Geometric ρ-mixing property of the interarrival times of a stationary Markovian Arrival Process

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Abstract

In this note, the sequence of the interarrivals of a stationary Markovian Arrival process is shown to be ρ-mixing with a geometric rate of convergence when the driving process is ρ-mixing. This provides an answer to an issue raised in the recent paper [4] on the geometric convergence of the autocorrelation function of the stationary Markovian Arrival process.

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1 Introduction

We provide a positive answer to a question raised in [4] on the geometric convergence of the autocorrelation function associated with the interarrival times of a stationary m-state Markovian Arrival Process (MAP). Indeed, it is shown in [3, Prop. 3.1] that the increment sequence $\{T_n := S_n - S_{n-1}\}_{n\geq 1}$ associated with a discrete time stationary Markov additive process $\{(X_n, S_n)\}_{n\in\mathbb{N}} \in \mathbb{X}\times\mathbb{R}^d$ is ρ-mixing with a geometric rate provided that the driving stationary Markov chain $\{X_n\}_{n\in\mathbb{N}}$ is ρ-mixing. There, $\mathbb{X}$ may be any measurable set. In the case where the increments $\{T_n\}_{n\geq 1}$ are non-negative random variables, $\{(X_n, S_n)\}_{n\in\mathbb{N}}$ is a Markov Renewal Process (MRP). Therefore, we obtain the expected answer to the question in [4] since such an MRP with $\{T_n\}_{n\geq 1}$ being the interarrival times can be associated with a m-state MAP and the ρ-mixing property of $\{T_n\}_{n\geq 1}$ with geometric rate ensures the geometric convergence of the autocorrelation function of $\{T_n\}_{n\geq 1}$. We refer to [1, Chap. XI] for basic properties of MAPs and Markov additive processes.

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2 Geometric ρ-mixing of the sequence of interarrivals of an MAP

Let us recall the definition of the ρ-mixing property of a (strictly) stationary sequence of random variables $\{T_n\}_{n\geq 1}$ (e.g. see [2]). The ρ-mixing coefficient with time lag $k>0$, denoted usually by $\rho(k)$, is defined by

$$\begin{aligned}\rho(k) \;:=\; & \sup_{n\geq 1}\sup_{m\in\mathbb{N}}\sup\Big\{\big|\mathrm{Corr}\big(f(T_1,\ldots,T_n);h(T_{n+k},\ldots,T_{n+k+m})\big)\big|,\\ & \qquad f,g\ \mathbb{R}\text{-valued functions such that}\\ & \mathbb{E}\big[|f(T_1,\ldots,T_n)|^2\big] \text{ and } \mathbb{E}\big[|h(T_{n+k},\ldots,T_{n+k+m})|^2\big] \text{ are finite}\Big\} \end{aligned} \tag{1}$$

where $\mathrm{Corr}\big(f(T_1,\ldots,T_n);h(T_{n+k},\ldots,T_{n+k+m})\big)$ is the correlation coefficient of the two square-integrable random variables. Note that $\{\rho(k)\}_{n\geq 1}$ is a non-increasing sequence. Then $\{T_n\}_{n\geq 1}$ is said to be ρ-mixing if

$$\lim_{k\to+\infty}\rho(k)=0.$$

When, for any $n\in\mathbb{N}$, the random variable T_n has a moment of order 2, the autocorrelation function of $\{T_n\}_{n\geq 1}$ as studied in [4], that is $\mathrm{Corr}(T_1;T_{k+1})$ as a function of the time lag k, clearly satisfies

$$\forall k\geq 1,\quad |\mathrm{Corr}(T_1;T_{k+1})|\leq \rho(k). \tag{2}$$

Therefore, any rate of convergence of the ρ-mixing coefficients $\{\rho(k)\}_{k\geq 1}$ is a rate of convergence for the autocorrelation function.

We only outline the main steps to obtain from [3, Prop. 3.1] a geometric convergence rate of $\{\rho(k)\}_{n\geq 1}$ for the m-state MRP $\{(X_n,S_n)\}_{n\in\mathbb{N}}$ associated with a m-state MAP. In [4, Section 2], the analysis of the autocorrelation function in the two-states case is based on such an MRP (notation and background in [4] are that of [5]). Recall that a m-state MAP is a bivariate continuous-time Markov process $\{(J_t,N_t)\}_{t\geq 0}$ on $\{1,\ldots,m\}\times\mathbb{N}$ where N_t represents the number of arrivals up to time t, while the states of the driving Markov process $\{J_t\}_{t\geq 0}$ are called phases. Let S_n be the time at the nth arrival ($S_0=0$ a.s.) and let X_n be the state of the driving process just after the nth arrival. Then $\{(X_n,S_n)\}_{n\in\mathbb{N}}$ is known to be an MRP with the following semi-Markov kernel Q on $\{1,\ldots,m\}\times[0,\infty)$

$$\forall (x_1,x_2)\in\{1,\ldots,m\}^2,\quad Q(x_1;\{x_2\}\times dy):=(e^{D_0y}D_1)(x_1,x_2)dy \tag{3}$$

parametrized by a pair of $m\times m$-matrices usually denoted by D_0 and D_1. The matrix D_0+D_1 is the infinitesimal generator of the background Markov process $\{J_t\}_{t\geq 0}$ which is always assumed to be irreducible, and D_0 is stable. The process $\{X_n\}_{n\in\mathbb{N}}$ is a Markov chain with state space $\mathbb{X}:=\{1,\ldots,m\}$ and transition probability matrix P:

$$\forall (x_1,x_2)\in\mathbb{X}^2,\quad P(x_1,x_2)=Q(x_1;\{x_2\}\times[0,\infty))=\big((-D_0)^{-1}D_1\big)(x_1,x_2). \tag{4}$$

$\{X_n\}_{n\in\mathbb{N}}$ has an invariant probability measure $\boldsymbol{\phi}$ (i.e. $\boldsymbol{\phi}P = \boldsymbol{\phi}$). It is well-known that, for $n \geq 1$, the interarrival time $T_n := S_n - S_{n-1}$ has a moment of order 2 (whatever the probability distribution of X_0). We refer to [1] for details about the above basic facts on an MAP and its associated MRP.

Let us introduce the $m \times m$-matrix

$$\Phi := \boldsymbol{e}^\top \boldsymbol{\phi} \tag{5}$$

when $\boldsymbol{e}$ is the m-dimensional row-vector with all components equal to 1. Any $\mathbb{R}$-valued function v on $\mathbb{X}$ may be identified to a $\mathbb{R}^m$-dimensional vector. We use the subordinate matrix norm induced by $\ell^2(\phi)$-norm $\|v\|_2 := \sqrt{\sum_{x\in\mathbb{X}} |v(x)|^2 \boldsymbol{\phi}(x)}$ on $\mathbb{R}^m$

$$\|M\|_2 := \sup_{v:\|v\|_2=1} \|Mv\|_2.$$

Let $\mathbb{E}_{\boldsymbol{\phi}}$ be the expectation with respect to the initial conditions $(X_0, S_0) \sim (\boldsymbol{\phi}, \delta_0)$. Recall that $T_n := S_n - S_{n-1}$ for $n \geq 1$. When $X_0 \sim \boldsymbol{\phi}$, we have (see [3, Section 3]):

1. if g is a $\mathbb{R}$-valued function such that $\mathbb{E}\big[|g(X_1, T_1, \ldots, X_n, T_n)|\big] < \infty$, then $\forall k \geq 0$, $\forall n \geq 1$

$$\begin{aligned} &\mathbb{E}[g(X_{k+1}, T_{k+1}, \ldots, X_{k+n}, T_{k+n}) \mid \sigma(X_l, T_l : l \leq k)] \\ &\quad = \int_{(\mathbb{X}\times[0,\infty))^n} Q(X_s; dx_1 \times dz_1) \prod_{i=2}^{n} Q(x_{i-1}; dx_i \times dz_i) g(x_1, z_1, \ldots, x_n, z_n) \\ &\quad = (Q^{\otimes n})(g)\big(X_k\big) \end{aligned} \tag{6}$$

   where $Q^{\otimes n}$ denotes the n-fold kernel product $\overset{n}{\underset{i=1}{\otimes}} Q$ of Q defined in (3).

2. Let f and h be two $\mathbb{R}$-valued functions such that $\mathbb{E}_{\boldsymbol{\phi}}\big[|f(T_1, \ldots, T_n)|^2\big] < \infty$ and $\mathbb{E}_{\boldsymbol{\phi}}\big[|h(T_{n+k}, \ldots, T_{n+k+m})|^2\big] < \infty$ for $(k, n) \in (\mathbb{N}^*)^2, m \in \mathbb{N}$. From (6) with $g(x_1, z_1, \ldots, x_{n+k+m}, z_{n+k+m}) \equiv f(z_1, \ldots, z_n)h(z_{n+k}, \ldots, z_{n+k+m})$, the process $\{T_n\}_{n\geq 1}$ is stationary and the following covariance formula holds (see [3, Lem. 3.3] for details)

$$\begin{aligned} &\mathrm{Cov}\big(f(T_1, \ldots, T_n); h(T_{n+k}, \ldots, T_{n+k+m})\big) \\ &\quad = \; \mathbb{E}_{\boldsymbol{\phi}}\left[f(T_1, \ldots, T_n)\,(P^{k-1} - \Phi)\big(Q^{\otimes m+1}(h)\big)(X_n)\right]. \end{aligned} \tag{7}$$

   where matrices P, Φ are defined in (4) and (5).

First, note that the random variables $f(\cdot)$ and $h(\cdot)$ in (1) may be assumed to be of $\mathbb{L}^2$-norm 1. Thus we just have to deal with covariances. Second, the Cauchy-Schwarz inequality and Formula (7) allow us to write

$$
\begin{aligned}
&\mathrm{Cov}\big(f(T_1,\dots,T_n);h(T_{n+k},\dots,T_{n+k+m})\big)^2\\
&\quad\leq\ \mathbb{E}_{\phi}\left[|f(T_1,\dots,T_n)|^2\right]\mathbb{E}_{\phi}\left[\left|(P^{k-1}-\Phi)\big(Q^{\otimes m+1}(h)\big)(X_n)\right|^2\right]\\
&\qquad=\mathbb{E}_{\phi}\left[\left|(P^{k-1}-\Phi)\big(Q^{\otimes m+1}(h)\big)(X_0)\right|^2\right]\qquad(\boldsymbol{\phi}\text{ is }P\text{-invariant})\\
&\qquad=\left\|(P^{k-1}-\Phi)\big(Q^{\otimes m+1}(h)\big)\right\|_2^2\\
&\quad\leq\ \|P^{k-1}-\Phi\|_2^2\left\|Q^{\otimes m+1}(h)\right\|_2^2\\
&\quad\leq\ \|P^{k-1}-\Phi\|_2^2\qquad(\text{since }\left\|Q^{\otimes m+1}(h)\right\|_2\leq 1).
\end{aligned}
$$

Therefore, we obtain from (1) and (2) that the autocorrelation coefficient $\mathrm{Corr}(T_1;T_{k+1})$ as studied in [4], satisfies

$$\forall k\geq 1,\quad |\mathrm{Corr}(T_1;T_{k+1})|\leq\rho(k)\leq\|P^{k-1}-\Phi\|_2^2. \tag{8}$$

The convergence rate to 0 of the sequence $\{\mathrm{Corr}(T_1;T_{k+1})\}_{n\geq 1}$ is bounded from above by that of $\{\|P^{k-1}-\Phi\|_2\}_{k\geq 1}$. Under usual assumptions on the MAP, $\{X_n\}_{n\in\mathbb{N}}$ is irreducible and aperiodic so that there exists $r\in(0,1)$ such that

$$\|P^k-\Phi\|_2=O(r^k) \tag{9}$$

with $r=\max(|\lambda|,\lambda$ is an eigenvalue of P such that $|\lambda|<1)$. For a stationary Markov chain $\{X_n\}_{n\in\mathbb{N}}$ with general state space, we know from [6, p 200,207] that Property (9) is equivalent to the ρ-mixing property of $\{X_n\}_{n\in\mathbb{N}}$.

3 Comments on [4]

In [4], the analysis is based on a known explicit formula of the correlation function in terms of the parameters of the m-state MRP (see [4, (2.6)]). Note that this formula can be obtained using $n=1, m=0$ and $f(T_1)=T_1, h(T_{1+k})=T_{1+k}$ in (7). When $m:=2$ and under standard assumptions on MAPs, matrix P is diagonalizable with two distinct real eigenvalues, 1 and $0<\lambda<1$ which has an explicit form in terms of entries of P. Then, the authors can analyze the correlation function with respect to the entries of matrix P [4, (3.4)-(3.7)]. As quoted by the authors, such an analysis would be tedious and difficult with $m>2$ due to the increasing number of parameters defining an m-state MAP. Note that Inequality (8) and Estimate (9) when $m:=2$ provide the same convergence rate as in [4], that is λ the second eigenvalue of matrix P.

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