assumptions ($\boldsymbol{A}$) with the atom $S = \{0\}$, $\nu = P(0, \cdot)$, and $\delta = 2\sqrt{pq}$. Here ($\boldsymbol{D}^{\boldsymbol{\alpha_0}}$) holds with $\alpha_0 = 1$ (atomic case, see Appendix A). Moreover the second eigenvalue ϱ_1 of P on $\mathcal{B}_V$ is given by (see [Bax05, Sec. 8.4], [HL14b, Prop. 4.1])

$$\varrho_1 = \frac{pq + (p - \varepsilon)^2}{p - \varepsilon} \quad \textit{if} \ \ \varepsilon < \frac{p - q}{1 + \sqrt{q/p}} \qquad \textit{and} \qquad \varrho_1 = 2\sqrt{pq} \quad \textit{otherwise.} \tag{32}$$

The exact value of ϱ_1 is obtained from algebraic computations in [Bax05, Sec. 8.4] and [HL14b, Prop. 4.1]. The spectral meaning of (32) is the following (see Theorem 2.1): in the first case P acting on $\mathcal{B}_V$ admits eigenvalues in the annulus $\{z \in \mathbb{C} : 2\sqrt{pq} < |z| < 1\}$ with maximal modulus given by $[pq + (p - \varepsilon)^2]/(p - \varepsilon)$; in the second case there is no eigenvalue in the previous annulus. From Theorem 3.1, Estimates (24a)-(24b) hold for every $\rho \in (\varrho_1, 1)$ with

$$\alpha_0 = 1, \quad \|1_S\|_V = 1, \quad \delta = 2\sqrt{pq} \quad \textit{and} \quad \nu(V) = \varepsilon + (1 - \varepsilon)(p/q)^{1/2}.$$

Moreover we deduce from [Bax05, Sec. 8.4] that for every $u \in \mathbb{C}$ such that $|u| < (2\sqrt{pq})^{-1}$

$$b(u) := \sum_{k=1}^{+\infty} \mathbb{P}_{s_0}\big(R_S = n\big)\, u^k = \varepsilon u + \frac{(1-\varepsilon)}{2q}\big[1 - (1-4pqu^2)^{1/2}\big]$$

where, for every $Z = re^{i\vartheta} \in \mathbb{C}$ with $r > 0$ and $\vartheta \in (-\pi/2, \pi/2)$, the complex number $Z^{1/2}$ is defined by $Z^{1/2} = \sqrt{r}\, e^{i\vartheta/2}$. Note that $Z = 1 - 4pqu^2$ is of the previous form. Then it follows from (30) that, for every $z \in \mathbb{C}$ such that $|z| > 2\sqrt{pq}$, we have $\mu_z(1_S) = b(z^{-1})$. Consequently, for every $\rho \in (\varrho_1, 1)$, the positive constant m_ρ in (24b) is given by

$$m_\rho := \min_{\theta \in [-\pi,\pi)} \big|1 - b\big(\rho^{-1}e^{-i\theta}\big)\big|.$$

For this discrete random walk, the numerical values in Table 1 of the multiplicative constant C_ρ in (24a)-(24b) are consistent.

		ε		
		0.05	0.25	0.5
$p = 0.6$	(δ, ϱ_1)	(0.9798,0.9864)	(0.9798,0.9798)	(0.9798,0.9798)
	(C_ρ, ρ)	(2.46×10^4,0.9932)	(3.19×10^4,0.9899)	(4.18×10^4,0.9899)
$p = 0.7$	(δ, ϱ_1)	(0.9165,0.9731)	(0.9165,0.9165)	(0.9165,0.9165)
	(C_ρ, ρ)	(1000.08,0.9866)	(840.49,0.9583)	(992.72,0.9583)
$p = 0.8$	(δ, ϱ_1)	(0.8000,0.9633)	(0.8000,0.8409)	(0.8000,0.8000)
	(C_ρ, ρ)	(187.29,0.9817)	(81.79,0.9204)	(93.68,0.9000)
$p = 0.9$	(δ, ϱ_1)	(0.6000,0.9559)	(0.6000,0.7875)	(0.6000,0.6250)
	(C_ρ, ρ)	(63.36,0.9779)	(19.99,0.8942)	(13.94,0.8125)

Table 1: Exact ϱ_1 with $\delta = 2\sqrt{pq}$ (see (32)), and estimate (24a) of $V-$geometric ergodicity with multiplicative constant C_ρ in (24b) and $\rho = (1+\varrho_1)/2$

4 A bound for the norm of solutions to Poisson's equation

Recall that the existence of Poisson's equation is studied under weak drift condition in [GM96] (also see [MT93, Th. 17.4.2]). In this section the solutions to Poisson's equation are easily obtained since we assume that P satisfies Assumptions $(\boldsymbol{A})$ of Section 1 which include the geometric drift condition $(\boldsymbol{D_{S^c}})$. Indeed assume that Assumptions $(\boldsymbol{A})$ holds and let $\alpha_0 \in (0,1]$ be given in $(\boldsymbol{D^{\alpha_0}})$. Then we know that P satisfies Inequality (2), from which we deduce that the operator $(I - P + \Pi)$ is invertible on $\mathcal{B}_{V^{\alpha_0}}$ with

$$(I - P + \Pi)^{-1} = \sum_{n=0}^{+\infty} (P - \Pi)^n = \sum_{n=0}^{+\infty} (P^n - \Pi) \tag{33}$$

where $\Pi := \pi(\cdot)1_{\mathbb{X}}$. Then, for any $f \in \mathcal{B}_{V^{\alpha_0}}$, it is easily checked that $\widehat{f} := (I - P + \Pi)^{-1} f$ is a solution to Poisson's equation on $\mathcal{B}_{V^{\alpha_0}}$

$$(I - P)\widehat{f} = f - \Pi f. \tag{34}$$

Note that $E_1 := \{h \in \mathcal{B}_{V^{\alpha_0}}, Ph = h\} = \mathbb{C} \cdot 1_{\mathbb{X}}$ from (2) (i.e. 1 is a simple eigenvalue of P) and that the difference of two solutions to Poisson's equation on $\mathcal{B}_{V^{\alpha_0}}$ belongs to E_1. Hence two solutions to Poisson's equation on $\mathcal{B}_{V^{\alpha_0}}$ differ by a constant function. Now, let $f \in \mathcal{B}_{V^{\alpha_0}}$ be such that $\pi(f) = 0$. In Theorem 4.1 below we prove that the function

$$\widetilde{f} := (I - R)^{-1} f = \sum_{n=0}^{+\infty} R^n f$$

is a solution in $\mathcal{B}_{V^{\alpha_0}}$ to Poisson's equation, where R is the non-negative operator of Section 2. Next, since $\pi(f) = 0$, the function

$$\widehat{f} = (I - P + \Pi)^{-1} f = \sum_{n=0}^{+\infty} P^n f$$

satisfies $\pi(\widehat{f}) = \pi(f) = 0$. In fact $\widehat{f}$ is the unique solution in $\mathcal{B}_{V^{\alpha_0}}$ to Poisson's equation which has a null $\pi-$integral. Finally we have $\widehat{f} = \widetilde{f} - \pi(\widetilde{f})1_{\mathbb{X}}$ since $\widetilde{f}$ and $\widehat{f}$ only differ by a constant function.

In Theorem 4.1 below we give a simple and explicit bound for $\|\widetilde{f}\|_{V^{\alpha_0}}$, which is relevant for the perturbation issue of Section 6. This allows us to derive an explicit bound for $\|\widehat{f}\|_{V^{\alpha_0}}$, that will be relevant in Section 5.

Theorem 4.1 *Assume that P satisfies Assumptions $(\boldsymbol{A})$. Let $\alpha_0 \in (0, 1]$ be given in $(\boldsymbol{D^{\alpha_0}})$. Then, for any $f \in \mathcal{B}_{V^{\alpha_0}}$ such that $\pi(f) = 0$, the following assertions hold.*

1. *$\widetilde{f} := (I - R)^{-1} f$ is a solution in $\mathcal{B}_{V^{\alpha_0}}$ to the Poisson equation (34), and*

$$\|\widetilde{f}\|_{V^{\alpha_0}} \leq \frac{1}{1 - \delta^{\alpha_0}} \|f\|_{V^{\alpha_0}}. \tag{35}$$

2. *$\widehat{f} := (I - P + \Pi)^{-1} f$ is the unique solution in $\mathcal{B}_{V^{\alpha_0}}$ to Poisson's equation (34) which has a null $\pi-$integral, and*

$$\|\widehat{f}\|_{V^{\alpha_0}} \leq \frac{1 + \pi(V^{\alpha_0})\|1_{\mathbb{X}}\|_{V^{\alpha_0}}}{1 - \delta^{\alpha_0}} \|f\|_{V^{\alpha_0}} \tag{36a}$$

$$\leq \frac{1 - \delta^{\alpha_0} + \nu(V^{\alpha_0})\|1_{\mathbb{X}}\|_{V^{\alpha_0}}}{(1 - \delta^{\alpha_0})^2} \|f\|_{V^{\alpha_0}}. \tag{36b}$$

Again, we may assume that $\alpha_0 = 1$ in $(\boldsymbol{D^{\alpha_0}})$ for the proof below. If $\alpha_0 < 1$, then replace V and δ with V^{α_0} and δ^{α_0} respectively.

Proof. Recall that $\|R^k\|_V \leq \delta^k$ (see (13)), so that $I - R$ is invertible on $\mathcal{B}_V$ with (see (25))

$$(I - R)^{-1} = \sum_{k=0}^{+\infty} R^k. \tag{37}$$

Next, we have

$$I - P = I - R - T = U \circ (I - R) \quad \text{with} \quad U := I - T \circ (I - R)^{-1}. \tag{38}$$

Let $f \in \mathcal{B}_V$ be such that $\pi(f) = 0$ and let $\widetilde{f} := (I - R)^{-1} f$. Then we obtain from (38)

$$(I - P)\widetilde{f} = \big(U \circ (I - R) \circ (I - R)^{-1}\big) f = Uf.$$

From $T = \nu(\cdot)1_S$ and from (26) applied to $z = 1$, we obtain that

$$Uf = f - \mu_1(f)1_S$$

where μ_1 is defined in (19). Moreover we know from Theorem 2.2 that μ_1 is a $P-$invariant positive finite measure, more precisely $\mu_1 = \mu_1(1_{\mathbb{X}})\pi$ (see also [HL20, HL22]). Hence we have $\mu_1(f) = 0$ since $\pi(f) = 0$, so that $Uf = f$. Thus $\widetilde{f}$ is a solution to the Poisson equation on $\mathcal{B}_V$. Moreover, it follows from (37) and $\|R^k\|_V \leq \delta^k$ that

$$\|\widetilde{f}\|_V = \|(I-R)^{-1}f\|_V = \Big\|\sum_{k=0}^{+\infty} R^k f\Big\|_V \leq \frac{1}{1-\delta}\,\|f\|_V. \tag{39}$$

The proof of the first assertion is complete.

Now let $f \in \mathcal{B}_V$ be such that $\pi(f) = 0$ and let $\widehat{f} := (I - P + \Pi)^{-1}f$. Recall that $\widehat{f} = \widetilde{f} - \pi(\widetilde{f})1_{\mathbb{X}}$. Hence

$$\begin{aligned}
\|\widehat{f}\|_V &\leq \|\widetilde{f}\|_V + |\pi(\widetilde{f})| \times \|1_{\mathbb{X}}\|_V && \text{(triangular inequality)}\\
&\leq \big(1 + \pi(V)\|1_{\mathbb{X}}\|_V\big)\|\widetilde{f}\|_V && \text{(since } |\widetilde{f}| \leq \|\widetilde{f}\|_V\, V)\\
&\leq \frac{1 + \pi(V)\|1_{\mathbb{X}}\|_V}{1-\delta}\,\|f\|_V && \text{(from (39))}.
\end{aligned}$$

This gives (36a). Finally (36b) follows from the inequality $\pi(V) \leq \nu(V)/(1-\delta)$ which can be easily derived from (20). The second assertion is proved. □

5 Bounds for the second eigenvalue of P

Using the results of Sections 2-3-4, we first present some results on the location of the eigenvalues of P on $B_{V^{\alpha_0}}$, from which bounds for the second eigenvalue ϱ_{α_0} (see (2)) can be deduced. For any $a \in \mathbb{C}$ and for any $r > 0$, define

$$D(a,r) := \{\lambda \in \mathbb{C} : |\lambda - a| < r\},\ \ C(a,r) := \{\lambda \in \mathbb{C} : |\lambda - a| = r\},\ \ \overline{D}(a,r) = D(a,r) \cup C(a,r).$$

Proposition 5.1 *Suppose that P satisfies Assumptions (**A**). Let $\alpha_0 \in (0,1]$ be given in ($\boldsymbol{D^{\alpha_0}}$), and define*

$$\widehat{r}_1 := \frac{1-\delta^{\alpha_0}}{1 + \pi(V^{\alpha_0})\|1_{\mathbb{X}}\|_{V^{\alpha_0}}}. \tag{40}$$

Then $\lambda = 1$ is the single spectral value of P on $\mathcal{B}_{V^{\alpha_0}}$ in the open disk $D(1, \widehat{r}_1)$, that is: for every $\lambda \in D(1, \widehat{r}_1) \setminus \{1\}$, the operator $\lambda I - P$ is invertible on $\mathcal{B}_{V^{\alpha_0}}$.

The real number $\widehat{r}_1$ in (40) satisfies

$$\widehat{r}_1 \geq \widetilde{r}_1 := \frac{(1-\delta^{\alpha_0})^2}{1 - \delta^{\alpha_0} + \nu(V^{\alpha_0})}$$

since $\pi(V^{\alpha_0}) \leq \nu(V^{\alpha_0})/(1-\delta^{\alpha_0})$. Therefore $\widehat{r}_1$ may be replaced with $\widetilde{r}_1$ in the conclusion of Proposition 5.1 when π is unknown.

Proof. Note that $\widehat{r}_1 < 1-\delta^{\alpha_0}$. Therefore, if $\lambda \in D(1,\widehat{r}_1)$, then $|\lambda| > \delta^{\alpha_0}$. Thus it follows from (17) that any spectral value of P on $\mathcal{B}_{V^{\alpha_0}}$ in $D(1,\widehat{r}_1)$ is actually an eigenvalue. Consequently we have to prove that $\lambda = 1$ is the single eigenvalue of P on $\mathcal{B}_{V^{\alpha_0}}$ in $D(1,\widehat{r}_1)$. Let $\lambda \in \mathbb{C}\setminus\{1\}$, be an eigenvalue of P on $\mathcal{B}_{V^{\alpha_0}}$, and let $f_\lambda \in \mathcal{B}_{V^{\alpha_0}}$ be such that $f_\lambda \neq 0$ and $Pf_\lambda = \lambda f_\lambda$. Then

$$(1-\lambda)f_\lambda = (I-P)f_\lambda. \tag{41}$$

Since $\lambda \neq 1$, we have $\pi(f_\lambda) = 0$. It follows that

$$(I-P+\Pi)^{-1} \circ (I-P)f_\lambda = (I-P+\Pi)^{-1} \circ (I-P+\Pi)f_\lambda = f_\lambda.$$

Then we obtain by composing to the left of (41) by $(I-P+\Pi)^{-1}$ that

$$(1-\lambda)\widehat{f_\lambda} = f_\lambda \quad \text{where} \quad \widehat{f_\lambda} := (I-P+\Pi)^{-1}f_\lambda$$

so that $\widehat{f_\lambda} = (1-\lambda)^{-1}f_\lambda$. It follows from (36a) applied to $f = f_\lambda$ that $|1-\lambda|^{-1} \leq \widehat{r}_1^{\,-1}$, thus $|1-\lambda| \geq \widehat{r}_1$. This proves the expected statement. □

Proposition 5.2 *Assume that P satisfies Assumptions ($\boldsymbol{A}$). Let $\alpha_0 \in (0,1]$ be given in ($\boldsymbol{D^{\alpha_0}}$). Let $z \in \mathbb{C}$ be such that $|z| = 1$, $z \neq 1$, and define*

$$r_z := \big(1-\delta^{\alpha_0}\big)\left(1 + \frac{\nu(V^{\alpha_0})\|1_S\|_{V^{\alpha_0}}}{\big|1-\mu_z(1_S)\big|\big(1-\delta^{\alpha_0}\big)}\right)^{-1}. \tag{42}$$

Then there is no spectral value of P on $\mathcal{B}_{V^{\alpha_0}}$ in the open disk $D(z,r_z)$, that is: $\forall \lambda \in D(z,r_z)$, the operator $\lambda I - P$ is invertible on $\mathcal{B}_{V^{\alpha_0}}$.

Proof. Let $z \in \mathbb{C}$ be such that $|z| = 1$, $z \neq 1$. Then $zI - P$ is invertible on $\mathcal{B}_{V^{\alpha_0}}$ from Theorem 3.1, so that $\mu_z(1_S) \neq 1$ due to Theorem 2.2. Since $r_z < 1-\delta^{\alpha_0}$ we have to prove that there is no eigenvalue of P on $\mathcal{B}_{V^{\alpha_0}}$ in $D(z,r_z)$ (as in the proof of Proposition 5.1). Let $\lambda \in \mathbb{C}$ be an eigenvalue of P on $\mathcal{B}_{V^{\alpha_0}}$ (thus $\lambda \neq z$), and let $f_\lambda \in \mathcal{B}_{V^{\alpha_0}}$ be such that $Pf_\lambda = \lambda f_\lambda$ and $\|f_\lambda\|_{V^{\alpha_0}} = 1$. We have $(zI-P)f_\lambda = (z-\lambda)f_\lambda$, so that $(zI-P)^{-1}f_\lambda = (z-\lambda)^{-1}f_\lambda$. Using $|z| = 1$, it follows from (23) applied to f_λ that $|z-\lambda|^{-1} \leq r_z^{-1}$, thus $|z-\lambda| \geq r_z$. This proves the desired statement.

□

Under Assumptions ($\boldsymbol{A}$) let $z_0 = e^{i\vartheta_0} \in \mathbb{C}$, $\vartheta_0 \in (0,\pi/2)$, be defined by

$$C(0,1) \cap C(1,\widehat{r}_1) = \{e^{i\vartheta_0}, e^{-i\vartheta_0}\} \tag{43}$$

with $\widehat{r}_1$ defined in (40), and let Γ_0 be the following closed subset of $C(0,1)$:

$$\Gamma_0 := \big\{z \in \mathbb{C} : z = e^{i\vartheta},\ \vartheta \in [\vartheta_0, 2\pi-\vartheta_0]\big\}.$$

Note that

$$m_0 := \min_{z\in\Gamma_0} \big|1-\mu_z(1_S)\big| > 0.$$

Indeed let $\alpha_0 \in (0,1]$ be given in ($\boldsymbol{D^{\alpha_0}}$). Then for every $z \in \Gamma_0$ we know that $zI - P$ is invertible on $\mathcal{B}_{V^{\alpha_0}}$ (Theorem 3.1), so that $\mu_z(1_S) \neq 1$ (Theorem 2.2). Then the positivity of m_0 follows from the continuity of the function $z \mapsto \mu_z(1_S) := \sum_{k=1}^{+\infty} z^{-k}\beta_k(1_S)$ on the compact set Γ_0. Finally let

$$\widehat{r}_0 := \big(1-\delta^{\alpha_0}\big)\left(1+\frac{\nu(V^{\alpha_0})\|1_S\|_{V^{\alpha_0}}}{m_0\big(1-\delta^{\alpha_0}\big)}\right)^{-1}. \tag{44}$$

Note that $\widehat{r}_0 \le r_{z_0}$ from the definition of m_0 and that $r_{z_0} \le \widehat{r}_1$ since $|z_0-1| = \widehat{r}_1$ and the eigenvalue 1 cannot belong to $D(z_0, r_{z_0})$ from Proposition 5.2. Consequently $\widehat{r}_0 \le \widehat{r}_1$, and we can define ξ_0 as the unique complex number such that

$$|\xi_0| < 1 \quad \text{and} \quad \xi_0 \in C(1,\widehat{r}_1) \cap C(z_0,\widehat{r}_0). \tag{45}$$

Corollary 5.1 *Assume that P satisfies Assumptions ($\boldsymbol{A}$). Let $\alpha_0 \in (0,1]$ be given in ($\boldsymbol{D^{\alpha_0}}$) and let ξ_0 be defined in (45). Then the second eigenvalue ϱ_{α_0} of P on $\mathcal{B}_{V^{\alpha_0}}$ (see (2)) is such that $\varrho_{\alpha_0} \le |\xi_0|$.*

Proof. From the definition of m_0 and from Proposition 5.2 we deduce that, for every $z \in \Gamma_0$, there is no spectral value of P on $\mathcal{B}_{V^{\alpha_0}}$ in the open disk $D(z,\widehat{r}_0)$, that is: $\forall z \in \Gamma_0$, $\forall \lambda \in D(z,\widehat{r}_0)$, the operator $\lambda I - P$ is invertible on $\mathcal{B}_{V^{\alpha_0}}$. Then Corollary 5.1 follows from Propositions 5.1 and from the spectral properties of Section 2. □

Note that the series $\sum_{k=1}^{+\infty}\beta_k(1_S)$ is convergent (see (19)) and that $\mu_1(1_S) = \sum_{k=1}^{+\infty}\beta_k(1_S) = 1$ since 1 is an eigenvalue of P (Theorem 2.2). Thus

$$1-\mu_{-1}(1_S) = \sum_{k=1}^{+\infty}\big(1-(-1)^k\big)\,\beta_k(1_S) = 2\sum_{k=1}^{+\infty}\beta_{2k-1}(1_S) \tag{46}$$

and $1-\mu_{-1}(1_S) \in [2\nu(1_S), 2]$ since we have $\beta_1(1_S) = \nu(1_S)$ and $\sum_{k=1}^{+\infty}\beta_k(1_S) = 1$. Recall that P is said to be reversible with respect to π if $\pi(dx)P(x,dy) = \pi(dy)P(y,dx)$. Under Assumptions ($\boldsymbol{A}$) and with $\alpha_0 \in (0,1]$ given in ($\boldsymbol{D^{\alpha_0}}$), define

$$\forall t > 0, \quad \psi(t) := \frac{\delta^{\alpha_0}(1-\delta^{\alpha_0})t + \nu(V^{\alpha_0})\|1_S\|_{V^{\alpha_0}}}{(1-\delta^{\alpha_0})t + \nu(V^{\alpha_0})\|1_S\|_{V^{\alpha_0}}}, \tag{47}$$

$$\forall n \ge 1, \quad \eta_n := 2\sum_{k=1}^{n}\beta_{2k-1}(1_S) \quad \text{and} \quad \eta_\infty := 2\sum_{k=1}^{+\infty}\beta_{2k-1}(1_S) = 1-\mu_{-1}(1_S). \tag{48}$$

Note that for any $n \ge 1$, $\eta_\infty \ge \eta_n \ge 2\beta_1(1_S) = 2\nu(1_S) > 0$.

Corollary 5.2 *Assume that P satisfies Assumptions ($\boldsymbol{A}$), and let $\alpha_0 \in (0,1]$ be given in ($\boldsymbol{D^{\alpha_0}}$). Moreover assume that P is reversible with respect to π, that $\pi(V^{2\alpha_0}) < \infty$, and that the following implication holds for every $\lambda \in \mathbb{C}$, $|\lambda| > \delta^{\alpha_0}$ and for every $f \in \mathcal{B}_{V^{\alpha_0}}$:*

$$Pf = \lambda f,\ f \ne 0 \implies \pi(|f|) \ne 0. \tag{49}$$

Then

$$\forall n \ge 1, \quad \varrho_{\alpha_0} \le \psi(\eta_\infty) \le \psi(\eta_n). \tag{50}$$

Recall that the bounds of ϱ_{α_0} in (50) can be used in the $\mathbb{L}^2(\pi)-$geometric ergodicity (see (3)). Also recall that $\lim_n \eta_n = \eta_\infty$ and $\eta_\infty \in [2\nu(1_S), 2]$. The second bound in (50) applied to $n = 1$ gives $\varrho_{\alpha_0} \le \psi(2\nu(1_S))$, but this bound is not accurate in general because $\nu(1_S)$ is small, so that the bound $\psi(\nu(1_S))$ is close to 1.

Proof. From reversibility we know that P is a self-adjoint bounded linear operator on $\mathbb{L}^2(\pi)$, and that the spectral values of P on $\mathbb{L}^2(\pi)$ are contained in $[-1, 1]$, e.g. see [RR97, Bax05]. Moreover note that every $f \in \mathcal{B}_{V^{\alpha_0}}$ is such that $\pi(|f|^2) < \infty$ from $\pi(V^{2\alpha_0}) < \infty$. Now let $\lambda \in \mathbb{C}$, $|\lambda| > \delta^{\alpha_0}$, be an eigenvalue of P on $\mathcal{B}_{V^{\alpha_0}}$, and let $f \in \mathcal{B}_{V^{\alpha_0}}$ be such that $Pf = \lambda f$ and $f \neq 0$. Then $\pi(|f|) \neq 0$ from (49), so that λ is an eigenvalue of P on $\mathbb{L}^2(\pi)$. Therefore every eigenvalue $\lambda \in \mathbb{C}$ of P on $\mathcal{B}_{V^{\alpha_0}}$ such that $|\lambda| > \delta^{\alpha_0}$ actually belongs to $(-1, -\delta^{\alpha_0}) \cup (\delta^{\alpha_0}, 1]$. Next, note that r_{-1} defined in (42) (with $z = -1$) satisfies $r_{-1} < 1 - \delta^{\alpha_0}$, thus $\delta^{\alpha_0} < 1 - r_{-1}$. Also observe that $1 - r_{-1} = \psi(\eta_\infty)$ from an easy computation, (46) and $1 - \mu_{-1}(1_S) > 0$. Thus, using Theorem 2.1, the first inequality in (50) holds if we establish that there is no eigenvalue of P on $\mathcal{B}_{V^{\alpha_0}}$ in $I_1 := (1 - r_{-1}, 1)$ and in $I_{-1} := [-1, -1 + r_{-1})$. This is true for I_1 since we know from Theorem 2.2 that $\lambda = 1$ is the unique solution to Equation $\sum_{k=1}^{+\infty} \lambda^{-k}\, \beta_k(1_S) = 1$ in the interval $(\delta^{\alpha_0}, 1]$. Moreover this is true for I_{-1} since we know from Proposition 5.2 applied to $z = -1$ that there is no spectral value of P on $\mathcal{B}_{V^{\alpha_0}}$ in the open disk $D(-1, r_{-1})$. Thus $\varrho_{\alpha_0} \le |-1 + r_{-1}| = 1 - r_{-1}$. Finally easy computations show that ψ in (47) is decreasing on $(0, +\infty)$. This proves the second inequality in (50) since $0 < \eta_n \le \eta_\infty$. □

Remark 5.1 *In the atomic case, Equality $\sum_{k=1}^{+\infty} \beta_k(1_S) = 1$ reads as $\mathbb{P}_{s_0}(R_S < \infty) = 1$, and $\eta_\infty = 2\mathbb{P}_{s_0}\big(R_S \in 2\mathbb{N} + 1\big)$ (see (30)). Moreover Conditions ($\boldsymbol{D}^{\boldsymbol{\alpha_0}}$) holds with $\alpha_0 = 1$ (see Appendix A). Consequently, when the assumptions of Corollary 5.2 hold with an atom S, then we have the following upper bound for the second eigenvalue ϱ_1 of P on $\mathcal{B}_V$:*

$$\varrho_1 \le \psi(2p_1) = \frac{2\delta(1-\delta)p_1 + \nu(V)\|1_S\|_V}{2(1-\delta)p_1 + \nu(V)\|1_S\|_V} \quad \text{with} \quad p_1 := \mathbb{P}_{s_0}\big(R_S \in 2\mathbb{N} + 1\big).$$

Remark 5.2 *Assumption (49) is used in the previous proof to ensure that every eigenvalue λ of P on $\mathcal{B}_{V^{\alpha_0}}$ such that $|\lambda| > \delta^{\alpha_0}$ is also an eigenvalue of P on $\mathbb{L}^2(\pi)$. If P is of the form $P(x, dy) = p(x, y)d\mu(y)$ where μ is a positive measure on $(\mathbb{X}, \mathcal{X})$ and if P admits an invariant probability measure $\pi(dx)$, then $\pi(dx)$ is absolutely continuous with respect to μ (i.e. $\pi(dx) = \pi(x)\,\mu(dx)$). If moreover the density function π is positive on $\mathbb{X}$, then Condition (49) holds. Indeed, if $f \in \mathcal{B}_{V^{\alpha_0}}$ is such that $\pi(|f|) = 0$, then $f = 0$ $\mu-$a.s., so that $\forall x \in \mathbb{X},\ (Pf)(x) = 0$. This proves (49). More generally note that Condition (49) is fulfilled when for every $f \in \mathcal{B}_{V^{\alpha_0}}$ we have:*

$$f = 0 \ \ \pi\text{-almost surely} \implies \forall x \in \mathbb{X},\ \exists n = n_x \ge 1,\ (P^n f)(x) = 0. \tag{51}$$

The bounds of the second eigenvalue in Corollaries 5.1-5.2 are quite simple, but their interest is essentially theoretical. In particular the bound (50) in the reversible case is in general inaccurate because of the form of the function ψ (even in the atomic case of Remark 5.1). By contrast, the following positive reversible case is much more favourable. Recall that a reversible Markov kernel P with respect to π is said to be positive if the following condition holds

$$\forall f \in \mathbb{L}^2(\pi), \quad \langle Pf, f\rangle_{\mathbb{L}^2(\pi)} = \int_{\mathbb{X}} (Pf)(x) f(x) \pi(dx) \ge 0. \tag{52}$$

Under this condition, every eigenvalue λ of P on $\mathbb{L}^2(\pi)$ is non-negative from (52). Consequently, if P satisfies the assumptions of Corollary 5.2 and if P is positive, then every eigenvalue λ of P on $\mathcal{B}_{V^{\alpha_0}}$ such that $|\lambda| > \delta^{\alpha_0}$ is actually positive. However, as already mentioned in the proof of Corollary 5.2, $\lambda \in (\delta^{\alpha_0}, 1)$ is not an eigenvalue of P on $\mathcal{B}_{V^{\alpha_0}}$ since $\sum_{k=1}^{+\infty} \lambda^{-k}\, \beta_k(1_S) > 1$ (Theorem 2.2). Thus the following statement holds.

Corollary 5.3 *Assume that P is a positive reversible Markov kernel with respect to π satisfying Assumptions ($\boldsymbol{A}$) and (49). Let $\alpha_0 \in (0,1]$ be given in ($\boldsymbol{D^{\alpha_0}}$), and assume that $\pi(V^{2\alpha_0}) < \infty$. Then we have*

$$\varrho_{\alpha_0} \leq \delta^{\alpha_0}. \tag{53}$$

Recall that $\alpha_0 = 1$ in the atomic case. If S is an atom, then the conclusion of Corollary 5.3 has been proved in [Bax05, Th. 1.3] where Condition (49) is not assumed to hold. Therefore, the previous corollary extends this result to the non-atomic case, provided that Condition (49) holds true and that the space $\mathcal{B}_V$ is replaced with $\mathcal{B}_{V^{\alpha_0}}$. The bound $\varrho_{\alpha_0} \leq \delta^{\alpha_0}$ can be used in (3) too.

If P is reversible with respect to π and if $\ell \geq 2$ is any even integer, then the $\ell-$th iterate P^ℓ of P is a positive reversible Markov kernel with respect to π. Moreover, if $\varrho(P^\ell)$ is the second eigenvalue of P^ℓ on $\mathcal{B}_W$ for some $W \geq 1$, then $\varrho(P^\ell)^{1/\ell}$ is the second eigenvalue of P on $\mathcal{B}_W$. Indeed, writing $n = k\ell + r$ (euclidean division) and defining $\Pi f = \pi(f)1_{\mathbb{X}}$, we have

$$P^n - \Pi = P^{k\ell+r} - \Pi = (P - \Pi)^r \left((P^\ell)^k - \Pi\right)$$

from which we easily deduce the desired result. Then the following statement follows from Corollary 5.3 applied to P^ℓ.

Corollary 5.4 *Assume that P is reversible with respect to π. Moreover assume that, for some even integer $\ell \geq 2$, the Markov kernel P^ℓ satisfies Assumptions ($\boldsymbol{A}$), so that P^ℓ satisfies Conditions ($\boldsymbol{D_{S^c}}$) and ($\boldsymbol{D^{\alpha_0}}$) with some $\delta(P^\ell) \in (0,1)$ and some $\alpha_0(P^\ell) \in (0,1]$. Finally suppose that P^ℓ satisfies (49) and that $\pi(V^{2\alpha_0(P^\ell)}) < \infty$. Then we have $\varrho_{\alpha_0} \leq \delta^{\alpha_0(P^\ell)/\ell}$.*

The use of Corollaries 5.3 and 5.4 are illustrated by the three following examples. Corollary 5.3 is applied in the first two. Moreover a simple condition to check Condition (49) in Metropolis-Hastings algorithms is provided (Remark 5.2 does not apply since the Metropolis kernel has a discrete part).

Example 5.1 (The Metropolis-Hastings algorithm) *The objective is to apply Corollary 5.3 in the context of Markov chain Monte Carlo (MCMC) algorithms where the reversibility property of the transition kernel is in force. Specifically, we consider the Metropolis-Hastings algorithm. Let π (the target density) be a positive distribution density function on $\mathbb{X} = \mathbb{R}^d$, and let $q(x,\cdot)$ be a proposal density on $\mathbb{X} = \mathbb{R}^d$ for any $x \in \mathbb{R}^d$. Let us introduce the acceptance probability*

$$\forall (x,y) \in \mathbb{R}^d \times \mathbb{R}^d, \quad p(x,y) := \begin{cases} \min\left(1\,,\ \frac{\pi(y)\,q(y,x)}{\pi(x)\,q(x,y)}\right) & \text{if} \quad \pi(x)\,q(x,y) > 0 \\ 1 & \text{if} \quad \pi(x)\,q(x,y) = 0. \end{cases}$$

Recall that the associated Metropolis-Hastings kernel is defined by

$$P(x,dy) := s(x)\,\delta_x(dy) + p(x,y)q(x,y)\,dy \quad \text{ with } \; s(x) := 1 - \int_{\mathbb{R}^d} p(x,z)q(x,z)\,dz, \tag{54}$$

where $\delta_x(dy)$ denotes the Dirac distribution at x. It is well-known that P is reversible with respect to π. The next step is to propose a criterion that ensures that Condition (49) is satisfied in this context. For any $x_0 \in \mathbb{R}^d$, let f_{x_0} be the Dirac function at x_0, that is: $f_{x_0}(x_0) = 1$ and $\forall x \neq x_0,\ f_{x_0}(x) = 0$. Then $Pf_{x_0} = s(x_0)f_{x_0}$, thus $s(x_0)$ is an eigenvalue of P on the space $\mathcal{B}_W$ for any function $W \geq 1$. Therefore a necessary condition for P to be $W-$geometrically ergodic is that $s_\infty := \sup_{x\in\mathbb{R}^d} s(x) < 1$. Now assume that $s_\infty < 1$. Moreover assume that the Metropolis-Hastings Markov kernel P defined in (54) satisfies Assumptions ($\boldsymbol{A}$), and let $\alpha_0 \in (0,1]$ be given in ($\boldsymbol{D^{\alpha_0}}$). Then P satisfies Condition (49) provided that

$$\forall x \in S, \quad s(x) \leq \delta^{\alpha_0}. \tag{55}$$

Indeed note that (55) is equivalent to the condition $s_\infty \leq \delta^{\alpha_0}$ since $\forall x \in S^c,\ s(x) \leq \delta^{\alpha_0}$ from

$$\forall x \in S^c, \quad s(x)V(x)^{\alpha_0} \leq (PV^{\alpha_0})(x) \leq \delta^{\alpha_0}V(x)^{\alpha_0}$$

(use the definition of P in (54) and ($\boldsymbol{D^{\alpha_0}}$)). Now assume that $s_\infty \leq \delta^{\alpha_0}$ and prove (49). Let $\lambda \in \mathbb{C}$, $|\lambda| > \delta^{\alpha_0}$, and let $f \in \mathcal{B}_{V^{\alpha_0}}$, $f \neq 0$, be such that $Pf = \lambda f$. We must prove that $\pi(|f|) \neq 0$. Suppose that $\pi(|f|) = 0$. Since $\pi > 0$, we have $f(y) = 0$ for almost every $y \in \mathbb{R}^d$ with respect to Lebesgue's measure on $\mathbb{R}^d$. Then

$$\forall x \in \mathbb{R}^d, \quad \int_{\mathbb{R}^d} f(y)\, p(x,y)q(x,y)\, dy = 0,$$

and $\forall x \in \mathbb{R}^d,\ \lambda f(x) = (Pf)(x) = s(x)\, f(x)$. Since $f \neq 0$, there exists $x_0 \in \mathbb{R}^d$ such that $f(x_0) \neq 0$, so that $s(x_0) = \lambda$. But this is impossible since $\forall x \in \mathbb{R}^d,\ s(x) \leq \delta^{\alpha_0} < |\lambda|$.

Next, we manage Assumptions ($\boldsymbol{A}$) on the specific Metropolis-Hastings algorithm for the standard Gaussian distribution π arising from the proposal Gaussian density $q(x,\cdot) = \mathcal{N}(x,1)$. The bounds of the second eigenvalue provided in [Bax05, Sec. 8.2] for this example can be compared with the bound (53). Let $\mathbb{X} := \mathbb{R}$. Since $q(x,y) = q(y,x)$, the acceptance probability is $p(x,y) = \min(\pi(y)/\pi(x), 1)$ and it may be checked that function $s(x) = P(x,\{x\})$ is

$$s(x) = \frac{3}{2} - \Phi(2|x|) - \frac{e^{x^2/4}}{\sqrt{2}}\Big(2 - \Phi(3|x|/\sqrt{2}) - \Phi(|x|/\sqrt{2})\Big)$$

and $s_\infty = \sup_{x\in\mathbb{R}} r(x) \leq 1/2$. Let $r, d > 0$ be two positive scalars. Set $V_r(x) := e^{r|x|}$ for any $x \in \mathbb{X}$ and $S_d := [-d,d]$. First, we know from [Bax05, Sect. 8.2] that P satisfies ($\boldsymbol{D_{S^c}}$) with

$$\begin{aligned}
\delta_{d,r} &:= \lambda(d,r) = \max_{|x|\geq d} \lambda(x,r) \\
\lambda(x,r) &= e^{r^2/2}\big[\,\Phi(-r) - \Phi(-r-x)\,\big] + \frac{1}{\sqrt{2}} e^{(x-r)^2/4}\,\Phi\big((r-x)/\sqrt{2}\,\big) + \\
&\quad e^{r^2/2-2rx}\big[\,\Phi(-x+r) - \Phi(-2x+r)\,\big] + \frac{1}{\sqrt{2}} e^{(x^2-6rx+r^2)/4}\,\Phi\big((r-3x)/\sqrt{2}\,\big) \\
&\quad +\Phi(0) + \Phi(-2x) - \frac{1}{\sqrt{2}} e^{x^2/4}\big[\,\Phi(-x/\sqrt{2}) + \Phi\big(-3x/\sqrt{2}\,\big)\,\big]
\end{aligned}$$

where Φ denotes the standard Gaussian distribution function. Second, we deduce from [Bax05, p. 726] that PV_r is bounded on $S = [-d,d]$ by $K_{d,r} := e^{d\times r}\,\lambda(d,r)$. Third, P satisfies ($\boldsymbol{S}$) with the two minorization measures $\nu_d^{(1)}(dx) = \big(e^{-d^2}/\sqrt{2\pi}\big)e^{-x^2}\,1_{[-d,d]}(x)\,dx$ and $\nu_d^{(2)}(dx) =$

$\inf_{y\in[-d,d]} p(x,y)\, q(x,y)\, dx$ *(see [Bax05, p. 727]). We know from [Bax05, Lem. 3.1]) that* P *is a positive reversible with respect to* π. *Using the value of* (r,d) *from which the best rate of convergence is obtained in [Bax05] for the measures* $\nu_d^{(i)}, i = 1,2$, *and moreover using the tuned value* (r,d) *providing the best rate (53), we obtain the numerical results of Table 2. In all these cases we get* $\alpha_0 = 1$ *from (70) in Appendix A since* $M_{r,d} := K_{r,d} - \nu_d^{(i)}(V_r)$ *with (see [Bax05, Sec. 8.3])*

$$\nu_d^{(1)}(V_r) = \sqrt{2}\, e^{-d^2+r^2/4}\big[\, \Phi\big(\sqrt{2}(d-r/2)\big) - \Phi\big(-r/\sqrt{2}\,\big)\,\big]$$
$$\nu_d^{(2)}(V_r) = 2\, e^{((r-d)^2-d^2)/2}\big[\, \Phi(2d-r) - \Phi(d-r)\,\big] + \sqrt{2}e^{(d-r)^2/4}\big[\, 1 - \Phi\big((3d-r)/\sqrt{2}\big)\,\big]$$

is such that $M_{r,d} - \delta_{r,d} \leq 0$. *Therefore all the convergence rates are for* $V-$*geometric ergodicity. It is clear that Condition (55) is satisfied and* $\pi(V_r) < \infty$.

	$\nu_d^{(1)}$					$\nu_d^{(2)}$			
	rate ϱ_1		d			rate ϱ_1		d	
r	[Bax05]	(53)	[Bax05]	(53)	r	[Bax05]	(53)	[Bax05]	(53)
0.16	0.9747	0.9634	1.10	1.39	0.22	0.9667	0.9480	1.11	1.49
0.36		0.9510		1.10	0.43		0.9344		1.2

Table 2: The convergence rates of [Bax05, Tab. 2,3, Th 1.3] and from Corollary 5.3

The rate 0.9344 *is the minimal value provided by (53) when tuning* (r,d). *Then it gives a bound for the second eigenvalue in the* V*-geometric ergodicity with* $V(x) = e^{0.43|x|}$. *Recall that Inequality (3) applies with the rate* 0.9344 *too.*

Example 5.2 (Gaussian autoregressive Markov chain) *Let us apply Corollary 5.3 to the autoregressive Gaussian Markov chain on* $\mathbb{X} = \mathbb{R}$ *associated with Gaussian transition kernel* $P(x,\cdot) = \mathcal{N}(\theta x, 1-\theta^2)$ *with* $\theta \in (-1,1)$. *The* P*-invariant distribution is* $\pi = \mathcal{N}(0,1)$ *for any* $\theta \in (-1,1)$. *This Markov model is also known as contracting normals if introduced as a component of a two-component Gibbs sampler. The convergence rate in* $V-$*geometric ergodicity is investigated in [Bax05, Sect. 8.3] with* $V(x) := 1 + x^2$ *and* $S := [-d,d]$. *When* $\theta > 0$, P *is positive reversible from [Bax05, Sect. 8.3] . As in [Bax05, Tab. 4], we only consider this case here. Moreover it follows from Remark 5.2 that* P *satisfies Condition (49). Then, if* $d > 1$, *we know from the computations in [Bax05, Sect. 8.3] that* P *satisfies* $(\boldsymbol{D_{S^c}})$ *with*

$$\delta_{d,\theta} = \theta^2 + 2\,\frac{1-\theta^2}{1+d^2} < 1$$

and Condition $(\boldsymbol{S})$ *with the minorization measure*

$$\nu_{d,\theta}(dy) = \min_{x\in[-d,d]} \frac{1}{\sqrt{2\pi(1-\theta^2)}} \exp\left(-\frac{(y-\theta\, x)^2}{2(1-\theta^2)}\right) 1_{[-d,d]}(y)\, dy.$$

We deduce from [Bax05, p. 728] that Condition $(\boldsymbol{K})$ *holds with*

$$K_{d,\theta} := \sup_{x\in[-d,d]} (PV)(x) = 2 + \theta^2(d^2-1).$$

The value of $\alpha_0 \in (0,1]$ *so that Inequality* ($\boldsymbol{D^{\alpha_0}}$) *holds true is obtained according to Appendix A. Note that* $\pi(V^{\alpha_0}) < \infty$ *for any* $\alpha_0 \in (0,1]$. *Therefore, Corollary 5.3 applies. Table 3 provides the rates of convergence from (53) in Corollary 5.3 and from [Bax05, Table 4, Th 1.3] which provided the best estimates, compared to previous works, for the* $V-$*geometric ergodicity (except for* $\theta := 1/2$ *where the exact rate is known to be 1/2).*

		Rate		
θ	d	ϱ_1 from [Bax05]	ϱ_{α_0} from (53)	α_0
0.5	1.5	0.897	0.892	0.336
	1.6		0.891	0.290
0.75	1.2	0.9847	0.9844	0.191
	1.3		0.9834	0.141
0.9	1.1	0.99948	0.99947	0.029
	1.14		0.99944	0.022

Table 3: The estimates of [Bax05, Table 4, Th 1.3] for $V-$geometric ergodicity and from Corollary 5.3 for $V^{\alpha_0}-$geometric ergodicity

The numerical findings in Tables 2 and 3 show that the bound $\varrho_{\alpha_0} \leq \delta^{\alpha_0}$ proved in Corollary 5.3 is slightly better (sometimes even quite significantly better in Table 2) than the bounds of the V-geometric rate of convergence obtained in [Bax05, Sec. 8.2-8.3]. Recall that our convergence rates in Table 3 hold for $V^{\alpha_0}-$geometric ergodicity. In any case, recall that the rates apply in the $\mathbb{L}^2(\pi)-$geometric ergodicity (3).

Example 5.3 *Let us give a simple example for which the second eigenvalue of* P *is known and compared with the bound provided by Corollary 5.4 (applied here with* $\ell = 2$). *Let* $P = (P(i,j))_{(i,j)\in\mathbb{N}^2}$ *be the reversible Markov kernel defined on* $\mathbb{X} = \mathbb{N}$ *by*

$$P(0,0) = 0.1,\ P(0,1) = 0.9 \quad \text{and} \quad \forall n \geq 1,\ P(n,n-1) := 0.6,\ P(n,n+1) := 0.4. \tag{56}$$

Define $\forall n \in \mathbb{N},\ V(n) = (0.6/0.4)^{n/2}$. *The second eigenvalue of* P *on* $\mathcal{B}_V$ *is* $\varrho_1 = 0.98$, *see [Bax05, Sec. 8.4], [HL14b, Prop. 4.1]. Note that* P *satisfies the assumptions of Corollary 5.4 with* $\ell = 2$. *Indeed,* P^2 *satisfies* ($\boldsymbol{S}$) *with* $S = \{0,1\}$, $\nu = \nu(0)\delta_0 + \nu(1)\delta_1$ *with*

$$\nu(0) = \min\left(P^2(0,0), P^2(1,0)\right) = 0.06, \quad \nu(1) = \min\left(P^2(0,1), P^2(1,1)\right) = 0.09,$$

and P^2 *satisfies* ($\boldsymbol{D_{S^c}}$) *with* V *as above defined and with* $\delta(P^2) = 4\times0.6\times0.4 = 0.96$. *Finally the real number* $\alpha_0(P^2) \in (0,1]$ *has to be chosen such that* P^2 *satisfies* ($\boldsymbol{D^{\alpha_0}}$), *that is*

$$\forall i \in \{0,1\},\ (P^2V^{\alpha_0(P^2)})(i) \leq 0.96^{\alpha_0(P^2)}\,(0.6/0.4)^{i\,\alpha_0(P^2)/2} + 0.06 + 0.09 \times (0.6/0.4)^{\alpha_0(P^2)/2}.$$

We find $\alpha_0(P^2) = 0.71$. *Consequently we deduce from Corollary 5.4 that*

$$\varrho_{\alpha_0} \leq \sqrt{0.96^{\alpha_{0,2}}} \leq \sqrt{0.9714} = 0.9856.$$

This bound is not very far from the exact value 0.98.

6 Application to perturbed Markov kernels

Recall that Liu and Li provide in [LL18] an interesting control on the invariant probability measure for truncated stochastic matrices, which can be thought of as the control of the invariant probability measure of a specific perturbation of a Markov kernel. Their bound is quite relevant in the atomic case (see Remark 6.1). Here, using Theorem 4.1, their result is extended to general perturbed Markov kernels in the non-atomic case.

Let $\{P_\theta\}_{\theta\in\Theta}$ be a family of transition kernels on $(\mathbb{X},\mathcal{X})$, where Θ is an open subset of some metric space. We assume that the family $\{P_\theta\}_{\theta\in\Theta}$ satisfies the following conditions: there exist $S\in\mathcal{X}$ and $\nu\in\mathcal{M}^+$ such that

$$\nu(1_S)>0 \quad \text{and} \quad \forall\theta\in\Theta,\ \forall x\in\mathbb{X},\ \forall A\in\mathcal{X},\quad P_\theta(x,A)\geq\nu(1_A)\,1_S(x) \qquad (\boldsymbol{S_\Theta})$$

and there exists a Lyapunov function $V:\mathbb{X}\to[1,+\infty)$ such that

$$\exists\delta\in(0,1),\ \forall\theta\in\Theta,\ \forall x\in S^c,\quad (P_\theta V)(x)\leq\delta\,V(x) \qquad (\boldsymbol{D_{\Theta,S^c}})$$

$$K:=\sup_{\theta\in\Theta}\sup_{x\in S}(P_\theta V)(x)<\infty. \qquad (\boldsymbol{K_\Theta})$$

Thus the whole family $\{P_\theta\}_{\theta\in\Theta}$ has is a small-set S with the same positive measure ν and satisfies the geometric drift conditions $(\boldsymbol{D_{S^c}})$-$(\boldsymbol{K})$ in a uniform way in $\theta\in\Theta$. Throughout this section, Assumptions $(\boldsymbol{A_\Theta})$ will stand for the set of Assumptions $(\boldsymbol{S_\Theta})$-$(\boldsymbol{D_{\Theta,S^c}})$-$(\boldsymbol{K_\Theta})$. Then for every $\theta\in\Theta$ there exists a unique $P_\theta-$invariant probability measure π_θ on $(\mathbb{X},\mathcal{X})$ such that $\pi_\theta(V)<\infty$. Moreover, under Assumptions $(\boldsymbol{A_\Theta})$, there exists $\alpha_0\in(0,1]$ such that

$$\forall\theta\in\Theta,\quad P_\theta V^{\alpha_0}\leq\delta^{\alpha_0}\,V^{\alpha_0}+\nu(V^{\alpha_0})1_S. \qquad (\boldsymbol{D_\Theta^{\alpha_0}})$$

In fact Property $(\boldsymbol{D_\Theta^{\alpha_0}})$ can be proved as for $(\boldsymbol{D^{\alpha_0}})$ (see Appendix A) since the data of Assumptions $(\boldsymbol{A_\Theta})$ are the same for every $\theta\in\Theta$. Now, let $\theta_0\in\Theta$ be fixed, and define

$$\forall\theta\in\Theta,\quad \Delta_{\theta,\alpha_0}(x):=\|P_\theta(x,\cdot)-P_{\theta_0}(x,\cdot)\|'_{V^{\alpha_0}}, \qquad (57)$$

that is: $\Delta_{\theta,\alpha_0}(x)$ is the V^{α_0}-weighted total variation norm of $P_\theta(x,\cdot)-P_{\theta_0}(x,\cdot)$. Next let us introduce the following condition:

$$\forall x\in\mathbb{X},\quad \lim_{\theta\to\theta_0}\Delta_{\theta,\alpha_0}(x)=0. \qquad (\boldsymbol{\Delta_\Theta^{\alpha_0}})$$

The stationary distribution π_{θ_0} of P_{θ_0} is supposed to be unknown and not directly computable, and P_θ for $\theta\neq\theta_0$ must be thought of as a perturbed Markov kernel of P_{θ_0}. Then, if the stationary distribution π_θ of P_θ is computable for $\theta\neq\theta_0$, Theorem 6.1 below provides an explicit control for the V^{α_0}-weighted total variation norm $\|\pi_\theta-\pi_{\theta_0}\|'_{V^{\alpha_0}}$, provided that the function Δ_{θ,α_0} in (57) is computable, so that the real number $\pi_\theta(\Delta_{\theta,\alpha_0})$ in (58a)-(58b) below is available. Provided that V is replaced with V^{α_0}, Inequalities (58a)-(58b) below extend the statement [LL18, Th. 2] to the above general perturbation context.

Theorem 6.1 *Suppose that the family $\{P_\theta\}_{\theta\in\Theta}$ satisfies Assumptions $(\boldsymbol{A_\Theta})$. Let $\alpha_0\in(0,1]$ be given in $(\boldsymbol{D_\Theta^{\alpha_0}})$, let $\theta_0\in\Theta$ and assume that Condition $(\boldsymbol{\Delta_\Theta^{\alpha_0}})$ holds. Then*

$$\lim_{\theta\to\theta_0}\|\pi_\theta-\pi_{\theta_0}\|'_{V^{\alpha_0}}=0 \quad \textit{and} \quad \lim_{\theta\to\theta_0}\pi_\theta\big(\Delta_{\theta,\alpha_0}\big)=0.$$

Moreover we have for every $\theta \in \Theta$

$$\big\|\pi_\theta - \pi_{\theta_0}\big\|'_{TV} \ \leq\ \big\|\pi_\theta - \pi_{\theta_0}\big\|'_{V^{\alpha_0}} \ \leq\ \frac{1+\pi_{\theta_0}(V^{\alpha_0})\|1_{\mathbb{X}}\|_{V^{\alpha_0}}}{1-\delta^{\alpha_0}} \times \pi_\theta\big(\Delta_{\theta,\alpha_0}\big) \tag{58a}$$

$$\leq\ \frac{1-\delta^{\alpha_0}+\nu(V^{\alpha_0})\|1_{\mathbb{X}}\|_{V^{\alpha_0}}}{(1-\delta^{\alpha_0})^2} \times \pi_\theta\big(\Delta_{\theta,\alpha_0}\big). \tag{58b}$$

Remark 6.1 (Comparison with [LL18]) *Let $P := (P(i,j))_{(i,j)\in\mathbb{N}^2}$ be a stochastic infinite matrix and for every $k \geq 1$ let P_k be the linear augmentation (e.g. in the last column) of the $(k+1)\times(k+1)$ northwest corner truncation of P. Hence P_k is a stochastic matrix of order $k+1$. Assume that P satisfies ($\boldsymbol{S}$) with an atom $S \subset \mathbb{N}$, and that there exist $b > 0$ and a Lyapunov function $V = (V(n))_{n\in\mathbb{N}}$ such that $PV \leq \delta V + b1_S$ and $\|1_{\mathbb{X}}\|_V = 1$. Let π (resp. π_k) be the invariant probability measure of P (resp. of P_k). For the sake of simplicity we also denote by P_k and π_k the natural extensions to $\mathbb{N}$ of P_k and π_k respectively. For every $k \in \mathbb{N}$ define*

$$\forall i = 0,\ldots,k, \quad \Delta_k(i,V) = \sum_{j>k} P(i,j)\big(V(k)+V(j)\big) \quad \textit{and} \quad \delta_k := \sum_{i=0}^{k} \pi_k(i)\,\Delta_k(i,V). \tag{59}$$

It is proved in [LL18, Th. 2] that

$$\big\|\pi - \pi_k\big\|'_V \ \leq\ \frac{1+\pi(V)}{1-\delta} \times \delta_k \tag{60a}$$

$$\leq\ \frac{1-\delta+b}{(1-\delta)^2} \times \delta_k. \tag{60b}$$

Let us show that Properties (60a)-(60b) and (58a)-(58b) coincide in this truncation and atomic context. Here $\Theta = \mathbb{N} \cup \{\infty\}$ with $P_\infty = P$ and $P_\theta = P_k$ for $\theta = k \in \mathbb{N}$. Moreover recall that in the atomic case we have $\alpha_0 = 1$ in ($\boldsymbol{D}_{\boldsymbol{\Theta}}^{\boldsymbol{\alpha_0}}$). Hence $\Delta_k(i,V)$ and δ_k in (59) are nothing else but the error term in (57) and the real number $\pi_\theta(\Delta_{\theta,\alpha_0})$ of Theorem 6.1. Moreover the constants in (60a)-(60b) are exactly those in (58a)-(58b) since we can choose $b = \nu(V)$ in the atomic case. This proves the claimed fact. Finally mention that the proof of the property $\lim_k \delta_k = 0$ in [LL18] is not complete because of the incorrect statement [LL18, lem. 1].

Again, we may suppose that $\alpha_0 = 1$ for the following proofs. If $\alpha_0 < 1$, replace V and δ with V^{α_0} and δ^{α_0} respectively. The next lemmas extend results in [Twe98, Sect. 3] and [LL18, Eq. (3)] proved for truncated discrete Markov kernels.

Lemma 6.1 *Under the assumptions of Theorem 6.1 we have: $\lim_{\theta\to\theta_0} \|\pi_\theta - \pi_{\theta_0}\|'_V = 0$.*

Proof. It follows from ($\boldsymbol{K_\Theta}$)-($\boldsymbol{D_{\Theta,S^c}}$) that

$$\forall \theta \in \Theta, \quad P_\theta V \leq cV \quad \text{with} \quad c := \delta + K. \tag{61}$$

Moreover we have for every $f \in \mathcal{B}_V$ such that $\|f\|_V \leq 1$

$$\forall n \geq 1,\ \forall x \in \mathbb{X}, \quad \big|(P_{\theta_0}^n f)(x) - (P_\theta^n f)(x)\big| \leq \sum_{j=0}^{n-1} c^{n-1-j}\big(P_{\theta_0}^j \Delta_{\theta,1}\big)(x). \tag{62}$$

Indeed, proceed by induction. Inequality (62) holds for $n = 1$ since we have from the definition of the V-weighted total variation norm

$$\big|(P_{\theta_0} f)(x) - (P_\theta f)(x)\big| \le \Delta_{\theta,1}(x).$$

Assume that (62) holds for some $n \ge 1$. Let $g \in \mathcal{B}_V$ be such that $\|g\|_V \le 1$. Then

$$\begin{aligned}\big|(P_{\theta_0}^{n+1} g)(x) - (P_\theta^{n+1} g)(x)\big| &\le \big|\big(P_{\theta_0}^n (P_{\theta_0} - P_\theta) g\big)(x)\big| + \big|\big((P_{\theta_0}^n - P_\theta^n) P_\theta g\big)(x)\big| \\ &\le \int_{\mathbb{X}} \big|(P_{\theta_0} g)(y) - (P_\theta g)(y)\big|\, P_{\theta_0}^n(x,dy) + \sum_{j=0}^{n-1} c^{n-j} \big(P_{\theta_0}^j \Delta_{\theta_0,1}\big)(x) \\ &\le \int_{\mathbb{X}} \Delta_{\theta_0,1}(y)\, P_{\theta_0}^n(x,dy) + \sum_{j=0}^{n-1} c^{n-j} \big(P_{\theta_0}^j \Delta_{\theta_0,1}\big)(x)\end{aligned}$$

using the triangular inequality, the fact that $\|P_\theta g\|_V \le c$ by (61) and the induction assumption, and finally the definition of $\Delta_{\theta,1}$. This gives (62) at order $n+1$. Now let $x_0 \in \mathbb{X}$ be fixed and define

$$\varepsilon_{n,\Theta} := \sup_{\theta\in\Theta} \big\|P_\theta^n(x_0,\cdot) - \pi_\theta\big\|_V'. \tag{63}$$

Let $f \in \mathcal{B}_V$ be such that $\|f\|_V \le 1$. Then we have from the definition of $\varepsilon_{n,\Theta}$ and from (62)

$$\begin{aligned}\big|\pi_{\theta_0}(f) - \pi_\theta(f)\big| &\le \big|\pi_{\theta_0}(f) - (P_{\theta_0}^n f)(x_0)\big| + \big|(P_{\theta_0}^n f)(x_0) - (P_\theta^n f)(x_0)\big| + \big|(P_\theta^n f)(x_0) - \pi_\theta(f)\big| \\ &\le 2\,\varepsilon_{n,\Theta} + \sum_{j=0}^{n-1} c^{n-1-j} \big(P_{\theta_0}^j \Delta_{\theta,1}\big)(x_0).\end{aligned}$$

Next fix $n \ge 1$. We have

$$\forall j = 0,\ldots,n-1, \quad \lim_{\theta\to\theta_0} \big(P_{\theta_0}^j \Delta_{\theta,1}\big)(x_0) = 0$$

from Lebesgue's theorem applied to the probability measure $P_{\theta_0}^j(x_0,\cdot)$ using Assumption $(\boldsymbol{\Delta}_{\boldsymbol{\Theta}}^{\boldsymbol{\alpha_0}})$ (with $\alpha_0 = 1$ here) and

$$\forall \theta \in \Theta, \quad \Delta_{\theta,1} \le 2cV \tag{64}$$

with c defined in (61). Hence

$$\forall n \ge 1, \quad \limsup_{\theta\to\theta_0} \big\|\pi_{\theta_0} - \pi_\theta\big\|_V' \le 2\,\varepsilon_{n,\Theta}.$$

Moreover we have from [Bax05, GP14]

$$\lim_n \varepsilon_{n,\Theta} = 0 \tag{65}$$

since Assumptions $(\boldsymbol{A_\Theta})$ are stated in a uniform way in $\theta \in \Theta$. Property (65) can be also derived from the results of Sections 2-3 when the parameter set Θ is assumed to be locally compact, see Appendix C. It follows that $\limsup_{\theta\to\theta_0} \big\|\pi_{\theta_0} - \pi_\theta\big\|_V' = 0$, hence the assertion of Lemma 6.1 holds. □

Lemma 6.2 *Suppose that the assumptions of Theorem 6.1 hold. For any $f \in \mathcal{B}_V$, let us introduce $f_0 := f - \pi_{\theta_0}(f)1_{\mathbb{X}}$. Set $\widetilde{f_0} := (I - R_{\theta_0})^{-1} f_0$ with $R_{\theta_0} := P_{\theta_0} - \nu(\cdot)1_S$. Then*

$$\pi_\theta(f) - \pi_{\theta_0}(f) = \pi_\theta\big(\Delta_\theta \widetilde{f_0}\big) \quad \textit{with} \quad \Delta_\theta := P_\theta - P_{\theta_0}.$$

Proof. Since $\pi_{\theta_0}(f_0) = 0$, we know from Theorem 4.1 applied to P_{θ_0} that $\widetilde{f}_0$ is a solution to Poisson's equation, that is $\widetilde{f}_0$ satisfies $\widetilde{f}_0 - P_{\theta_0}\widetilde{f}_0 = f_0$, or $P_{\theta_0}\widetilde{f}_0 = \widetilde{f}_0 - f_0$. Then, it follows that

$$\begin{aligned}\pi_\theta\big(\Delta_\theta \widetilde{f}_0\big) &= \pi_\theta\big(P_\theta \widetilde{f}_0 - P_{\theta_0}\widetilde{f}_0\big) = \pi_\theta(\widetilde{f}_0) + \pi_\theta\big(-\widetilde{f}_0 + f_0\big) \\ &= \pi_\theta(f_0) = \pi_\theta(f) - \pi_{\theta_0}(f) \qquad \text{(from the definition of } f_0).\end{aligned}$$

□

Proof of Theorem 6.1. That $\lim_{\theta\to\theta_0} \|\pi_\theta - \pi_{\theta_0}\|'_V = 0$ is proved in Lemma 6.1. Next we get from (64)

$$\pi_\theta(\Delta_{\theta,1}) \leq \big|\pi_\theta(\Delta_{\theta,1}) - \pi_{\theta_0}(\Delta_{\theta,1})\big| + \pi_{\theta_0}(\Delta_{\theta,1}) \leq 2c\|\pi_\theta - \pi_{\theta_0}\|'_V + \pi_{\theta_0}(\Delta_{\theta,1}).$$

Moreover we obtain that $\lim_{\theta\to\theta_0} \pi_{\theta_0}(\Delta_{\theta,1}) = 0$ from Lebesgue's dominated convergence theorem with respect to the probability measure π_{θ_0} using Assumption ($\mathbf{\Delta_\Theta^{\alpha_0}}$) (with $\alpha_0 = 1$ here), (64) and $\pi_{\theta_0}(V) < \infty$. We have proved that $\lim_{\theta\to\theta_0} \pi_\theta\big(\Delta_{\theta,1}\big) = 0$. Now let $f \in \mathcal{B}_V$ be such that $\|f\|_V \leq 1$. Define $f_0 := f - \pi_{\theta_0}(f)1_{\mathbb{X}}$ and $\widetilde{f}_0 := (I - R_{\theta_0})^{-1} f_0$ as in Lemma 6.2. Apply Theorem 4.1 to the Markov kernel P_{θ_0} to obtain that

$$\begin{aligned}\big|\pi_\theta(f) - \pi_{\theta_0}(f)\big| &\leq \int_{\mathbb{X}} \big|(P_\theta \widetilde{f}_0)(x) - (P_{\theta_0}\widetilde{f}_0)(x)\big|\, \pi_\theta(dx) \qquad \text{(from Lemma 6.2)} \\ &\leq \|\widetilde{f}_0\|_V \int_{\mathbb{X}} \Delta_{\theta,1}(x)\, \pi_\theta(dx) \qquad \text{(from the definition of } \Delta_{\theta,1}) \\ &\leq \frac{1}{1-\delta}\|f_0\|_V \times \pi_\theta\big(\Delta_{\theta,1}\big) \qquad \text{(from (35))} \\ &\leq \frac{1 + \pi_{\theta_0}(V)||1_X||_V}{1-\delta} \times \pi_\theta\big(\Delta_{\theta,1}\big) \qquad \text{(from the definition of } f_0) \\ &\leq \frac{1 - \delta + \nu(V)\|1_{\mathbb{X}}\|_V}{(1-\delta)^2} \times \pi_\theta\big(\Delta_{\theta,1}\big) \qquad \text{(from } \pi_{\theta_0}(V) \leq \nu(V)/(1-\delta)).\end{aligned}$$

The proof of Theorem 6.1 is complete. □

Remark 6.2 *As introduced in [Twe98] for discrete set $\mathbb{X}$, Condition ($\mathbf{\Delta_\Theta^{\alpha_0}}$) is the expected continuity assumption in order to study the V^{α_0}-weighted total variation distance between π_θ and π_{θ_0}. When this condition is satisfied, not only the bound (58a) in Theorem 6.1 has the expected form, but also the constant in (58a) is simple (and moreover explicit in (58b)). Let us discuss Condition ($\mathbf{\Delta_\Theta^{\alpha_0}}$) and alternative assumptions used in prior works.*

- *The standard operator norm continuity assumption introduced in [Kar86] writes as $\lim_{\theta\to\theta_0} \|P_\theta - P_{\theta_0}\|_{V^{\alpha_0}} = 0$, namely*

$$\lim_{\theta\to\theta_0} \sup_{x\in\mathbb{X}} \frac{\Delta_{\theta,\alpha_0}(x)}{V(x)^{\alpha_0}} = 0.$$

  This condition is clearly much more restrictive than Condition ($\mathbf{\Delta_\Theta^{\alpha_0}}$).

- *The weak operator norm continuity assumptions used in [SS00, FHL13, HL14a, RS18, MARS20] requires that*

$$\lim_{\theta\to\theta_0} \sup_{x\in\mathbb{X}} \frac{\|P_\theta(x,\cdot) - P_{\theta_0}(x,\cdot)\|'_{TV}}{V(x)^{\alpha_0}} = 0. \tag{66}$$

To understand the difference between Conditions ($\boldsymbol{\Delta_\Theta^{\alpha_0}}$) and (66), consider the following simple example derived from perturbed linear autoregressive models:

$$\forall\theta \in (0,1),\ \forall x \in \mathbb{X} = \mathbb{R},\ \forall A \in \mathcal{X}, \quad P_\theta(x, A) := \int_{\mathbb{R}} 1_A(y)\, \nu(y - \theta x)\, dy,$$

where $\mathcal{X}$ is here the Borel $\sigma-$algebra on $\mathbb{R}$ and where ν is some probability density function (p.d.f.) with respect to Lebesgue's measure on $\mathbb{R}$. Let $\widehat{\theta} \in (0,1)$. It is well-known that, under moment conditions on the p.d.f. ν, the family $\{P_\theta\}_{\theta\in(0,\widehat{\theta})}$ satisfies Assumptions ($\boldsymbol{A_\Theta}$) (e.g. see [RS18, HL23]). Here we only focus on Conditions ($\boldsymbol{\Delta_\Theta^{\alpha_0}}$) and (66). Let $\theta_0 \in (0, \widehat{\theta})$ be fixed. Condition ($\boldsymbol{\Delta_\Theta^{\alpha_0}}$) writes as follows

$$\forall x \in \mathbb{R}, \quad \lim_{\theta \to \theta_0} \int_{\mathbb{X}} V(y)^{\alpha_0} \big|\nu(y - \theta x) - \nu(y - \theta_0 x)\big| dy = 0, \tag{67}$$

while Condition (66) is:

$$\lim_{\theta \to \theta_0} \sup_{x\in\mathbb{R}} \frac{\int_{\mathbb{X}} \big|\nu(z - \theta x) - \nu(z - \theta_0 x)\big| dz}{V(x)^{\alpha_0}} = 0. \tag{68}$$

Actually Conditions (67) and (68) are quite different. In (67) the convergence is simple in $x \in \mathbb{R}$, but the presence of $V(y)$ in the integral may be problematic. In (68) the absence of the function V in the integral is of course an advantage, but the convergence has to be uniform on $\mathbb{R}$ (actually it has to be uniform on every compact of $\mathbb{R}$ thanks to the division by $V(x)$). In this example Condition (68) is satisfied thanks to the continuity of $t \mapsto \nu(\cdot - t)$ from $\mathbb{R}$ to the Lebesgue space $\mathbb{L}^1(\mathbb{R})$ (see [HL23]). Consequently, if the rate ρ and the associated multiplicative constant C_ρ in (2) are known, then the bounds obtained in [RS18, HL23] for $\|\pi_\theta - \pi_{\theta_0}\|'_{TV}$ hold and are explicit. Otherwise, the explicit bounds (58a) and (58b) which depend neither on the rate ρ nor on the constant C_ρ may be used provided that the p.d.f. ν satisfies Condition (67).

A Complement on the real number α_0

Let $\alpha \in (0,1]$. If $x \in \mathbb{X} \setminus S$, then we have $(PV^\alpha)(x) \le \delta^\alpha V(x)^\alpha$ from ($\boldsymbol{D_{S^c}}$) and Jensen's inequality. Recall that $K := \sup_{x\in S}(PV)(x)$ (see ($\boldsymbol{K}$)). We have $1 \le \sup_{x\in S}(PV^\alpha)(x) \le K^\alpha$ from $1_{\mathbb{X}} \le V^\alpha$ and $PV^\alpha \le (PV)^\alpha$ using again Jensen's inequality. Finally we get from $1_{\mathbb{X}} \le V$

$$\forall x \in S, \quad (PV^\alpha)(x) - \delta^\alpha V(x)^\alpha - \nu(V^\alpha) \le K^\alpha - \delta^\alpha - \nu(1_{\mathbb{X}}).$$

Passing to the limit when $\alpha \to 0$ provides the existence of $\alpha_0 \in (0,1]$ such that ($\boldsymbol{D^{\alpha_0}}$) holds since $\nu(1_{\mathbb{X}}) > 0$. Note that, if Condition ($\boldsymbol{S}$) is fulfilled with an atom S and with $\nu(\cdot) := P(a_0, \cdot)$ for some (any) $a_0 \in S$, then ($\boldsymbol{D^{\alpha_0}}$) holds with $\alpha_0 = 1$. Indeed we then have

$$\forall x \in S, \quad PV(x) - \delta\, V(x) - \nu(V) = -\delta\, V(x) \le 0.$$

Since under Assumption ($\boldsymbol{D_{S^c}}$) we have $PV^\alpha \le \delta^\alpha V^\alpha$ on $\mathbb{X} \setminus S$ for any $\alpha \in (0,1]$, the computation of α_0 in ($\boldsymbol{D^{\alpha_0}}$) only concerns the elements $x \in S$. Under Assumption ($\boldsymbol{S}$) define $\sigma := 1 - \nu(1_{\mathbb{X}}) \in [0,1)$. The value $\sigma = 0$ corresponds to the atomic case for which $\alpha_0 = 1$. If $\alpha_0 = 1$ does not work, the following statement is useful to find an explicit value for $\alpha_0 \in (0,1)$ in ($\boldsymbol{D^{\alpha_0}}$).

Proposition A.1 *Assume that P satisfies Condition ($\boldsymbol{S}$) with S that is not an atom, so that $\sigma \in (0,1)$. Then we have for any Lyapunov function V:*

$$\forall \alpha \in (0,1],\ \forall x \in S, \quad (PV^\alpha)(x) - \nu(V^\alpha) \leq \frac{\sigma}{\sigma^\alpha} \big[(PV)(x) - \nu(V)\big]^\alpha. \tag{69}$$

Proof. Let $x \in S$. Note that $\sigma_x(\cdot) := P(x,\cdot) - \nu(\cdot)$ is a non-negative measure on $(\mathbb{X}, \mathcal{X})$ from Assumption ($\boldsymbol{S}$), and that $\sigma_x(1_{\mathbb{X}}) = 1 - \nu(1_{\mathbb{X}}) = \sigma$ does not depend on x. Define the following probability measure on $(\mathbb{X}, \mathcal{X})$: $\widehat{\sigma}_x(\cdot) = \sigma_x(\cdot)/\sigma$. Let $\alpha \in (0,1]$. It follows from Jensen's inequality that

$$\frac{(PV^\alpha)(x) - \nu(V^\alpha)}{\sigma} = \widetilde{\sigma}_x(V^\alpha) \leq \big[\widetilde{\sigma}_x(V)\big]^\alpha = \frac{\big[(PV)(x) - \nu(V)\big]^\alpha}{\sigma^\alpha},$$

from which we deduce (69).

□

The real number α_0 can be computed as follows thanks to Proposition A.1. Let $M := K - \nu(V)$ with K given in ($\boldsymbol{K}$). Then

$$\forall \alpha \in (0,1],\ \forall x \in S, \quad (PV^\alpha)(x) - \nu(V^\alpha) - \delta^\alpha V(x)^\alpha \leq \frac{\sigma}{\sigma^\alpha} M^\alpha - \delta^\alpha \tag{70}$$

since $V \geq 1_{\mathbb{X}}$. Then $\alpha_0 \in (0,1]$ can be chosen such that $\frac{\sigma}{\sigma^{\alpha_0}} M^{\alpha_0} - \delta^{\alpha_0} \leq 0$ since

$$\lim_{\alpha \to 0} \Big[\frac{\sigma}{\sigma^\alpha} M^\alpha - \delta^\alpha\Big] = \sigma - 1 < 0.$$

B Order of the eigenvalues of P

Under Assumptions ($\boldsymbol{A}$) we deduce from Property (13) that $z \mapsto \mu_z$ given in (19) is derivable on the domain $D_0 = \{z \in \mathbb{C} : |z| > \delta^{\alpha_0}\}$ with $\alpha_0 \in (0,1]$ given in ($\boldsymbol{D^{\alpha_0}}$), and that its derivative is given by

$$\forall z \in D_0, \quad \mu'_z := -\sum_{k=1}^{+\infty} k\, z^{-(k+1)}\, \beta_k \tag{71}$$

which is absolutely convergent in $\mathcal{B}'_{V^{\alpha_0}}$.

Proposition B.1 *Assume that P satisfies ($\boldsymbol{A}$), and let $\alpha_0 \in (0,1]$ be given in ($\boldsymbol{D^{\alpha_0}}$). Let $\lambda \in D_0$ be an eigenvalue of P on $\mathcal{B}_{V^{\alpha_0}}$ (equivalently $\mu_\lambda(1_S) = 1$ from Theorem 2.2). Then the two following assertions are equivalent:*

(i) λ is of order one, that is $\mathrm{Ker}(P - \lambda I)^2 = \mathrm{Ker}(P - \lambda I)$ or equivalently $\mathrm{Ker}(P^* - \lambda I)^2 = \mathrm{Ker}(P^* - \lambda I)$;

(ii) $\mu'_\lambda(1_S) \neq 0$.

Moreover, if we have $\mu'_\lambda(1_S) = 0$, then the system $\{\mu_\lambda, \mu'_\lambda\}$ form a basis of the subspace $\mathrm{Ker}(P^* - \lambda I)^2 := \{\psi \in \mathcal{B}'_{V^{\alpha_0}} : \psi \circ (P - \lambda I)^2 = 0\}$.

Proof. Again we suppose that $\alpha_0 = 1$ in $(\boldsymbol{D^{\alpha_0}})$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of P on $\mathcal{B}_V$ such that $|\lambda| > \delta$. Assume that $\mu'_\lambda(1_S) = 0$. From (21) we obtain the following equality in $\mathcal{B}'_V$

$$\mu'_\lambda \circ P = \mu_\lambda + \lambda\mu'_\lambda + \mu'_\lambda \circ T$$

$$\text{with} \quad \forall f \in \mathcal{B}_V, \quad (\mu'_\lambda \circ T)(f) = -\sum_{k=1}^{+\infty} k\,\lambda^{-(k+1)}\,\beta_k(Tf) \;=\; -\nu(f)\sum_{k=1}^{+\infty} k\,\lambda^{-(k+1)}\,\beta_k(1_S) \;=\; -\nu(f)\,\mu'_\lambda(1_S) = \;0.$$

Hence $\mu'_\lambda \circ P = \lambda\mu'_\lambda + \mu_\lambda$. Recall that $\mu_\lambda \in \mathrm{Ker}(P^* - \lambda I)$ (Theorem 2.2) with $\mu_\lambda \neq 0$ since $\mu_\lambda(1_S) = 1$. Thus μ'_λ is nonzero and satisfies $\mu'_\lambda \circ (P - \lambda I) = \mu_\lambda \in \mathrm{Ker}(P^* - \lambda I)$, so that $\mu'_\lambda \in \mathrm{Ker}(P^* - \lambda I)^2 \setminus \mathrm{Ker}(P^* - \lambda I)$. We have proved the implication $(i) \Rightarrow (ii)$. Conversely, assume that there exists $\psi \in \mathcal{B}'_V$, $\psi \neq 0$, such that $\psi \circ (P - \lambda I)^2 = 0$ and $\psi \circ (P - \lambda I) \neq 0$. Since $\phi := \psi \circ (P - \lambda I) \in \mathrm{Ker}(P^* - \lambda I)$, we deduce from the last assertion of Theorem 2.2 that $\phi = c\,\mu_\lambda$ for some $c \in \mathbb{C}$. Obviously we may suppose that $c = 1$ (replacing ψ with ψ/c). Hence $\psi \circ P = \lambda\psi + \mu_\lambda$, and an easy induction gives

$$\forall n \geq 0, \quad \psi \circ P^n = \lambda^n\psi + n\lambda^{n-1}\mu_\lambda.$$

Next, composing on the left by ψ in (22), we obtain the following equalities in $\mathcal{B}'_V$

$$\lambda^n\psi + n\lambda^{n-1}\mu_\lambda - \psi(1_S)\sum_{k=1}^{n}\lambda^{n-k}\,\beta_k - \mu_\lambda(1_S)\sum_{k=1}^{n}(n-k)\lambda^{n-k-1}\,\beta_k = \mathrm{O}(\delta^n).$$

Using $\mu_\lambda(1_S) = 1$ we deduce that

$$\psi - \psi(1_S)\sum_{k=1}^{n}\lambda^{-k}\,\beta_k + \sum_{k=1}^{n}k\lambda^{-(k+1)}\,\beta_k + n\lambda^{-1}\left(\mu_\lambda - \sum_{k=1}^{n}\lambda^{-k}\,\beta_k\right) = \mathrm{o}(1).$$

When $n \to +\infty$ we obtain that

$$\psi = \psi(1_S)\mu_\lambda + \mu'_\lambda$$

since $\mu_\lambda - \sum_{k=1}^{n}\lambda^{-k}\,\beta_k = \mathrm{O}((\delta/|\lambda|)^n)$ with $|\lambda| > \delta$. Applying the above equality to the function 1_S gives $\mu'_\lambda(1_S) = 0$ since $\mu_\lambda(1_S) = 1$. We have proved the implication $(ii) \Rightarrow (i)$, as well as the last assertion of Proposition B.1.

□

Under Assumptions $(\boldsymbol{A})$ define for every $z \in \mathbb{C}$ such that $|z| > \delta^{\alpha_0}$

$$\chi_S(z) = \mu_z(1_S) - 1 = \sum_{k=1}^{+\infty} z^{-k}\,\beta_k(1_S) \; - 1.$$

We know from Theorem 2.2 that $\lambda \in \mathbb{C}$ such that $\delta^{\alpha_0} < |\lambda| \leq 1$ is an eigenvalue of P on $\mathcal{B}_{V^{\alpha_0}}$ if, and only if, $\chi_S(\lambda) = 0$. Moreover, from Proposition B.1, such an eigenvalue λ is of order one if, and only if, $\chi'_S(\lambda) \neq 0$. An easy extension of Proposition B.1 shows that, for every $p \geq 2$, λ is of order p if, and only if, $\forall i = 0, \ldots, p-1$, $\chi_S^{(i)}(\lambda) = 0$ and $\chi_S^{(p)}(\lambda) \neq 0$.

C Proof of (65) when Θ is locally compact

The following statement shows that, under Assumptions ($\boldsymbol{A_\Theta}$), the family $\{P_\theta\}_{\theta\in\Theta}$ satisfies (2) in a uniform way in θ when Θ is locally compact, so that Property (65) holds. Under Assumptions ($\boldsymbol{A_\Theta}$), we denote by $\varrho_{\alpha_0}^{(\theta)}$ the second eigenvalue of P_θ on $\mathcal{B}_{V^{\alpha_0}}$.

Proposition C.1 *Assume that $\{P_\theta\}_{\theta\in\Theta}$ satisfies Assumptions ($\boldsymbol{A_\Theta}$). Let $\alpha_0 \in (0,1]$ be given in ($\boldsymbol{D_\Theta^{\alpha_0}}$), let $\theta_0 \in \Theta$ and suppose that Assumption ($\boldsymbol{\Delta_\Theta^{\alpha_0}}$) holds. Moreover suppose that Θ is locally compact. Then there exists a compact neighbourhood $\mathcal{V}_{\theta_0}$ of θ_0 in Θ such that*

$$\forall\theta \in \mathcal{V}_{\theta_0}, \quad \varrho_{\alpha_0}^{(\theta)} \le \max(\delta^{\alpha_0}, \varrho_{\alpha_0}^{(\theta_0)}). \tag{72}$$

Moreover, for every $\rho \in \big(\max(\delta^{\alpha_0}, \varrho_{\alpha_0}^{(\theta_0)}), 1\big)$, we have

$$\forall\theta \in \mathcal{V}_{\theta_0}, \quad \|P_\theta^n f - \pi_\theta(f)1_{\mathbb{X}}\|_{V^{\alpha_0}} \le \frac{\rho}{2\pi(\rho-\delta^{\alpha_0})}\left(1 + \frac{\nu(V^{\alpha_0})\|1_S\|_{V^{\alpha_0}}}{m_{\rho,\Theta}\big(\rho-\delta^{\alpha_0}\big)}\right)\rho^n \tag{73}$$

$$\textit{with} \qquad m_{\rho,\Theta} := \min\big\{\big|1-\mu_z^{(\theta)}(1_S)\big| \,:\, z\in\mathbb{C} : |z| = \rho,\ \theta \in \mathcal{V}_{\theta_0}\big\} > 0. \tag{74}$$

Proof. Again we suppose that $\alpha_0 = 1$. Note that under Assumptions ($\boldsymbol{A_\Theta}$) we have

$$\forall k \ge 1,\ \forall\theta\in\Theta, \quad \|R_\theta^k\|_V \le \delta^k \quad \text{with} \quad R_\theta := P_\theta - \nu(\cdot)1_S \tag{75}$$

from (13) and from the uniformity of ($\boldsymbol{A_\Theta}$) in $\theta\in\Theta$. Define

$$\forall\theta\in\Theta,\ \forall k\ge 1, \quad \beta_k^{(\theta)} = \nu\circ R_\theta^{k-1}. \tag{76}$$

Hence

$$\forall k\ge 1,\ \forall\theta\in\Theta, \quad \beta_k^{(\theta)}(1_S) \le \nu(V)\|1_S\|_V\,\delta^{k-1}. \tag{77}$$

For every $z\in\mathbb{C}$ such that $|z|>\delta$ and for every $\theta\in\Theta$ we define

$$\mu_z^{(\theta)}(1_S) := \sum_{k=1}^{+\infty} z^{-k}\beta_k^{(\theta)}(1_S). \tag{78}$$

Let $f\in\mathcal{B}_V$ such that $\|f\|_V \le 1$. Observing that $\Delta_{\theta,1}(x) := \|R_\theta(x,\cdot) - R_{\theta_0}(x,\cdot)\|_V$ and that $R_\theta V \le V$ from ($\boldsymbol{D_\Theta^{\alpha_0}}$) (with $\alpha_0 = 1$ here), we can prove as in (62) that

$$\forall k\ge 1,\ \forall x\in\mathbb{X}, \quad \big|(R_{\theta_0}^k f)(x) - (R_\theta^k f)(x)\big| \le \sum_{j=0}^{k-1}\big(R_{\theta_0}^j\Delta_{\theta,1}\big)(x) \le \sum_{j=0}^{k-1}\big(P_{\theta_0}^j\Delta_{\theta,1}\big)(x).$$

Note that $\beta_1^{(\theta)} = \nu$. Then, using the definition (76) of $\beta_k^{(\theta)}$, we have for every $k\ge 2$

$$\big|\beta_k^{(\theta)}(f) - \beta_k^{(\theta_0)}(f)\big| \le \int_{\mathbb{X}}\big|(R_\theta^{k-1}f)(x) - (R_{\theta_0}^{k-1}f)(x)\big|\,d\nu(x) \le \sum_{j=0}^{k-2}\nu\big(P_{\theta_0}^j\Delta_{\theta,1}\big).$$

Moreover we have

$$\forall j = 0,\ldots,k-2, \quad \lim_{\theta\to\theta_0}\nu\big(P_{\theta_0}^j\Delta_{\theta,1}\big) = 0$$

from Lebesgue's dominated convergence theorem with respect to the positive measure $\nu P_{\theta_0}^j$ using Assumption $(\mathbf{\Delta}_{\mathbf{\Theta}}^{\boldsymbol{\alpha_0}})$, (64) and $\nu(P_{\theta_0}^j V) < \infty$ (use (61)). This proves that

$$\forall k \geq 1, \quad \lim_{\theta \to \theta_0} \|\beta_k^{(\theta)} - \beta_k^{(\theta_0)}\|_V' = 0. \tag{79}$$

To simplify, for every $\theta \in \Theta$ and for every $z \in \mathbb{C}$ such that $|z| > \delta$, we set $\phi(\theta, z) := \mu_z^{(\theta)}(1_S)$ (see (78)). We easily deduce from (77) and (79) that ϕ is continuous on $\Theta \times \{z \in \mathbb{C} : |z| > \delta\}$ (note that θ_0 has been arbitrarily chosen in Θ). Let $\rho \in \big(\max(\delta, \varrho^{(\theta_0)}), 1\big)$, where $\varrho^{(\theta_0)}$ denotes the second eigenvalue of P_{θ_0} on $\mathcal{B}_V$. Let $\gamma \in (0, 1-\rho)$, and finally let $\mathcal{D}_{\rho,\gamma}$ be the following compact subset of $\mathbb{C}$:

$$\mathcal{D}_{\rho,\gamma} := \big\{z \in \mathbb{C} : |z| \geq \rho,\ |z-1| \geq \gamma\big\}.$$

We know from the definition of $\varrho^{(\theta_0)}$ and from Theorem 2.2 that

$$\forall z \in \mathcal{D}_{\rho,\gamma}, \quad \phi(\theta_0, z) \neq 1. \tag{80}$$

Let us prove that there exists a neighbourhood $\mathcal{V}_{\theta_0} \equiv \mathcal{V}_{\theta_0}(\rho, \gamma)$ of θ_0 in Θ such that

$$\forall z \in \mathcal{D}_{\rho,\gamma},\ \forall \theta \in \mathcal{V}_{\theta_0}, \quad \phi(\theta, z) \neq 1. \tag{81}$$

Assume that such a neighbourhood does not exist. Then there exists a sequence $(\vartheta_n)_{n\geq 1} \in \Theta^{\mathbb{N}}$ and a sequence $(z_n)_{n\geq 1} \in \mathcal{D}_{\rho,\gamma}^{\mathbb{N}}$ such that $\lim \vartheta_n = \theta_0$ and $\forall n \geq 1,\ \phi(\vartheta_n, z_n) = 1$. Up to select a subsequence we can suppose that $\lim_n z_n = u$ for some u in the compact set $\mathcal{D}_{\rho,\gamma}$. Then we deduce from the continuity of ϕ that

$$\phi(\theta_0, u) = \lim_n \phi(\vartheta_n, z_n) = 1.$$

This contradicts Property (80). Hence (81) is proved. Next let $\widehat{r}_1$ be defined in (40) (with $\alpha_0 = 1$ here), let $\gamma \in (0, \min(1-\rho, \widehat{r}_1/2))$ and let $\mathcal{V}_{\theta_0} \equiv \mathcal{V}_{\theta_0}(\rho, \gamma)$ such that (81) holds. Let us prove that

$$\forall z \in \mathbb{C},\ |z| \geq \rho,\ z \neq 1,\ \forall \theta \in \mathcal{V}_{\theta_0}, \quad \phi(\theta, z) \neq 1. \tag{82}$$

First it follows from the uniformity in $\theta \in \Theta$ of Assumptions $(\boldsymbol{A}_{\mathbf{\Theta}})$ and from Proposition 5.1 that, for every $\theta \in \Theta$, $\lambda = 1$ is the single spectral value of P_θ on $\mathcal{B}_V$ in the open disk $D(1, \widehat{r}_1)$. Thus we have

$$\forall \theta \in \Theta,\ \forall z \in D(1, \widehat{r}_1),\ z \neq 1, \quad \phi(\theta, z) \neq 1 \tag{83}$$

from Theorem 2.2. Then (82) follows from (81) and (83) since $\gamma < \widehat{r}_1/2$.

Now we can complete the proof of Proposition C.1. Let $\rho \in \big(\max(\delta, \varrho^{(\theta_0)}, 1\big)$. Using the spectral properties of Section 2 and Theorem 2.2 we deduce from (82) that, for every $\theta \in \mathcal{V}_{\theta_0}$, the spectral gap $\varrho^{(\theta)}$ of P_θ on $\mathcal{B}_V$ is less than ρ. In fact this gives (72) since ρ is arbitrarily close to $\max\big(\delta, \varrho^{(\theta_0)}\big)$. Next note that the neighbourhood $\mathcal{V}_{\theta_0}$ of θ_0 in (82) can be assumed to be compact since Θ is locally compact. Then (74) follows from the continuity of ϕ on the compact set $\mathcal{H} := \mathcal{V}_{\theta_0} \times \{z \in \mathbb{C} : |z| = \rho\}$ since we know from (82) that $\forall (\theta, z) \in H,\ \phi(\theta, z) \neq 1$. Finally (73) follows from Theorem 3.1 applied to P_θ, $\theta \in \mathcal{V}_{\theta_0}$. $\square$

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