ON THE RECURRENCE SET OF PLANAR MARKOV RANDOM WALKS

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Abstract. In this paper, we investigate properties of recurrent planar Markov random walks. More precisely, we study the set of recurrence points with the use of local limit theorems. The Nagaev-Guivarc'h spectral method provides several examples for which these local limit theorems are satisfied as soon as some (standard or non-standard) central limit theorem and some non-sublattice assumption hold.

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INTRODUCTION

Let $\mathbb{X}$ be a measurable space endowed with a σ-algebra $\mathcal{X}$. Let $(X_n, S_n)_{n\in\mathbb{N}}$ be a Markov random walk (MRW) with state space $\mathbb{X}\times\mathbb{R}^2$, i.e. a Markov chain such that the distribution of $(X_{n+1}, S_{n+1}-S_n)$ depends on the past only through X_n. Namely: $(X_n, S_n)_{n\in\mathbb{N}}$ is a Markov chain with transition kernel P satisfying, for any set $A\in\mathcal{X}$ and any Borel subset S of $\mathbb{R}^2$, the following additive property (in the second component):

$$\forall (x,s)\in\mathbb{X}\times\mathbb{R}^2,\quad P\big((x,s);A\times S\big)=P\big((x,0);A\times(S-s)\big). \tag{1}$$

From this definition, the first component $(X_n)_{n\in\mathbb{N}}$ is a Markov chain, called the driving Markov chain of the MRW. We suppose that $S_0=0$. Given any distribution μ on $(\mathbb{X},\mathcal{X})$ (corresponding to the distribution of X_0), notation $\mathbb{P}_{(\mu,0)}$ refers to the distribution of $(X_n,S_n)_{n\in\mathbb{N}}$ with initial distribution $\mu\otimes\delta_0$. This notation takes the usual sense when $(X_n,S_n)_{n\in\mathbb{N}}$ is the canonical version defined on $(\mathbb{X}\times\mathbb{R}^2)^{\mathbb{N}}$. In this work, the last assumption may be assumed without loss of generality. The transition kernel of $(X_n)_{n\in\mathbb{N}}$ is denoted by Q.

Throughout the paper, we assume that Q admits an invariant probability measure on $\mathbb{X}$, called π, and that S_1 is $\mathbb{P}_{(\pi,0)}$ centered, namely: S_1 is $\mathbb{P}_{(\pi,0)}$-integrable and $\mathbb{E}_{(\pi,0)}[S_1]=0$. Moreover we suppose that there exists a two-dimensional closed subgroup $\mathbb{S}$ in $\mathbb{R}^2$ such that we have

$$\forall x\in\mathbb{X},\ \forall n\in\mathbb{N},\quad \mathbb{P}_{(x,0)}\big(S_n\in\mathbb{S}\big)=1. \tag{2}$$

Let $|\cdot|$ denote the euclidean norm on $\mathbb{R}^2$. Let us recall that $(S_n)_n$ is said to be recurrent if $\forall\varepsilon>0,\ \mathbb{P}_\pi\big(|S_n|<\varepsilon\text{ i.o.}\big)=1$, with the usual notation $[A_n\text{ i.o.}]:=\cap_{n\ge1}\cup_{k\ge n}A_k$ ("i.o." meaning infinitely often). Contrarily to the one-dimensional case, the strong law of large numbers (i.e. $S_n/n\to0$ a.s.) is not sufficient in dimension 2 to obtain the recurrence property for $(S_n)_n$. This is true even in the independent case (which is a special instance of MRW): if $(X_n)_n$ is a sequence of $\mathbb{R}^2$-valued independent identically distributed (i.i.d.) centered random variables (r.v.), then $S_n=X_1+\ldots+X_n$ is recurrent if and only if $\sum_n\mathbb{P}(|S_n|<\varepsilon)=\infty$ for every $\varepsilon>0$. Hence, in the i.i.d. case, if the distribution of X_1 is in the domain of attraction of a stable distribution of index α, then $\alpha=2$ is required. In other words, in this case, a central limit theorem (CLT) with a good normalization is needed for $S_n=X_1+\ldots+X_n$ to be recurrent, see [9, Sect. 3.2].

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Recurrence property of d-dimensional random walks is investigated in many papers. This study is well-known for i.i.d. increments, see for instance [9]. In the dependent case, let us mention in particular [3, 7, 6, 30, 29, 31] for random walks with stationary increments, [13] for MRWs (case $d \geq 3$) associated with uniformly ergodic Markov chains, [12] for MRWs associated with strongly ergodic Markov chains, and [19] for additive functionals of Harris recurrent Markov chains.

For general stationary $\mathbb{R}^2$-valued random walks, the link between CLT and recurrence of $(S_n)_n$ has been investigated by Conze in [6] and by Schmidt in [30] in the situation when the CLT holds with the standard normalization in $\sqrt{n}$. The methods used in these two works do not extend directly to other normalizations.

Transience/recurrence properties of MRWs (with $\mathbb{R}^d$-valued second component) have been investigated in [12] on the basis of a local limit theorem (LLT) obtained via the standard Nagaev-Guivarc'h spectral method. This method combined with the use of the Kochen-Stone adaptation of the Borel-Cantelli lemma (see (LLa) and (LLb) below) has also been used by Szász and Varjú for particular planar stationary walks with standard normalization in $\sqrt{n}$ in [32] as well as with a non-standard normalization in $\sqrt{n \log n}$ in [33].

Our work uses a similar approach to that of [12, 32, 33], but it goes beyond the question of recurrence. In fact we want to investigate the set of recurrence points $\mathcal{R}_{(\mu,0)}$, also called *recurrence set*, defined by

$$\mathcal{R}_{(\mu,0)} := \Big\{ s \in \mathbb{S} \, : \, \forall \varepsilon > 0, \ \mathbb{P}_{(\mu,0)}\big(|S_n - s| < \varepsilon \text{ i.o.}\big) = 1 \Big\}.$$

We simply write $\mathcal{R}_{(x,0)}$ when μ is the Dirac distribution δ_x at $x \in \mathbb{X}$. We describe situations in which we prove that $\mathcal{R}_{(\mu,0)} = \mathbb{S}$ for every initial distribution μ.

The recurrence set is well-known in the i.i.d. case (e.g. see [9, Sect. 3.2]), and it has been fully investigated in [1] for one-dimensional MRW (i.e. S_n is real-valued). However, to the best of our knowledge, the recurrence set has not been investigated for planar MRWs.

In dimension 2, whereas some recurrence results are only based on the CLT, the study of the recurrence set requires some assumption ensuring (roughly speaking) that $\mathbb{S}$ is the smallest lattice in $\mathbb{R}^2$ satisfying (2). Note that such a lattice-type assumption is also the additional condition to pass from CLT to LLT. Therefore, it is not surprising that local limit theorems will play here an important role in the study of the set $\mathcal{R}_{(\mu,0)}$ for planar MRWs. The LLTs involved in this work are obtained by using the weak Nagaev-Guivarc'h spectral method developed in [18].

In Section 1, we state our main results and give applications. First we state two key theorems (Theorems I-II) giving $\mathcal{R}_{(\mu,0)} = \mathbb{S}$ under conditions related to LLTs. Second we state two theorems (Theorems III-IV) giving these local limit conditions, under general assumptions, as soon as $(S_n)_n$ satisfies a (standard or non-standard) central limit theorem with suitable normalization as well as a non-sublattice condition in $\mathbb{S}$. We illustrate our general results with classes of models.

In Section 2, we make some simple remarks on the subgroup $\mathbb{S}$ appearing in (2), on the non-sublattice condition in $\mathbb{S}$, and on the special case when S_n is an additive functional of $(X_n)_{n\in\mathbb{N}}$.

In Section 3, we prove Theorems I-II by using classical arguments derived from the i.i.d. case [9] and the Kochen and Stone adaptation of the Borel-Cantelli lemma [24]. In Sections 4 and 5, we prove Theorems III-IV thanks to the weak Nagaev-Guivarc'h spectral method and Fourier techniques. In Section 6, we detail our applications (that have been shortly introduced in Section 1). These applications are obtained thanks to recent works [5, 11, 17, 18]. Some complements concerning spectral method are given in appendices.

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1. Main results and applications

Let $(X_n, S_n)_{n\in\mathbb{N}}$ be a Markov random walk with state space $\mathbb{X}\times\mathbb{R}^2$, let $\mathbb{S}$ be a two-dimensional closed subgroup of $\mathbb{R}^2$ satisfying (2). The Haar measure on $\mathbb{S}$ is called $m_{\mathbb{S}}$. We denote by $B(s,\varepsilon)$ the open ball in $\mathbb{R}^2$ centered at s with radius ε. We denote by $\mathcal{B}(\mathbb{R}^2)$ the Borel σ-algebra of $\mathbb{R}^2$.

We start by stating two key results giving two different approaches to prove $\mathcal{R}_{(\mu,0)} = \mathbb{S}$. The first theorem borrows classical arguments derived from the i.i.d. case [9]. Recall that $(X_n)_{n\in\mathbb{N}}$ is said to be Harris recurrent if, for any set $B \in \mathcal{X}$ such that $\pi(B) > 0$, for every $x \in \mathbb{X}$, we have $\mathbb{P}_{(x,0)}\big(X_k \in B \text{ i.o.}\big) = 1$.

Theorem I. *Assume that there exist $\varepsilon_{\mathbb{S}} > 0$ and a sequence $(a_n)_{n\ge1}$ of positive real numbers satisfying $\sum_{n\ge1} a_n = \infty$ such that, for every $(s,\varepsilon) \in \mathbb{S}\times(0;\varepsilon_{\mathbb{S}})$, for every bounded measurable function $f : \mathbb{X} \to [0,+\infty)$, the following local limit property holds with $B := B(s,\varepsilon)$:*

$$\mathbb{E}_{(\pi,0)}\big[f(X_n)\,\mathbf{1}_B(S_n)\big] \sim a_n\,\pi(f)\,m_{\mathbb{S}}(B) \quad \text{when } n\to+\infty. \tag{LL0}$$

Then the following assertions hold true:

- **(a)** $\mathcal{R}_{(\pi,0)} = \mathbb{S}$*;*
- **(b)** *if in addition $(X_n)_{n\in\mathbb{N}}$ is Harris recurrent, then $\mathcal{R}_{(\mu,0)} = \mathbb{S}$ for every probability measure μ on $(\mathbb{X},\mathcal{X})$.*

Theorem I is direct and quite natural but it requires a local limit estimate for every bounded measurable functions. Moreover it needs Harris-recurrence hypothesis to obtain the non-stationary result **(b)**. These two assumptions appear to be not satisfied on some classes of models. For this reason, we give another theorem based on another approach but still related to local limit theorems. Theorem II below involves the Kochen and Stone adaptation of the Borel-Cantelli lemma.

Theorem II. *Assume that there exist $\varepsilon_{\mathbb{S}} > 0$ and $d > 0$ such that, for every $\varepsilon \in (0;\varepsilon_{\mathbb{S}})$, for every $(x,s) \in \mathbb{X}\times\mathbb{S}$, we have*

$$\sum_{n\ge1} \mathbb{P}_{(x,0)}\big(|S_n - s| < \varepsilon\big) = \infty, \tag{KSa}$$

$$\liminf_{N\to+\infty} \frac{\sum_{n,m=1}^{N} \mathbb{P}_{(x,0)}\big(|S_n - s| < \varepsilon,\, |S_{n+m} - s| < \varepsilon\big)}{\Big(\sum_{n=1}^{N} \mathbb{P}_{(x,0)}(|S_n - s| < \varepsilon)\Big)^2} \le d. \tag{KSb}$$

Then $\mathcal{R}_{(\mu,0)} = \mathbb{S}$ for every probability measure μ on $(\mathbb{X},\mathcal{X})$.

It is not difficult to prove (see Lemma 4.1) that the Kochen-Stone Conditions (KSa)-(KSb) are implied by the two following local limit properties (the second being a bivariate local limit theorem):

$$\mathbb{P}_{(x,0)}\big(S_n \in B\big) \sim D\,a_n m_{\mathbb{S}}(B), \tag{LLa}$$

$$\mathbb{P}_{(x,0)}\big((S_n, S_{n+m}) \in B^2\big) \sim D^2\,a_n a_m\,m_{\mathbb{S}}(B)^2, \tag{LLb}$$

where $B := B(s,\varepsilon)$, $D \in (0,+\infty)$ and where $(a_n)_{n\ge1}$ is a sequence of positive numbers satisfying $\sum_{n\ge1} a_n = \infty$.

Now let us present, and illustrate by classes of models, our general operator-type strategy providing (LL0) and (LLa)-(LLb), and so conclusions of Theorems I and II. To that effect, we consider the Fourier operators $Q(t)$ $(t \in \mathbb{R}^2)$, associated with the MRW, acting (in a first step) on the space of bounded measurable functions $f : \mathbb{X} \to \mathbb{C}$, as follows:

$$\forall t \in \mathbb{R}^2,\ \forall x \in \mathbb{X},\quad \big(Q(t)f\big)(x) := \mathbb{E}_{(x,0)}\big[e^{i\langle t,S_1\rangle} f(X_1)\big]. \tag{3}$$

If $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is a Banach space, $\mathcal{L}(\mathcal{Y})$ denotes the space of linear continuous endomorphisms of $\mathcal{Y}$. The associated operator norm is also denoted by $\|\cdot\|_{\mathcal{Y}}$.

Let $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ and $(\widehat{\mathcal{B}}, \|\cdot\|_{\widehat{\mathcal{B}}})$ be two complex Banach spaces composed of π-integrable $\mathbb{C}$-valued functions on $\mathbb{X}$ (or of classes modulo π of such functions). We assume that $\mathcal{B} \subset \widehat{\mathcal{B}}$ and that

$$\exists c \in (0,+\infty),\ \forall f \in \mathcal{B},\ \|f\|_{\widehat{\mathcal{B}}} \le c\|f\|_{\mathcal{B}} \quad \text{and} \quad \exists d \in (0,+\infty),\ \forall f \in \widehat{\mathcal{B}},\ \int |f|d\pi \le d\|f\|_{\widehat{\mathcal{B}}}. \tag{4}$$

For $\mathcal{Y} = \mathcal{B}$ or $\widehat{\mathcal{B}}$ and for any probability measure ν on $\mathbb{X}$, we write $\nu \in \mathcal{Y}'$ when the linear map $f \mapsto \nu(f) := \int_{\mathbb{X}} f\, d\nu$ is well-defined and continuous on $\mathcal{Y}$. The conditions in (4) imply that $\pi \in \mathcal{B}' \cap (\widehat{\mathcal{B}})'$. We denote by $\|\cdot\|_{\mathcal{B},\widehat{\mathcal{B}}}$ the operator norm of bounded linear operators from $\mathcal{B}$ to $\widehat{\mathcal{B}}$.

The essential spectral radius of any $T \in \mathcal{L}(\mathcal{B})$ is defined by $r_{ess}(T) := \lim_n(\inf \|T^n - K\|_{\mathcal{B}})^{1/n}$ where the infimum is taken over the ideal of compact endomorphisms K on $\mathcal{B}$, see e.g. [34]. The following assumptions will be used to check hypotheses of Theorems I-II.

Operator-type assumptions. *Function $\mathbf{1}_{\mathbb{X}}$ (or its class) is in $\mathcal{B}$; for every $t \in \mathbb{R}^2$ we have $Q(t) \in \mathcal{L}(\mathcal{B}) \cap \mathcal{L}(\widehat{\mathcal{B}})$. Moreover:*

- **(A1)** *Q is strongly ergodic on $\mathcal{B}$, that is: $\lim_n Q^n = \pi$ in $\mathcal{L}(\mathcal{B})$,*
- **(A2)** $\forall t \in \mathbb{R}^2,\quad \lim_{h\to 0} \|Q(t+h) - Q(t)\|_{\mathcal{B},\widehat{\mathcal{B}}} = 0,$
- **(A3)** *for every compact K in $\mathbb{R}^2$, there exist $\kappa \in (0;1)$, $C \in (0;+\infty)$ such that, for every $t \in K$, the essential spectral radius of $Q(t)$ on $\mathcal{B}$ satisfies $r_{ess}(Q(t)) \le \kappa$, and*
$$\forall n \ge 1,\ \forall f \in \mathcal{B},\quad \|Q(t)^n f\|_{\mathcal{B}} \le C\kappa^n\|f\|_{\mathcal{B}} + C\|f\|_{\widehat{\mathcal{B}}}, \tag{5}$$
- **(A4)** *for every $\lambda \in \mathbb{C}$ such that $|\lambda| \ge 1$ and for every nonzero element $f \in \mathcal{B}$, the implication*
$$[\exists n_0,\ \forall n \ge n_0,\ |\lambda|^n|f| \le Q^n|f|] \ \Rightarrow\ [|\lambda| = 1 \text{ and } |f| \le \pi(|f|)]$$
holds true.

These operator-type assumptions put all together correspond to Conditions $(\widehat{K})$ and (P) of [18, p. 434-436]. They are the key assumptions to use the weak Nagaev-Guivarc'h method. Further comments on these hypotheses are presented at the end of Section 4.

Probabilistic-type assumptions:

- **(A5)** *There exist a function $L : (0;+\infty) \to (0;+\infty)$ assumed to be slowly varying at ∞ (i.e. $\forall k > 0$, $\lim_{x\to+\infty} L(kx)/L(x) = 1$) and a sequence $(A_n)_n$ of positive real numbers satisfying $A_n^2 \sim nL(A_n)$ and $\sum_{n\ge1} A_n^{-2} = \infty$ such that, under $\mathbb{P}_{(\pi,0)}$, $(S_n/A_n)_n$ converges in distribution to a non-degenerate Gaussian law $\mathcal{N}(0,\Gamma)$,*

- **(A6)** $(S_n)_n$ *is* **non-sublattice in** $\mathbb{S}$*; namely there exists no $(\mathbb{S}_0, \chi, (\beta_t)_{t\in\mathbb{S}_0^*})$ with $\mathbb{S}_0$ a closed proper subgroup of $\mathbb{S}$, $\chi : \mathbb{X} \to \mathbb{R}^2$ a bounded measurable function, $(\beta_t)_{t\in\mathbb{S}_0^*}$ a family of real numbers indexed by the dual group $\mathbb{S}_0^*$ of $\mathbb{S}_0$, satisfying the following property for π-almost every $x \in \mathbb{X}$:*
$$\forall t \in \mathbb{S}_0^*,\ \forall n \ge 1,\quad \big\langle t, S_n + \chi(X_n) - \chi(x)\big\rangle \in n\beta_t + 2\pi\mathbb{Z} \ \ \mathbb{P}_{(x,0)} - a.s.. \tag{6}$$

Theorem III. *Assume that assumptions (A1) to (A6) hold true with $\mathcal{B}$ containing all the non-negative bounded measurable functions. Then (LL0) is fulfilled, and so conclusions of Theorem I apply.*

Let us observe that the assumption on $\mathcal{B}$ in Theorem III is not fulfilled if $\mathcal{B}$ is defined as some space of regular functions. In this case, the next statement is relevant, and it is worth noticing that it does not require the Harris-recurrence hypothesis.

In Theorem IV below, we suppose that $\widehat{\mathcal{B}}$ is composed of functions (not of classes), so that the Dirac distribution at any $x \in \mathbb{X}$, called δ_x, is defined on $\widehat{\mathcal{B}}$ (i.e. $\delta_x(f) := f(x)$). Note that $\delta_x \in (\widehat{\mathcal{B}})'$ means that there exists $c_x > 0$ such that: $\forall f \in \widehat{\mathcal{B}},\ |f(x)| \leq c_x \|f\|_{\widehat{\mathcal{B}}}$.

Theorem IV. *Assume that assumptions (A1) to (A6) hold true. Suppose that, for every $x \in \mathbb{X}$, $\delta_x \in (\widehat{\mathcal{B}})'$. Then (LLa) and (LLb) are fulfilled, thus the conclusion of Theorem II holds.*

Our method enables the study of recurrence set for every model satisfying our general assumptions **(A1)-(A4)**. The development of the use of the Nagaev-Guivarc'h method offers large perspectives of applications. To fix ideas, we give now particular applications of Theorems III and IV to classes of models. These applications are proved and detailed in Section 6.

The following result will be derived from Theorem III.

Application 1 (ρ-mixing driving chain)**.** *Assume that the driving Markov chain $(X_n)_n$ is ρ-mixing, that S_1 is centered and square integrable under $\mathbb{P}_{(\pi,0)}$, that $(S_n)_n$ is non-sublattice in $\mathbb{S}$ and that the limit covariance matrix Γ of $(S_n/\sqrt{n})_n$ is positive definite. Then $\mathcal{R}_{(\pi,0)} = \mathbb{S}$. If in addition $(X_n)_{n\in\mathbb{N}}$ is Harris recurrent, then $\mathcal{R}_{(\mu,0)} = \mathbb{S}$ for any initial distribution μ.*

The three following results will be derived from Theorem IV. They concern the special case when S_n is defined as a univariate or bivariate additive functional (AF) of $(X_n)_n$.

Application 2 (AF of V-geometrically ergodic Markov chain)**.** *Let us assume that $(X_n)_n$ is V-geometrically ergodic for some $V : \mathbb{X} \to [1, +\infty)$ and that*

(a) *either $S_n = \sum_{k=1}^n \xi(X_k)$ for some π-centered $\xi : \mathbb{X} \to \mathbb{R}^2$ such that $|\xi|^2/V$ is bounded,*

(b) *or $S_n = \sum_{k=1}^n \xi(X_{k-1}, X_k)$ for some $\xi : \mathbb{X} \times \mathbb{X} \to \mathbb{R}^2$ such that $\xi(X_0, X_1)$ is $\mathbb{P}_\pi$-centered and $\sup_{x,y} |\xi(x,y)|^{2+\varepsilon}/(V(x) + V(y))$ is finite for some $\varepsilon > 0$.*

If $(S_n)_n$ is non-sublattice in $\mathbb{S}$ and the limit covariance matrix Γ of $(S_n/\sqrt{n})_n$ is positive definite, then, for every initial distribution μ of $(X_n)_{n\in\mathbb{N}}$, the recurrence set $\mathcal{R}_{(\mu,0)}$ of $(S_n)_n$ satisfies $\mathcal{R}_{(\mu,0)} = \mathbb{S}$.

Application 3 (AF of Lipschitz iterative models)**.** *Let $b \geq 0$. Suppose that (X_n) is a random walk on $\mathbb{X} = \mathbb{R}^D$ given by $X_n = F(X_{n-1}, \vartheta_n),\ \ n \geq 1$, with $F : \mathbb{X} \times V \to \mathbb{X}$ a measurable function and with $(\vartheta_n)_{n\geq 1}$ a sequence of i.i.d. random variables independent of X_0. Assume that, almost surely, $F(\cdot, \vartheta_1)$ has Lipschitz constant strictly less than 1, that $\mathbb{E}[d\big(F(0,\vartheta_1), 0\big)^{2(b+1)}] < \infty$. Take $S_n = \sum_{k=1}^n \xi(X_k)$ for some π-centered $\xi : \mathbb{X} \to \mathbb{R}^2$. Finally assume that $(S_n)_n$ is non-sublattice in $\mathbb{S}$ and that*

$$\exists C \geq 0, \quad \forall (x,y) \in \mathbb{X}^2, \quad \big|\xi(x) - \xi(y)\big| \leq C\, d(x,y)\, \big[1 + d(x,x_0) + d(y,x_0)\big]^b. \tag{7}$$

Then $\mathcal{R}_{(\mu,0)} = \mathbb{S}$ for every initial distribution μ on $\mathbb{X}$.

The previous applications involve the standard CLT. In fact it is not so easy to find examples of MRWs, even in case of additive functionals, for which S_n satisfies a non-standard CLT. Such instances can be found in [5], see also [26]. The following application, based on [5], shows how our results apply to affine recursions which are special instances of Lipschitz iterative models.

Application 4 (Affine recursion with non-standard CLT). *Suppose that $(X_n)_n$ is a Markov chain on $\mathbb{R}^2$ given by $X_n = A_n X_{n-1} + B_n$, where $(B_n, A_n)_n$ is a sequence of i.i.d. $\mathbb{R}^2 \rtimes Sim(\mathbb{R}^2)$-valued random variables ($Sim(\mathbb{R}^2)$ being the similarity group of $\mathbb{R}^2$) independent of X_0. Assume*

$$\mathbb{E}[|A_1|^2] = 1, \quad \mathbb{E}[|A_1|^2 \log |A_1|] < \infty, \quad \mathbb{E}[|B_1|^2] < \infty.$$

Under some additional conditions (to be specified in Subsection 6.4) on the support of the distribution of (B_1, A_1) and of the invariant probability measure, it is proved in [5] that there exist $m_0 \in \mathbb{R}^2$ and a gaussian random variable Z such that, for every $x \in \mathbb{R}^2$, $(S_n/\sqrt{n \log(n)})_n$ converges in distribution (under $\mathbb{P}_x$) to Z, with $S_n := \sum_{k=1}^n (X_k - m_0)$.

Then, if $(S_n)_n$ is non-sublattice in $\mathbb{R}^2$, we have $\mathcal{R}_{(\mu,0)} = \mathbb{R}^2$ for every initial measure μ.

2. Preliminary remarks on Hypotheses (2) and **(A6)**

Given any subgroup H of $\mathbb{R}^2$, its dual subgroup is defined as

$$H^* := \{s \in \mathbb{R}^2 \ : \ \forall t \in H, \ \langle s, t\rangle \in 2\pi\mathbb{Z}\}. \tag{8}$$

H^* is a subgroup of $\mathbb{R}^2$, and the dual subgroup of H^* (i.e. the bidual of H) coincides with H. These properties are classical, anyway they can easily be proved in Cases *(H1) (H2) (H3)* below.

• *Remarks on (2).* Theorems I and II are valid for d-dimensional MRW. However, in practice, the condition $\sum_{n\geq 1} a_n = \infty$ involved in (LL0) and (LLa)-(LLb) is only fulfilled in dimension $d = 1$ or 2. The one-dimensional cases $\mathbb{S} = \mathbb{R}\,\vec{u}$ and $\mathbb{S} = \mathbb{Z}\,\vec{u}$ $(\vec{u} \in \mathbb{R}^2)$ are not investigated here since the recurrence set of S_n can be deduced from [1] (thanks to the strong law of large numbers).

Consequently, throughout the paper, the subgroups $\mathbb{S}$ of interest in (2) are the two-dimensional closed subgroups of $\mathbb{R}^2$, which correspond to the three following cases:

(H1) $\mathbb{S} = \mathbb{R}^2$. *We have* $\mathbb{S}^* = \{0\}$. *We set* $\varepsilon_{\mathbb{S}} := 1$.

(H2) There exists $(b, \vec{u}, \vec{v}) \in (0, +\infty) \times \mathbb{R}^2 \times \mathbb{R}^2$ such that: $\mathbb{S} = b\,\mathbb{Z}\,\vec{u} \oplus \mathbb{R}\,\vec{v}$. We suppose, without loss of generality, that $(\vec{u}, \vec{v})$ is an orthonormal basis of $\mathbb{R}^2$. We set $\varepsilon_{\mathbb{S}} := b$. Note that, for every $(s, \varepsilon) \in \mathbb{S} \times (0; \varepsilon_{\mathbb{S}})$, we have $B(s, \varepsilon) \cap \mathbb{S} = \{s + w\vec{v} \ : \ w \in \mathbb{R}, |w| < \varepsilon\}$, and that $\mathbb{S}^* = a\,\mathbb{Z}\,\vec{u}$ with $a = 2\pi/b$.

(H3) There exists some real-valued invertible $2 \times 2-$matrix B such that: $\mathbb{S} = B\,\mathbb{Z}^2$. We set $\varepsilon_{\mathbb{S}} := \min\{|s|; \ s \in \mathbb{S} \setminus \{0\}\}$. Note that, for every $(s, \varepsilon) \in \mathbb{S} \times (0; \varepsilon_{\mathbb{S}})$, $B(s, \varepsilon) \cap \mathbb{S} = \{s\}$, and that $\mathbb{S}^* = A\,\mathbb{Z}^2$ with $A = 2\pi\,(B^*)^{-1}$, where B^* is the transpose matrix of B.

• *Remarks on non-sublattice condition* **(A6)**. Since the dual subgroup of $\mathbb{S}_0^*$ is $\mathbb{S}_0$, one can easily check that, if $(s, s') \in \mathbb{R}^2$ is such that $\langle t, s\rangle \in n\beta_t + 2\pi\mathbb{Z}$ and $\langle t, s'\rangle \in n\beta_t + 2\pi\mathbb{Z}$ for every $t \in \mathbb{S}_0^*$, then $s - s' \in \mathbb{S}_0$, namely: s and s' belong to the same class modulo $\mathbb{S}_0$. Therefore the non-sublattice assumption **(A6)** is equivalent to the nonexistence of $(\mathbb{S}_0, (b_n)_n, \chi)$ with $\mathbb{S}_0$ a proper subgroup of $\mathbb{S}$, $(b_n)_n$ a sequence of vectors in $\mathbb{R}^2$, $\chi : \mathbb{X} \to \mathbb{R}^2$ a bounded measurable function such that:

$$\forall t \in \mathbb{S}_0^*, \ \exists \beta_t \in \mathbb{R}, \ \forall n \geq 1, \quad \langle t, b_n\rangle \in n\beta_t + 2\pi\mathbb{Z} \tag{9}$$

and such that, for π-almost every $x \in \mathbb{X}$, we have

$$\forall n \geq 1, \quad S_n + \chi(X_n) - \chi(x) \in b_n + \mathbb{S}_0 \ \ \mathbb{P}_{(x,0)} - a.s.. \tag{10}$$

Hence, a sufficient condition for $(S_n)_n$ to be non-sublattice in $\mathbb{S}$ is that there exists no $(a_1, \mathbb{S}_0, \chi(\cdot))$ with $a_1 \in \mathbb{R}^2$, $\mathbb{S}_0$ a proper subgroup of $\mathbb{S}$, χ a bounded measurable function from $\mathbb{X}$ to $\mathbb{R}^2$, satisfying for π-almost every $x \in \mathbb{X}$,

$$S_1 + \chi(X_1) - \chi(x) \in a_1 + \mathbb{S}_0 \ \ \mathbb{P}_{(x,0)} - a.s..$$

In some cases (such as additive functionals, or general MRW with $\mathcal{B}$ in **(A1)-(A4)** composed of classes of functions modulo π), the last condition is equivalent to the non-sublattice condition, see Remark B.5.

• *Remarks for Markov additive functionals.* Let $(X_n)_{n\in\mathbb{N}}$ be a Markov chain with state space $\mathbb{X}$, transition kernel Q, invariant distribution π, and initial distribution μ. Here, given $\xi = (\xi_1, \xi_2) : \mathbb{X} \to \mathbb{R}^2$ a π-centered function (i.e. for $i = 1, 2$, ξ_i is π-integrable and $\pi(\xi_i) = 0$), we consider the classical MRW $(X_n, S_n)_{n\in\mathbb{N}}$ defined by $S_0 = 0$ and $\forall n \geq 1$:

$$S_n := \sum_{k=1}^{n} \xi(X_k). \tag{11}$$

The sequence $(S_n)_n$ is called an *additive functional* (AF) of $(X_n)_n$. In this case, the two following remarks are of interest.

Remark 2.1. *(Reduction of* **(A6)***.)*
Condition (2) holds if and only if $\xi(\mathbb{X}) \subset \mathbb{S}$. Under this assumption, $(S_n)_n$ is non-sublattice in $\mathbb{S}$ if and only if there exists no $(a_0, \mathbb{S}_0, A, \chi)$ with $a_0 \in \mathbb{R}^2$, $\mathbb{S}_0$ a proper closed subgroup in $\mathbb{S}$, $A \in \mathcal{X}$ a π-full Q-absorbing set (i.e. such that $\pi(A) = 1$ and $Q(z, A) = 1$ for all $z \in A$), $\chi : \mathbb{X} \to \mathbb{R}^2$ a bounded measurable function, such that

$$\forall x \in A, \quad \xi(y) + \chi(y) - \chi(x) \in a_0 + \mathbb{S}_0 \ \ Q(x, dy) - a.s..$$

This statement has been proved in [18, Section 5.2] *when $\mathbb{S} = \mathbb{R}^2$, extension to Cases (H2)-(H3) is easy.*

Remark 2.2. *If $(S_n)_n$ satisfies the standard CLT ($A_n = \sqrt{n}$) and $(S_n)_n$ is non-sublattice in $\mathbb{S}$, then the covariance matrix Γ of the CLT is automatically positive definite, see e.g.* [18, Section 5.2].

3. Proof of Theorems I-II

Let $(X_n, S_n)_{n\in\mathbb{N}}$ be a Markov random walk with state space $\mathbb{X} \times \mathbb{R}^2$, and let $\mathbb{S}$ be a two-dimensional closed subgroup of $\mathbb{R}^2$ satisfying (2). We use the notations of Section 1.

The first assertion of Theorem I is established in Subsection 3.1, the second one in Subsection 3.2. Theorem II is proved in Subsection 3.3. Auxiliary statements of interest are also presented in these subsections.

3.1. **Recurrence set in the stationary case (proof of Theorem I-(a)).** To prove Theorem I, we define the r.v. $\xi_0 = 0$ and $\xi_k = S_k - S_{k-1}$ for $k \geq 1$. From the additive property (1), it can be easily seen that the distribution of $((X_{n+k}, \xi_{n+k}))_{k\geq 1}$ given $\{X_n = x,\ S_n = s\}$ is equal to the distribution of $((X_k, \xi_k))_{k\geq 1}$ under $\mathbb{P}_{(x,0)}$. We assume (without loss of generality) that $((X_n, \xi_n))_{n\geq 0}$ is the canonical Markov chain with transition kernel $\tilde{P}((x, s); \cdot) := P((x, 0); \cdot)$. Hence, defining the σ-algebra $\mathcal{F}_n = \sigma(\xi_k,\ 0 \leq k \leq n)$ and writing θ for the usual shift operator on $\Omega = (\mathbb{X} \times \mathbb{R}^2)^{\mathbb{N}}$, we obtain for every bounded measurable function $F : \Omega \to \mathbb{R}$ and for every $x \in \mathbb{X}$: $\mathbb{E}_{(x,0)}[F \circ \theta^n \,|\, \mathcal{F}_n\,] = \mathbb{E}_{(X_n,0)}[F]$.

Remark 3.1. *For $A \in \mathcal{B}(\mathbb{R}^2)$, $k \in \mathbb{N}^*$, set $Y_k = \prod_{j=k}^{+\infty} \mathbf{1}_A(S_j)$, and $f_k(x) = \mathbb{E}_{(x,0)}[Y_k]$ $(x \in \mathbb{X})$. Then, for any $B \in \mathcal{B}(\mathbb{R}^2)$ and $n \in \mathbb{N}^*$, we have*

$$\mathbb{P}_{(\pi,0)}\Big(S_n \in B, \quad S_{n+j} - S_n \in A, \ \forall j \geq k\Big) = \mathbb{E}_{(\pi,0)}\big[\mathbf{1}_B(S_n)\, f_k(X_n)\big].$$

Note that, for any $A \in \mathcal{B}(\mathbb{R}^2)$, the corresponding function f_k in Remark 3.1 is nonnegative, bounded and measurable. We start by proving the recurrence of $(S_n)_n$.

Lemma 3.2. *We have: $0 \in \mathcal{R}_{(\pi,0)}$.*

Proof. Let $\varepsilon > 0$, $k \geq 1$. Let us prove that $\mathbb{P}_\pi(\,\exists j \geq k \,:\, |S_j| < 2\varepsilon\,) = 1$. For any $n \geq 1$, set

$$A_n^{(k)} = \Big[\,|S_n| < \varepsilon,\ |S_{n+j}| \geq \varepsilon,\ \forall j \geq k\Big].$$

If $|n - n'| \geq k$, then $A_n^{(k)} \cap A_{n'}^{(k)} = \emptyset$. Hence we have $\sum_{n\geq 1} \mathbb{P}_{(\pi,0)}(A_n^{(k)}) \leq k$. Moreover we have

$$\mathbb{P}_{(\pi,0)}(A_n^{(k)}) \geq \mathbb{P}_{(\pi,0)}\Big(\,|S_n| < \varepsilon,\ |S_{n+j} - S_n| \geq 2\varepsilon,\ \forall j \geq k\,\Big).$$

Then, applying Remark 3.1 with $B = \{z \in \mathbb{R}^2 : |z| < \varepsilon\}$ and $A = \{z \in \mathbb{R}^2 : |z| \geq 2\varepsilon\}$, we obtain $\sum_{n\geq 1} \mathbb{E}_{(\pi,0)}\big[\mathbf{1}_B(S_n)\, f_k(X_n)\big] \leq k$. But (LL0) gives as $n \to +\infty$:

$$\mathbb{E}_{(\pi,0)}[\mathbf{1}_B(S_n)\, f_k(X_n)] \sim a_n\, \pi(f_k) m_{\mathbb{S}}(B).$$

Since $0 \in \mathbb{S}$ and B is centered at 0, we have $m_{\mathbb{S}}(B) > 0$. Finally the fact that $\sum_{n\geq 1} a_n = \infty$ implies $\pi(f_k) = \mathbb{P}_{(\pi,0)}(Y_k = 1) = 0$. □

Proof of assertion (a) in Theorem I. Let $s \in \mathbb{S}$. Let us show that

$$\forall \varepsilon > 0,\ \forall k \geq 1,\ \ \mathbb{P}_{(\pi,0)}\big(\,|S_j - s| \geq 2\varepsilon,\ \forall j \geq k\,\big) = 0.$$

Let $\varepsilon > 0$ and $k \geq 1$ be fixed. Set $B' = \{z \in \mathbb{R}^2 : |z + s| < \varepsilon\}$ and $A' = \{z \in \mathbb{R}^2 : |z - s| \geq 2\varepsilon\}$, and denote by Y'_k and f'_k the elements associated to A' as in Remark 3.1. Then, according to Lemma 3.2 and Remark 3.1, we have for $n \geq 1$

$$\begin{aligned} 0 = \mathbb{P}_{(\pi,0)}\big(\,|S_{n+j}| \geq \varepsilon,\ \forall j \geq k\,\big) \ &\geq\ \mathbb{P}_{(\pi,0)}\big(\,S_n \in B',\, S_{n+j} - S_n \in A',\, \forall j \geq k\,\big) \\ &=\ \mathbb{E}_{(\pi,0)}\big[\mathbf{1}_{B'}(S_n)\, f'_k(X_n)\big]. \end{aligned}$$

Hence $\mathbb{E}_{(\pi,0)}\big[\mathbf{1}_{B'}(S_n)\, f'_k(X_n)\big] = 0$. From (LL0) and $m_{\mathbb{S}}(B') > 0$ (since B' is centered at $-s \in \mathbb{S}$), it then follows that $\pi(f'_k) = \mathbb{P}_\pi(Y'_k = 1) = 0$. □

3.2. **From stationarity to non-stationarity under Harris recurrence (proof of Theorem I-(b)).** Let us define the following subset of $\mathbb{X}$:

$$\mathcal{A} := \{x \in \mathbb{X} : \mathcal{R}_{(x,0)} = \mathbb{S}\}.$$

Property $\mathcal{R}_{(\pi,0)} = \mathbb{S}$ implies that $\pi(\mathcal{A}) = 1$ (since $\mathbb{S}$ is separable). Of course, if $\mathcal{A} = \mathbb{X}$, we obtain $\mathcal{R}_{(\mu,0)} = \mathbb{S}$ for every initial distribution μ of the driving Markov chain $(X_n)_{n\in\mathbb{N}}$.

The second assertion of Theorem I follows from the following statement.

Proposition 3.3. *The following assertions hold:*

(i) If $x \in \mathbb{X}$ is such that $\mathbb{P}_{(x,0)}\big(X_n \in \mathcal{A}$ i.o.$\big) = 1$, then $x \in \mathcal{A}$ (i.e. $\mathcal{R}_{(x,0)} = \mathbb{S}$).

(ii) If the driving Markov chain $(X_n)_{n\in\mathbb{N}}$ is Harris recurrent and if $\mathcal{R}_{(\pi,0)} = \mathbb{S}$, then $\mathcal{A} = \mathbb{X}$. In this case, we have $\mathcal{R}_{(\mu,0)} = \mathbb{S}$ for every initial distribution μ of $(X_n)_{n\in\mathbb{N}}$.

Lemma 3.4. *Let $(s,\varepsilon)\in\mathbb{S}\times(0,1)$ and $E:=[|S_n-s|<\varepsilon$ i.o.$]$. For every $x\in\mathbb{X}$, we have for $\mathbb{P}_{(x,0)}-$almost every $\omega\in\Omega$:*

$$\lim_{k\to+\infty}\mathbb{P}_{(X_k(\omega),0)}\Big(\big|S_n-\big(s-S_k(\omega)\big)\big|<\varepsilon \text{ i.o.}\Big)=\mathbf{1}_E(\omega).$$

Proof of Lemma 3.4. Let $x\in\mathbb{X}$. According to a classical argument due to Doob (see [27, Prop. V-2.4]), we have for $\mathbb{P}_{(x,0)}-$almost every $\omega\in\Omega$: $\lim_{k\to+\infty}\mathbb{P}_{(X_k(\omega),S_k(\omega))}(E)=\mathbf{1}_E(\omega)$. Then the desired property easily follows from the additive property (1). □

Proof of Proposition 3.3. We suppose that $(X_n,S_n)_n$ is the canonical version defined on the set $\Omega:=(\mathbb{X}\times\mathbb{R}^2)^{\mathbb{N}}$. Let us fix any $(s,\varepsilon)\in\mathbb{S}\times(0,1)$, and set $E:=\big[|S_n-s|<\varepsilon \text{ i.o.}\big]$. Using the assumption in *(i)*, Lemma 3.4 and Lebesgue's theorem, using finally the definition of $\mathcal{A}$ and the fact that $S_k-s\in\mathbb{S}$ $\mathbb{P}_{(x,0)}-$a.s. (use (2)), we obtain the following property

$$\mathbb{P}_{(x,0)}(E)=\lim_k\int_{\{\omega:X_n(\omega)\in\mathcal{A}\text{ i.o.}\}}\mathbb{P}_{(X_k(\omega),0)}\big(\big|S_n-\big(s-S_k(\omega)\big)\big|<\varepsilon\text{ i.o.}\big)\,d\mathbb{P}_{(x,0)}(\omega)=1,$$

from which we deduce $\mathcal{R}_{(x,0)}=\mathbb{S}$. Now, if $\mathcal{R}_{(\pi,0)}=\mathbb{S}$, then $\pi(\mathcal{A})=1$, so that the Harris recurrence of $(X_n)_{n\in\mathbb{N}}$ gives $\mathbb{P}_{(x,0)}\big(X_k\in\mathcal{A}\text{ i.o.}\big)=1$ for all $x\in\mathbb{X}$. Thus (ii) follows from (i). □

Lemma 3.4, based on both Markov and additive properties of $(X_n,S_n)_n$, plays an important role in the previous proof, as well as in the main statement (Proposition 3.7) of the next section.

3.3. **Borel-Cantelli adaptation of Kochen and Stone (Proof of Theorem II).** We present now a general strategy to obtain $\mathcal{R}_{(x,0)}=\mathbb{S}$ for every $x\in\mathbb{X}$, even when the driving Markov chain is not Harris-recurrent. In particular Theorem II directly follows from the next Corollary 3.6 and Proposition 3.7.

The following Proposition 3.5, as well as its Corollary 3.6, are true for any sequence $(Y_n)_{n\in\mathbb{N}}$ of r.v. defined on some probability space $(\Omega,\mathcal{F},\mathbb{P})$ and taking their values in $\mathbb{R}^2$.

Proposition 3.5 ([24])**.** *Let $(s,\varepsilon)\in\mathbb{R}^2\times(0,1]$. Assume that there exists $c\in[1,+\infty)$ such that*

$$\sum_{n\geq1}\mathbb{P}\big(|Y_n-s|<\varepsilon\big)=\infty \tag{12a}$$

$$\liminf_{N\to+\infty}\frac{\sum_{n,m=1}^N\mathbb{P}\big(|Y_n-s|<\varepsilon,\ |Y_m-s|<\varepsilon\big)}{\Big(\sum_{n=1}^N\mathbb{P}\big(|Y_n-s|<\varepsilon\big)\Big)^2}\leq c. \tag{12b}$$

Then we have: $\mathbb{P}\big(|Y_n-s|<\varepsilon\text{ i.o.}\big)\geq\dfrac{1}{c}$.

Corollary 3.6. *Let $(s,\varepsilon)\in\mathbb{R}^2\times(0,1]$. Assume that Condition (12a) is fulfilled and that there exists $d\in(0,+\infty)$ such that*

$$\liminf_{N\to+\infty}\frac{\sum_{n,m=1}^N\mathbb{P}\big(|Y_n-s|<\varepsilon,\,|Y_{n+m}-s|<\varepsilon\big)}{\Big(\sum_{n=1}^N\mathbb{P}(|Y_n-s|<\varepsilon)\Big)^2}\leq d. \tag{13}$$

Then we have: $\mathbb{P}\big(|Y_n-s|<\varepsilon\text{ i.o.}\big)\geq\dfrac{1}{2d}$.

Proof of Corollary 3.6. Let us define $p_{n,m} := \mathbb{P}(|Y_n - s| < \varepsilon,\ |Y_{n+m} - s| < \varepsilon)$. Observe that $p_{n,0} := \mathbb{P}(|Y_n - s| < \varepsilon)$. We have

$$\sum_{n,m=1}^{N} \mathbb{P}(|Y_n - s| < \varepsilon,\ |Y_m - s| < \varepsilon) \le 2\sum_{n=1}^{N}\sum_{m=n}^{N} p_{n,m-n} \le 2\sum_{n=1}^{N}\sum_{m=0}^{N} p_{n,m} = 2\left(\sum_{n,m=1}^{N} p_{n,m} + \sum_{n=1}^{N} p_{n,0}\right).$$

From (12a) and the previous inequality, we obtain (12b) with $c = 2d$. □

Notice that, even in the i.i.d. case, Corollary 3.6 does not give $p_n := \mathbb{P}(|Y_n - s| < \varepsilon \text{ i.o.}) = 1$ as expected, but only $p_n \ge 1/2$ (since constant d is equal to 1). Therefore, further arguments (here based on the additive property (1)) must be exploited to deduce the recurrence set from Corollary 3.6. The next proposition gives such a result for general Markov random walks.

Again $(X_n, S_n)_{n\in\mathbb{N}}$ denotes a MRW with state space $\mathbb{X} \times \mathbb{R}^2$, and $\mathbb{S}$ is given in (2).

Proposition 3.7. *Let $\varepsilon > 0$. Assume that there exists a real number $e_\varepsilon > 0$ such that*

$$\forall (x', s') \in \mathbb{X} \times \mathbb{S}, \quad \mathbb{P}_{(x',0)}\big(|S_n - s'| < \varepsilon \ i.o.\big) \ge e_\varepsilon. \tag{14}$$

Then, for every $(x, s) \in \mathbb{X} \times \mathbb{S}$, we have: $\mathbb{P}_{(x,0)}\big(|S_n - s| < \varepsilon \ i.o.\big) = 1$. In particular, if (14) is fulfilled for every $\varepsilon \in (0; 1)$, then we have for every $x \in \mathbb{X}$: $\mathcal{R}_{(x,0)} = \mathbb{S}$.

Proof. Suppose that $(X_n, S_n)_n$ is the canonical version defined on $\Omega := (\mathbb{X} \times \mathbb{R}^2)^{\mathbb{N}}$. Let us fix $(x, s) \in \mathbb{X} \times \mathbb{S}$, and set $S := [|S_n - s| < \varepsilon \text{ i.o.}]$. Then, from Lemma 3.4, (2) and (14), it follows that, for $\mathbb{P}_{(x,0)}-$almost every $\omega \in \Omega$, we have: $\mathbf{1}_S(\omega) \ge e_\varepsilon$. Hence: $\mathbf{1}_S = 1$ $\mathbb{P}_{(x,0)}$-a.s.. □

4. LLTs under Hypotheses **(A1)-(A6)** (Proof of Theorems III-IV)

Let $(X_n, S_n)_{n\in\mathbb{N}}$ be a Markov random walk with state space $\mathbb{X}\times\mathbb{R}^2$, let $\mathbb{S}$ be a two-dimensional closed subgroup of $\mathbb{R}^2$ satisfying (2). Hypotheses of Theorems I-II involve some local limit properties. This is obvious for Theorem I since Hypothesis (LL0) directly writes as a local limit property. The next lemma shows that this is also true for Theorem II, more precisely: Conditions (KSa) and (KSb) of Section 1 are implied by the limit properties (LLa) and (LLb).

The notations $u_n \sim v_n$ or $u_n \sim_n v_n$ refer to the usual equivalence relation between two sequences as $n \to +\infty$. We write $u_{n,m} \sim_{(n,m)} v_{n,m}$ when, for n, m large enough, we have $u_{n,m} = v_{n,m}(1 + \eta_{n,m})$ for some bounded $(\eta_{n,m})_{n,m}$ such that $\eta_{n,m} \to 0$ when $\min(n, m) \to +\infty$.

Lemma 4.1. *Let μ be a probability measure on $\mathbb{X}$ and let B be a ball in $\mathbb{R}^2$. Assume that there exist a constant $D > 0$, a sequence $(a_n)_{n\ge1}$ of positive numbers such that $\sum_{n\ge1} a_n = \infty$ and:*

$$\mathbb{P}_{(\mu,0)}\big(S_n \in B\big) \sim_n D\, a_n\, m_{\mathbb{S}}(B) \tag{15a}$$

$$\mathbb{P}_{(\mu,0)}\big((S_n, S_{n+m}) \in B^2\big) \sim_{(m,n)} D^2\, a_n a_m\, m_{\mathbb{S}}(B)^2. \tag{15b}$$

Then we have

$$\sum_{n\ge1} \mathbb{P}_{(\mu,0)}\big(S_n \in B\big) = \infty,$$

$$\lim_{N\to+\infty} \frac{\sum_{n,m=1}^{N} \mathbb{P}_{(\mu,0)}\big((S_n, S_{n+m}) \in B^2\big)}{\Big(\sum_{n=1}^{N} \mathbb{P}_{(\mu,0)}\big(S_n \in B\big)\Big)^2} = 1.$$

Proof. Set $p_{n,m} := \mathbb{P}_{(\mu,0)}((S_n, S_{n+m}) \in B^2)$. Note that $p_{n,0} := \mathbb{P}_{(\mu,0)}\big(S_n \in B\big)$. We have when $N \to +\infty$:

$$\sum_{n=1}^{N} p_{n,0} \sim_N D\, m_{\mathbb{S}}(B) \sum_{n=1}^{N} a_n \qquad \text{and} \qquad \sum_{n,m=1}^{N} p_{n,m} \sim_N D^2\, m_{\mathbb{S}}(B)^2 \left(\sum_{n=1}^{N} a_n\right)\left(\sum_{m=1}^{N} a_m\right),$$

from which we deduce the desired statement. □

In the next Propositions 4.2-4.3, the local limit properties (LL0) and (15a)-(15b) are obtained under Hypotheses **(A1)-(A6)**. Theorems III-IV are then deduced from Theorems I-II. Another interesting application to recurrence is presented in Corollary 4.4.

Recall that $B := B(s,\varepsilon)$ is the open ball in $\mathbb{R}^2$, centered at s with radius ε. The sequence $(A_n)_n$ is given in **(A5)**. Let us define the following positive constant: $D_{\mathbb{S}} := (2\pi)^{-1}\, c_{\mathbb{S}}\, (\det \Gamma)^{-1/2}$, where $c_{\mathbb{S}} = \varepsilon_{\mathbb{S}}$ for (H1) (H2), and $c_{\mathbb{S}} = |\det B|$ for (H3), where Cases (H1) (H2) (H3) are described in Section 2.

Proposition 4.2. *Assume that Hypotheses* **(A1)-(A6)** *hold true and that* $\mu \in (\widehat{\mathcal{B}})'$. *Then, for every* $(s,\varepsilon) \in \mathbb{S} \times (0;\varepsilon_{\mathbb{S}})$, *for every bounded nonnegative* $f \in \mathcal{B}$, *we have:*

$$\mathbb{E}_{(\mu,0)}\big[f(X_n)\, \mathbf{1}_{B(s,\varepsilon)}(S_n)\big] \sim_n D_{\mathbb{S}}\, \pi(f)\, A_n^{-2}\, m_{\mathbb{S}}(B(s,\varepsilon)). \tag{16}$$

Proposition 4.3. *Assume that Hypotheses* **(A1)-(A6)** *hold true and that* $\mu \in (\widehat{\mathcal{B}})'$. *Then, for every* $(s,\varepsilon) \in \mathbb{S} \times (0;\varepsilon_{\mathbb{S}})$, *we have (15a)-(15b) with* $B := B(s,\varepsilon)$, $a_n := A_n^{-2}$ *and* $D := D_{\mathbb{S}}$.

Propositions 4.2 and 4.3 are proved in Section 5.

Proof of Theorem III-IV. Condition (LL0) of Theorem I is nothing else but (16) stated with $\mu = \pi$ and for every nonnegative bounded measurable function $f : \mathbb{X} \to \mathbb{R}$. Note that $\pi \in (\widehat{\mathcal{B}})'$ from the continuous inclusion $\widehat{\mathcal{B}} \subset \mathbb{L}^1(\pi)$. Consequently Theorem III follows from Proposition 4.2. Theorem IV follows from Proposition 4.3, Lemma 4.1 and Theorem II. □

When the assumption of Theorem IV on Dirac distributions is not fulfilled, the following corollary may also be of interest. It follows from Proposition 4.3, Lemma 4.1 and Corollary 3.6.

Corollary 4.4. *Assume that Hypotheses* **(A1)-(A6)** *hold true, that* $\mu \in (\widehat{\mathcal{B}})'$. *Then we have for every* $(s,\varepsilon) \in \mathbb{S} \times (0;\varepsilon_{\mathbb{S}}]$: $\mathbb{P}_{(\mu,0)}\big(|S_n - s| < \varepsilon \text{ i.o.}\big) \geq 1/2$.

We present now some remarks concerning the operator-type Hypotheses **(A1)-(A4)** of Section 1. Further comments can be found in [18, Sect. 4-5]. Actually Hypotheses **(A1)** and **(A2)-(A3)** are the key assumptions of the weak Nagaev-Guivarc'h spectral method presented in [18, Sect. 4-5], which is used in Section 5 to prove Propositions 4.2 and 4.3.

• **Comments on Hypotheses (A1)-(A4).** The strong ergodicity condition **(A1)** only involves the driving Markov chain $(X_n)_{n\in\mathbb{N}}$ of the MRW. More specifically, defining the following rank-one projection in $\mathcal{L}(\mathcal{B})$,

$$\forall f \in \mathcal{B}, \quad \Pi f = \pi(f)\mathbf{1}_{\mathbb{X}}, \tag{17}$$

Condition **(A1)** writes as: $\lim_n \|Q^n - \Pi\|_{\mathcal{B}} = 0$. This can be easily seen that the last condition is equivalent to $\|Q^n - \Pi\|_{\mathcal{B}} = O(\kappa^n)$ for some $\kappa \in (0,1)$. Under Condition **(A1)**, the technical condition **(A4)** is satisfied in many cases, see [18, p. 436]. Mention that Inequalities $|\lambda|^n|f| \leq Q^n|f|$ and $|f| \leq \pi(|f|)$ in **(A4)** must be understood as follows: they hold, either everywhere on $\mathbb{X}$ if $\mathcal{B}$ is a space of functions, or π-almost everywhere on $\mathbb{X}$ if $\mathcal{B}$ is a space of classes modulo π.

The condition $Q(t) \in \mathcal{L}(\mathcal{Y})$ for $\mathcal{Y} = \mathcal{B}$ or $\mathcal{Y} = \widehat{\mathcal{B}}$ means that, for every $f \in \mathcal{Y}$, function $\mathbb{X} \ni x \mapsto \big(Q(t)f\big)(x)$ (or its class mod. π) belongs to $\mathcal{Y}$, and that $f \mapsto Q(t)f$ is in $\mathcal{L}(\mathcal{Y})$.

Hypotheses **(A2)**-**(A3)** enable the use of the Keller-Liverani perturbation theorem in the Nagaev-Guivarc'h spectral method. Note that Hypotheses **(A2)**-**(A3)** involve not only the transition kernel Q of the driving Markov chain $(X_n)_{n\in\mathbb{N}}$, but also the additive component S_n of the MRW. For instance, if $S_n := \sum_{k=1}^n \xi(X_k)$ is an additive functional, then **(A2)**-**(A3)** mainly focus on the function $\xi : \mathbb{X} \to \mathbb{R}^2$.

Hypothesis **(A2)** is a continuity condition involving two different spaces $\mathcal{B} \subset \widehat{\mathcal{B}}$. This condition is much weaker than in the usual perturbation theorem involving a single space (i.e. $\widehat{\mathcal{B}} = \mathcal{B}$): for instance, as illustrated in [18, Sect. 3], Hypothesis **(A2)** does not hold in general with $\widehat{\mathcal{B}} = \mathcal{B}$ in the classical Markov models considered in Applications 1-4 of Section 1. Condition (5) in **(A3)** is the so-called Doeblin-Fortet inequality: here it is required for all the $Q(t)$ in a uniform way on compact sets of $\mathbb{R}^2$.

Finally, concerning the notion of essential spectral radius, recall that $T \in \mathcal{L}(\mathcal{B})$ is said to be quasi-compact if there exist $r_0 \in (0,1)$, $m \in \mathbb{N}^*$, $\lambda_i \in \mathbb{C}$, $p_i \in \mathbb{N}^*$ $(i = 1,\dots,m)$ such that:

$$\mathcal{B} = \overset{m}{\underset{i=1}{\oplus}} \ker(T - \lambda_i I)^{p_i} \oplus H,$$

where $|\lambda_i| \geq r_0$, $1 \leq \dim \ker(T - \lambda_i I)^{p_i} < \infty$, and H is a closed T-invariant subspace such that $\sup_{h\in H,\, \|h\|_{\mathcal{B}}\leq 1} \|T^n h\|_{\mathcal{B}} = O({r_0}^n)$. If T is quasi-compact, then $r_{ess}(T)$ is the infimum bound of the real numbers r_0 such that the last conditions hold. If T is not quasi-compact, then $r_{ess}(T)$ is equal to the spectral radius of T. For further details on the essential spectral radius, in particular for the link with the Doeblin-Fortet inequalities, see [14, 15].

5. Proof of Propositions 4.2 and 4.3

5.1. **Spectral properties of $Q(t)$ under Hypotheses (A1)-(A6).** The Fourier kernels $Q(t)$ are defined in (3). Note that $Q(0) = Q$, and $Q(t+g) = Q(t)$ for every $(t,g) \in \mathbb{R}^2 \times \mathbb{S}^*$. The positive definite symmetric 2×2-matrix Γ and the slowly varying function $L(\cdot)$ used below are defined in **(A5)**. Recall that we have set in (17): $\forall f \in \mathcal{B},\ \Pi f = \pi(f)\mathbf{1}_{\mathbb{X}}$.

Proposition 5.1. *Under Hypotheses* **(A1)**-**(A6)**, *the following assertions hold true:*

(a) There exist two real numbers $\alpha > 0$ and $\kappa \in [0,1)$, a function $t \mapsto \lambda(t)$ from $B(0,\alpha)$ into $\mathbb{C}$, a bounded map $t \mapsto \Pi(t)$ from $B(0,\alpha)$ into $\mathcal{L}(\mathcal{B})$ such that $\lambda(0) = 1$, $\Pi(0) = \Pi$ and

$$\sup_{t\in B(0,\alpha)} \|Q(t)^n - \lambda(t)^n \Pi(t)\|_{\mathcal{B}} = O(\kappa^n), \tag{18}$$

$$\lambda(t) = 1 - \frac{1}{2}\langle t, \Gamma t\rangle L(|t|^{-1})\big(1 + \varepsilon(t)\big), \tag{19}$$

(b) If $\mu \in (\widehat{\mathcal{B}})'$, then

$$\forall f \in \mathcal{B},\ \lim_{t\to 0} \mu\big(\Pi(t)f\big) = \pi(f) \tag{20}$$

$$\lim_{(u,v)\to 0} \mu\big(\Pi(u)\Pi(v)\mathbf{1}_{\mathbb{X}}\big) = 1. \tag{21}$$

(c) For any compact subset K of $\mathbb{R}^2 \setminus \mathbb{S}^*$, there exists $\rho = \rho(K) \in [0,1)$ such that

$$\sup_{t\in K} \|Q(t)^n\|_{\mathcal{B}} = O(\rho^n). \tag{22}$$

Proof. Property (18) is presented in [18, Sect. 4]. Property (19) is established in [16, lem. 4.2] when the standard CLT holds in **(A5)**, see also [18, Lem. 5.2]. Extension to non-standard CLT is easy, see Appendix A. Property (22) is established in [18, Sect. 5] when $\mathbb{S} = \mathbb{R}^2$. Extension to a proper closed subgroup $\mathbb{S}$ of $\mathbb{R}^2$ is simple, see Appendix B. To obtain (20)-(21), recall that the main argument in the proof of (18) is the Keller-Liverani perturbation theorem [23], which also gives the following properties (see [18, Sect. 4] for details):

$$M := \sup_{t\in B(0,\alpha)} \|\Pi(t)\|_{\mathcal{B}} < \infty \quad \text{and} \quad \lim_{t\to 0} \|\Pi(t)-\Pi\|_{\mathcal{B},\widehat{\mathcal{B}}} = 0.$$

Then (20) follows from $\mu \in (\widehat{\mathcal{B}})'$ and the last property. In particular, since $\pi \in (\widehat{\mathcal{B}})'$, (20) holds with $\mu = \pi$. Next, using $\Pi(\Pi(v)\mathbf{1}_{\mathbb{X}}) = \pi(\Pi(v)\mathbf{1}_{\mathbb{X}})\mathbf{1}_{\mathbb{X}}$, $\Pi\mathbf{1}_{\mathbb{X}} = \mathbf{1}_{\mathbb{X}}$ and $\mu(\mathbf{1}_{\mathbb{X}}) = 1$, we obtain:

$$\begin{aligned} \left|\mu\big(\Pi(u)\Pi(v)\mathbf{1}_{\mathbb{X}}\big) - 1\right| &\le \mu\Big(\big|\big(\Pi(u)-\Pi\big)\Pi(v)\mathbf{1}_{\mathbb{X}}\big|\Big) + \left|\pi\big(\Pi(v)\mathbf{1}_{\mathbb{X}}\big) - 1\right| \\ &\le \|\mu\|_{\widehat{\mathcal{B}}}\, \big\|\Pi(u)-\Pi\big\|_{\mathcal{B},\widehat{\mathcal{B}}}\, \big\|\Pi(v)\mathbf{1}_{\mathbb{X}}\big\|_{\mathcal{B}} + \left|\pi\big(\Pi(v)\mathbf{1}_{\mathbb{X}}\big) - 1\right| \\ &\le \|\mu\|_{\widehat{\mathcal{B}}}\, \big\|\Pi(u)-\Pi\big\|_{\mathcal{B},\widehat{\mathcal{B}}}\, M\, \|\mathbf{1}_{\mathbb{X}}\|_{\mathcal{B}} + \left|\pi\big(\Pi(v)\mathbf{1}_{\mathbb{X}}\big) - 1\right|. \end{aligned}$$

Hence we have (21). □

5.2. **Preliminary lemmas.** Let f be a $\mathbb{C}$-valued bounded measurable function on $\mathbb{X}$.

Lemma 5.2. *We have for every $x \in \mathbb{X}$, every $(u,v) \in \mathbb{R}^2$ and every $(n,m) \in \mathbb{N}^2$:*

$$\mathbb{E}_{(x,0)}\big[e^{i\langle u,S_n\rangle}\, e^{i\langle v,S_{n+m}-S_n\rangle}\, f(X_{n+m})\big] = \big(Q(u)^n Q(v)^m f\big)(x).$$

Consequently, we have for any initial distribution μ on $\mathbb{X}$:

$$\mathbb{E}_{(\mu,0)}\big[e^{i\langle u,S_n\rangle}\, e^{i\langle v,S_{n+m}-S_n\rangle}\, f(X_{n+m})\big] = \mu\big(Q(u)^n Q(v)^m f\big).$$

Proof of Lemma 5.2. Using additivity property (1) (see Subsection 3.1), we obtain

$$\begin{aligned} \mathbb{E}_{(x,0)}\big[e^{i\langle u,S_n\rangle}\, e^{i\langle v,S_{n+m}-S_n\rangle}\, f(X_{n+m})\big] &= \mathbb{E}_{(x,0)}\Big[e^{i\langle u,S_n\rangle}\, \mathbb{E}_{(x,0)}\big[e^{i\langle v,S_{n+m}-S_n\rangle}\, f(X_{n+m}) \,\big|\, \mathcal{F}_n\big]\Big] \\ &= \mathbb{E}_{(x,0)}\Big[e^{i\langle u,S_n\rangle}\, \mathbb{E}_{(X_n,0)}\big[f(X_m)\, e^{i\langle v,S_m\rangle}\big]\Big]. \end{aligned} \tag{23}$$

Applying (23) with $m = 1$ and $u = v$, and according to definition (3) of Fourier maps, we obtain for every $n \ge 0$,

$$\mathbb{E}_{(x,0)}\big[e^{i\langle v,S_{n+1}\rangle}\, f(X_{n+1})\big] = \mathbb{E}_{(x,0)}\Big[e^{i\langle v,S_n\rangle}\, \mathbb{E}_{(X_n,0)}\big[f(X_1)\, e^{i\langle v,S_1\rangle}\big]\Big] = \mathbb{E}_{(x,0)}\big[e^{i\langle v,S_n\rangle}\big(Q(v)f\big)(X_n)\big].$$

We deduce by induction that we have for all $v \in \mathbb{R}^2$, $k \ge 1$, and for all $\mathbb{C}$-valued bounded measurable function g on $\mathbb{X}$:

$$\mathbb{E}_{(x,0)}\big[e^{i\langle v,S_k\rangle}\, g(X_k)\big] = \big(Q(v)^k g\big)(x).$$

Next, by applying (23) (with $m \ge 1$ and $u, v \in \mathbb{R}^2$) and using the previous equality (first with $g = f$, second with $g = Q(v)^m f$), we obtain

$$\mathbb{E}_{(x,0)}\big[e^{i\langle u,S_n\rangle}\, e^{i\langle v,S_{n+m}-S_n\rangle}\, f(X_{n+m})\big] = \mathbb{E}_{(x,0)}\big[e^{i\langle u,S_n\rangle}\, \big(Q(v)^m f\big)(X_n)\big] = \big(Q(u)^n Q(v)^m f\big)(x).$$

□

For any Lebesgue-integrable function $h : \mathbb{R}^2 \to \mathbb{C}$, we define its Fourier transform $\hat{h}$ by $\hat{h}(u) := \int_{\mathbb{R}^2} h(t)e^{-i\langle t,u\rangle}\, dt$, and we set

$$\forall t \in \mathbb{R}^2, \quad P_h(t) := \sum_{g\in\mathbb{S}^*} \hat{h}(t+g).$$

Let $\mathcal{D}$ be the fundamental domain of $\mathbb{R}^2/\mathbb{S}^*$, namely:

- $\mathcal{D} := \mathbb{R}^2$ in Case *(H1)*,
- $\mathcal{D} := [-\frac{a}{2}, \frac{a}{2}] \times \mathbb{R}$ in Case *(H2)*, with $a = 2\pi/b$,

- $\mathcal{D} := A([-\frac{1}{2}, \frac{1}{2}]^2)$ in Case *(H3)*, with $A := 2\pi(B^*)^{-1}$.

Lemma 5.3. *Let h_1 and h_2 be $\mathbb{C}$-valued Lebesgue-integrable functions on $\mathbb{R}^2$ such that their Fourier transforms are Lebesgue-integrable on $\mathbb{R}^2$. Then we have for any probability measure μ on $\mathbb{X}$ and for every $(n,m) \in \mathbb{N}^2$:*

$$\mathbb{E}_{(\mu,0)}\big[h_1(S_n)\, f(X_n)\big] = \frac{1}{(2\pi)^2}\int_{\mathcal{D}} P_{h_1}(u)\, \mu\big(Q(u)^n f\big)\, du.$$

$$\mathbb{E}_{(\mu,0)}\big[h_1(S_n)\, h_2(S_{n+m}-S_n)\, f(X_{n+m})\big] = \frac{1}{(2\pi)^4}\int_{\mathcal{D}\times\mathcal{D}} P_{h_1}(u)\, P_{h_2}(v)\, \mu\big(Q(u)^n Q(v)^m f\big)\, dudv.$$

Proof of Lemma 5.3. We easily obtain the first formula by applying the inverse Fourier formula to h_1, Lemma 5.2 (with $m=0$), and finally the fact that $Q(\cdot)$ and P_{h_1} are $\mathbb{S}^*$-periodic. The second formula can be proved similarly. □

Lemma 5.4. *Up to a reduction of the positive real number α of Proposition 5.1, there exists $\tilde{a} > 0$ such that, for every $t \in B(0,\alpha)$, we have $|\lambda(t)| \le e^{-\tilde{a}|t|^2 L(|t|^{-1})}$, and for all $n \ge 1$*

$$\left|\lambda\left(\frac{u}{A_n}\right)\right|^n \mathbf{1}_{B(0,\alpha A_n)}(u) \le \mathbf{1}_{B(0,1)}(u) + e^{-\frac{\tilde{a}}{4}|u|}\, \mathbf{1}_{\{u\,:\,1\le|u|\le\alpha A_n\}}(u). \tag{24}$$

Proof of Lemma 5.4. From (19) and the fact that Γ is positive definite, there exists $\tilde{a} > 0$ such that, for every $t \in B(0,\alpha)$ (with α possibly reduced), we have

$$|\lambda(t)| \le 1 - \tilde{a}|t|^2 L(|t|^{-1}) \le e^{-\tilde{a}|t|^2 L(|t|^{-1})}.$$

Since $L(\cdot)$ is slowly varying, we know (see [22] or [10], p. 282) that there exist two functions $\ell(\cdot)$ and $\tilde{\varepsilon}(\cdot)$ such that $\lim_{x\to+\infty} \ell(x)$ exists in $(0,+\infty)$ and $\lim_{x\to+\infty}\tilde{\varepsilon}(x) = 0$, and such that

$$L(x) = \ell(x)\exp\left(\int_1^x \frac{\tilde{\varepsilon}(y)}{y}\, dy\right). \tag{25}$$

Using this representation of L, it is easy to see that there exists n_0 such that, for any $n \ge n_0$ and any u such that $1 \le |u| \le \alpha A_n$, we have :

$$\frac{1}{2}|u|^{-1} \le \frac{L(A_n|u|^{-1})}{L(A_n)}.$$

From $A_n^2 \sim nL(A_n)$, one can also assume that, for every $n \ge n_0$ (up to a change of n_0), we have $n/A_n^2 \ge \frac{1}{2L(A_n)}$. Therefore we have: $\forall u \in B(0,\alpha A_n)$, $\forall n \ge n_0$,

$$\begin{aligned}\left|\lambda\left(\frac{u}{A_n}\right)\right|^n \mathbf{1}_{B(0,\alpha A_n)}(u) &\le e^{-n\,\tilde{a}\frac{|u|^2}{A_n^2} L(A_n|u|^{-1})}\, \mathbf{1}_{B(0,\alpha A_n)}(u)\\ &\le \mathbf{1}_{B(0,1)}(u) + e^{-\frac{\tilde{a}}{4}|u|}\, \mathbf{1}_{B(0,\alpha A_n)\setminus B(0,1)}(u).\end{aligned}$$

□

5.3. **Proof of Proposition 4.2.** Let $\mathcal{H}_2$ denote the space of Lebesgue-integrable continuous functions on $\mathbb{R}^2$ having a compactly supported Fourier transform. Let $f \in \mathcal{B}$, $f \ge 0$ be fixed. Property (16) will be proved if we establish that we have for every $h \in \mathcal{H}_2$:

$$\lim_n 2\pi\, A_n^2\, \mathbb{E}_{(\mu,0)}\big[f(X_n)\, h(S_n)\big] = c_{\mathbb{S}}\, (\det\Gamma)^{-1/2}\, \pi(f)\, m_{\mathbb{S}}(h). \tag{26}$$

Indeed, (26) ensures that the sequence $(\nu_n)_n$ of positive measures defined by

$$\forall C \in \mathcal{B}(\mathbb{R}^2),\quad \nu_n(C) := 2\pi\, A_n^2\, \mathbb{E}_{(\mu,0)}\big[f(X_n)\, \mathbf{1}_C(S_n)\big],$$

converges weakly to measure $\nu(\cdot) := c_{\mathbb{S}}\, (\det\Gamma)^{-1/2}\, \pi(f)\, m_{\mathbb{S}}(\cdot)$, see [4]. Since the boundary of $B := B(s,\varepsilon)$ has zero $\nu-$measure when $\varepsilon \in (0,\varepsilon_{\mathbb{S}})$, we have: $\lim_n \nu_n(B) = \nu(B)$, which is (16).

Proof of (26). Note that

- $m_{\mathbb{S}}(h) := \int_{\mathbb{R}^2} h(t)dt$ in Case *(H1)*,
- $m_{\mathbb{S}}(h) := \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} h(bn, y)\, dy$ in Case *(H2)*,
- $m_{\mathbb{S}}(h) := \sum_{\eta\in\mathbb{S}} h(\eta)$ in Case *(H3)*.

Let $h \in \mathcal{H}_2$. Let β be a positive real number such that Supp$(\hat{h}) \subset B(0,\beta)$. Without loss of generality, one can suppose that the positive real numbers β and α (of (18)) are such that $\alpha < a/2 < \beta$ in Case *(H2)*, and $B(0,\alpha) \subset A([-\frac{1}{2}, \frac{1}{2}]^2) \subset B(0,\beta)$ in Case *(H3)*. We set

$$K := \big(\overline{B}(0,\beta) \setminus B(0,\alpha)\big) \cap \mathcal{D}. \tag{27}$$

Observe that K is a compact subset of $\mathbb{R}^2 \setminus \mathbb{S}^*$. Let $\rho \in (0;1)$ be defined in (22) w.r.t. K, and set $r := \max(\kappa, \rho)$, where κ is defined in (18). Using (18) and (22), we abuse the notation $O(r^n)$ for $Q(u)^n - \lambda(u)^n\Pi(u)$ when $u \in B(0,\alpha)$, and for $Q(u)^n$ when $u \in K$. So we have:

$$\forall u \in B(0,\beta), \quad Q(u)^n = \mathbf{1}_{B(0,\alpha)}(u)\, \lambda(u)^n\, \Pi(u) + O(r^n) \ \text{ in } \mathcal{L}(\mathcal{B}), \tag{28}$$

where $\Pi(\cdot)$ is the $\mathcal{L}(\mathcal{B})$-valued bounded function in (18). Recall that, by hypothesis, $f \in \mathcal{B}$, $\mu \in (\widehat{\mathcal{B}})'$. Since $\hat{h}$ is integrable, we then deduce from Lemma 5.3 and (28) that

$$\begin{aligned}(2\pi)^2\, \mathbb{E}_{(\mu,0)}\big[f(X_n)\, h(S_n)\big] &= \int_{B(0,\alpha)} P_h(u)\, \lambda(u)^n \mu\big(\Pi(u) f\big)\, du + O(r^n)\\ &= \frac{1}{A_n^2} \int_{B(0,\alpha A_n)} P_h\left(\frac{u}{A_n}\right) \lambda\left(\frac{u}{A_n}\right)^n \mu\left(\Pi\left(\frac{u}{A_n}\right) f\right) du + O(r^n).\end{aligned}$$

Next, from (19), $A_n^2 \sim nL(A_n)$ and from the fact that L is slowly varying, it can be easily seen that $\lim_n \lambda\big(\frac{u}{A_n}\big)^n = e^{-\langle u,\Gamma u\rangle/2}$. Moreover we know by (20) that $\lim_n \mu(\Pi(u/A_n)f) = \pi(f)$. By using (24), Lebesgue's theorem gives:

$$\begin{aligned}\lim_{n\to+\infty} \int_{B(0,\alpha A_n)} P_h\left(\frac{u}{A_n}\right) \lambda\left(\frac{u}{A_n}\right)^n \mu\left(\Pi\left(\frac{u}{A_n}\right) f\right) du &= \pi(f)\, P_h(0) \int_{\mathbb{R}^2} e^{-\langle u,\Gamma u\rangle/2}\, du\\ &= 2\pi(\det \Gamma)^{-1/2}\, \pi(f)\, P_h(0).\end{aligned}$$

Finally, the Poisson summation formula yields $P_h(0) := \sum_{g\in\mathbb{S}^*} \hat{h}(g) = c_{\mathbb{S}}\, m_{\mathbb{S}}(h)$. $\square$

5.4. **Proof of Proposition 4.3.** Proposition 4.2 applied to $f = \mathbf{1}_{\mathbb{X}}$ gives (15a). To prove (15b), let us first state a lemma concerning the sequence $(S_n, S_{n+m} - S_n)_{n,m}$.

Lemma 5.5. *The sequence $(\nu_{n,m})_{n,m}$ of positive measures on $\mathbb{R}^4$ defined by*

$$\forall C \in \mathcal{B}(\mathbb{R}^4), \quad \nu_{n,m}(C) := (2\pi)^2 A_n^2 A_m^2\, \mathbb{E}_{(\mu,0)}\big[\mathbf{1}_C(S_n, S_{n+m} - S_n)\big],$$

converges weakly, as $\min(n,m) \to +\infty$, to the measure ν defined by: $\nu(C) := c_{\mathbb{S}}^2\, (det(\Gamma))^{-1}\, m_{\mathbb{S}} \otimes m_{\mathbb{S}}(C)$.

Before proving this lemma, let us first show how it is used to give (15b). Let T be the linear (invertible) endomorphism on $\mathbb{R}^4$ defined by: $Tw := (u, u+v)$, where we write $w = (u,v) \in \mathbb{R}^4$, with u and v in $\mathbb{R}^2$. From Lemma 5.5, the family of measures $\tilde{\nu}_{n,m}$ on $\mathbb{R}^4$ defined by

$$\tilde{\nu}_{n,m}(C) := \nu_{n,m}(\mathbf{1}_C \circ T) = (2\pi)^2 A_n^2 A_m^2\, \mathbb{P}_{(\mu,0)}\big((S_n, S_{n+m}) \in C\big)$$

converges weakly to $\tilde{\nu}(C) := \nu(\mathbf{1}_C \circ T)$ when $\min(n,m) \to +\infty$. But, from Fubini's theorem and since $m_{\mathbb{S}}$ is the Haar measure, we have $\tilde{\nu} = \nu$. Now set $B := B(s,\varepsilon)$ for $(s,\varepsilon) \in \mathbb{S} \times (0;\varepsilon_{\mathbb{S}})$. Since the boundary of $B \times B$ has zero $\nu-$measure, we obtain the following convergence when $\min(n,m) \to +\infty$: $\lim \tilde{\nu}_{n,m}(B \times B) = \nu(B \times B) = c_{\mathbb{S}}^2\, (det(\Gamma))^{-1}\, m_{\mathbb{S}}(B)^2$, which is (15b).

Proof of Lemma 5.5. Observe that there exists a continuous $m_{\mathbb{S}}$-integrable function $h > 0$ on $\mathbb{R}^2$ having a compactly supported Fourier transform, see [4, Section 10.2]. Define the following function on $\mathbb{R}^4$: $G(w) := h(u)h(v)$, where $w = (u,v) \in \mathbb{R}^4$, with u and v in $\mathbb{R}^2$. Then the Fourier transform of G is compactly supported on $\mathbb{R}^4$, and we have $G(w)\, e^{i\langle w,c\rangle} = h(u)\, e^{i\langle u,a\rangle}\, h(v)\, e^{i\langle v,b\rangle}$ for any $c = (a,b) \in \mathbb{R}^4$, with a and b in $\mathbb{R}^2$. Therefore, using again classical properties on convergence of positive measures [4], Lemma 5.5 will be established provided that we prove the following: $\forall (h_1,h_2) \in \mathcal{H}_2 \times \mathcal{H}_2$,

$$\lim_{n\to+\infty} (2\pi)^2 A_n^2 A_m^2 \mathbb{E}_{(\mu,0)}\left[h_1(S_n)\, h_2(S_{n+m} - S_n)\right] = c_{\mathbb{S}}^2\, (det(\Gamma))^{-1}\, m_{\mathbb{S}}(h_1)\, m_{\mathbb{S}}(h_2). \tag{29}$$

Let $h_1, h_2 \in \mathcal{H}_2$ be fixed, and let $\beta > 0$ be such that both $\mathrm{Supp}(\hat{h}_1)$ and $\mathrm{Supp}(\hat{h}_2)$ are contained in $B(0,\beta)$. Real numbers α, κ in (18), and ρ in (22), are chosen as in the previous proof, and again we set $r := \max(\kappa,\rho)$. We obtain by using (28): $\forall (u,v) \in B(0,\beta)^2$,

$$Q(u)^n\, Q(v)^m = \mathbf{1}_{B(0,\alpha)}(u)\, \mathbf{1}_{B(0,\alpha)}(v)\, \lambda(u)^n\, \lambda(v)^m\, \Pi(u)\, \Pi(v) + O(r^{\min(n,m)}) \quad \text{in } \mathcal{L}(\mathcal{B}).$$

Using the second formula of Lemma 5.3 (with $f = \mathbf{1}_{\mathbb{X}}$) and Property (21), the arguments used to prove (26) can be easily extended to prove (29). □

6. Proof (and complements) for applications 1 to 4 of Section 1

6.1. ρ-mixing case (Proof of Application 1 of Section 1). For $p \in \mathbb{N}^*$ and $q \in \mathbb{N}^* \cup \{\infty\}$ with $p \le q$, let $\mathcal{G}_p^q$ denote the σ-algebra $\sigma(X_p,\ldots,X_q)$ generated by $X_p,\ldots,X_q$. The ρ-mixing coefficient of $(X_n)_{n\in\mathbb{N}}$ at horizon $k \ge 1$ is defined by

$$\rho(k) := \sup_{j\in\mathbb{N}^*} \sup \left\{ |\mathrm{Corr}(f;h)|,\ f \in \mathbb{L}^2(\mathcal{G}_1^j),\, h \in \mathbb{L}^2(\mathcal{G}_{k+j}^\infty) \right\}. \tag{30}$$

where $\mathrm{Corr}(f;h)$ is the correlation coefficient of the two random variables f and g.

In Application 1 of Section 1, the driving Markov chain $(X_n)_{n\in\mathbb{N}}$ of the MRW $(X_n,S_n)_{n\in\mathbb{N}}$ is assumed to be ρ-mixing, namely

$$\lim_{k\to+\infty} \rho(k) = 0.$$

The previous property is equivalent to the following spectral gap property of the transition kernel Q of $(X_n)_n$ with respect to the Lebesgue space $\mathbb{L}^2(\pi)$, see [28]:

$$\lim_{n\to+\infty} \sup \left\{ \|Q^n f - \pi(f)\|_{\mathbb{L}^2(\pi)},\ f \in \mathbb{L}^2(\pi),\, \|f\|_2 \le 1 \right\} = 0.$$

Classical Markov models satisfying this property are reviewed in [11].

Proof of Application 1 of Section 1. The operator-type hypotheses **(A1)-(A4)** hold with $\mathcal{B} := \mathbb{L}^2(\pi)$ and $\widehat{\mathcal{B}} := \mathbb{L}^1(\pi)$: this is established in [18, Sec. 4-5] for additive functionals. Extension to general MRW is straightforward. Moreover, since by hypothesis $\mathbb{E}_{(\pi,0)}[|S_1|^2] < \infty$ and $\mathbb{E}_{(\pi,0)}[S_1] = 0$, $(S_n/\sqrt{n})_n$ converges in distribution under $\mathbb{P}_{(\pi,0)}$ to a Gaussian distribution $\mathcal{N}(0,\Gamma)$, see [11, Th. 1]. Then Application 1 of Section 1 follows from Theorem III. □

Let us mention that the convergence to stable distributions of additive functionals associated with ρ-mixing Markov chains is investigated in [20]. Unfortunately the non-standard CLT is not studied in [20].

6.2. **V-geometrical ergodicity case (Proof of Application 2 of Section 1).** Given some unbounded function $V : \mathbb{X} \to [1, +\infty)$, $(X_n)_{n\in\mathbb{N}}$ is assumed to be V-geometrically ergodic, namely we have $\pi(V) < \infty$ and there exists $\kappa \in (0,1)$ such that we have:

$$\sup_{|f|\le V} \sup_{x\in\mathbb{X}} \frac{\big|\mathbb{E}_x[f(X_n)] - \pi(f)\big|}{V(x)} = O(\kappa^n),$$

where functions $f : \mathbb{X} \to \mathbb{C}$ are assumed to be measurable. The V-geometrical ergodicity condition can be investigated with the help of the so-called drift conditions. For this fact and for classical examples of such models, we refer to [25].

Corollary 6.1. *Let $\xi : \mathbb{X} \to \mathbb{R}^2$ be a π-centered function taking values in a two-dimensional closed subgroup $\mathbb{S}$ of $\mathbb{R}^2$. Set $S_n := \sum_{k=1}^n \xi(X_k)$.*

Under Hypotheses **(A5)**-**(A6)***, if $|\xi|^\alpha/V$ is bounded for some $\alpha > 0$, then for every initial distribution μ of $(X_n)_{n\in\mathbb{N}}$, the recurrence set $\mathcal{R}_{(\mu,0)}$ of $(S_n)_n$ satisfies $\mathcal{R}_{(\mu,0)} = \mathbb{S}$.*

Proof of Corollary 6.1. For $\gamma \in (0,1]$, denote by $(\mathcal{B}_\gamma, \|\cdot\|_\gamma)$ the space of measurable $\mathbb{C}$-valued functions f on $\mathbb{X}$ such that $\|f\|_\gamma = \sup_{x\in E} |f(x)|/V(x)^\gamma < \infty$. Let $\gamma_0 \in (0,1)$. From [18, Sect. 10], operator-type assumptions **(A1)** to **(A4)** are fulfilled with $\mathcal{B} = \mathcal{B}_{\gamma_0}$ and $\widehat{\mathcal{B}} = \mathcal{B}_1$ (this is proved in [18, Lem. 10.1] with $\widehat{\mathcal{B}} := \mathbb{L}^1(\pi)$; the case $\widehat{\mathcal{B}} = \mathcal{B}_1$ is similar, use [18, Lem. 10.4]). The assumption of Theorem IV concerning the δ_x's is obviously fulfilled. Then Corollary 6.1 follows from Theorem IV. □

Proof of Application 2(a) of Section 1. If $|\xi|/\sqrt{V}$ is bounded on $\mathbb{X}$, then the domination assumption of Corollary 6.1 is fulfilled, and $(S_n)_n$ satisfies the standard CLT. If moreover ξ is non-sublattice in $\mathbb{S}$, then the covariance matrix Γ of the CLT is automatically positive definite from Remark 2.2. The last remarks together with Corollary 6.1 give Application 2(a) of Section 1. □

Alternative conditions for the CLT can be found in [21]. To the best of our knowledge, the non-standard CLT has not been investigated for V-geometrically ergodic Markov chains.

Proof of Application 2(b) of Section 1. If $\sup_{x,y} |\xi(x,y)|^{2+\varepsilon}/(V(x) + V(y)) < \infty$, then **(A1)**-**(A4)** are fulfilled with $\mathcal{B} = \mathcal{B}_{\gamma_0}$ (for some $\gamma_0 \in (0,1)$) and $\widehat{\mathcal{B}} = \mathcal{B}_1$ (use [18, Lem. 10.1] and [17, Lemma B.2]). Moreover, from [17, Lemma 1] and Levy's theorem, it can be easily seen that $S_n := \sum_{k=1}^n \xi(X_{k-1}, X_k)$ satisfies the standard CLT. We conclude thanks to Theorem IV. □

6.3. **Case of Lipschitz iterative models (Proof of Application 3 of Section 1).** Here $(\mathbb{X}, d)$ is a non-compact metric space in which every closed ball is compact. $\mathbb{X}$ is endowed with its Borel σ-field $\mathcal{X}$. Let $(V, \mathcal{V})$ be a measurable space, let $(\vartheta_n)_{n\ge1}$ be a i.i.d. sequence of random variables taking values in V, let $F : \mathbb{X} \times V \to \mathbb{X}$ be a measurable function. Given a $\mathbb{X}$-valued r.v. X_0 independent of $(\vartheta_n)_{n\ge1}$, the random iterative model associated to $((\vartheta_n)_{n\ge1}, F, X_0)$ is defined by (see [8])

$$X_n = F(X_{n-1}, \vartheta_n), \quad n \ge 1.$$

Let us consider the two following random variables which are classical in these models (see [8]) :

$$\mathcal{C} := \sup\left\{ \frac{d\big(F(x,\vartheta_1), F(y,\vartheta_1)\big)}{d(x,y)},\ x, y \in \mathbb{X},\ x \ne y \right\} \quad \text{and} \quad \mathcal{M} = 1 + \mathcal{C} + d\big(F(x_0,\vartheta_1), x_0\big)$$

where x_0 is some fixed point in $\mathbb{X}$.

Corollary 6.2. *Assume that $\mathcal{C} < 1$ almost surely, that $\mathbb{E}[\mathcal{M}^\delta] < \infty$ for some arbitrary small $\delta > 0$, and finally that ξ satisfies the following weighted-Lipschitz condition:*

$$\exists C \geq 0,\ \exists b \geq 0,\ \forall (x,y) \in \mathbb{X}^2,\ \ \big|\xi(x) - \xi(y)\big| \leq C\, d(x,y)\, \big[1 + d(x,x_0) + d(y,x_0)\big]^b, \tag{31}$$

Then, under Hypotheses **(A5)**-**(A6)**, *the AF $(S_n)_n$ defined in (11) satisfies $\mathcal{R}_{(\mu,0)} = \mathbb{S}$ for every initial distribution μ on $\mathbb{X}$.*

Proof. Set: $\forall x \in \mathbb{X},\ p(x) = 1 + d(x,x_0)$. For $0 < \alpha \leq 1$ and $\gamma > 0$, let $\mathcal{B}_{\alpha,\gamma}$ be the space of all $\mathbb{C}$-valued functions on $\mathbb{X}$ satisfying the following condition

$$m_{\alpha,\gamma}(f) = \sup\left\{\frac{|f(x) - f(y)|}{d(x,y)^\alpha\, p(x)^{\alpha\gamma}\, p(y)^{\alpha\gamma}},\ x,y \in \mathbb{X},\ x \neq y\right\} < +\infty.$$

Set $|f|_{\alpha,\gamma} = \sup_{x\in\mathbb{X}} |f(x)|/p(x)^{\alpha(\gamma+1)}$ and $\|f\|_{\alpha,\gamma} = m_{\alpha,\gamma}(f) + |f|_{\alpha,\gamma}$. Then $(\mathcal{B}_{\alpha,\gamma}, \|\cdot\|_{\alpha,\gamma})$ is a Banach space (this corresponds to the space $\mathcal{B}_{\alpha,\beta,\gamma}$ of [18, Sect. 11] in case $\beta = \gamma$). Now let us assume that $\gamma > b+1$ and $2\alpha\gamma < \delta$, and consider $\gamma' > \gamma$ such that $b + 1 + (\gamma' - \gamma) \leq \gamma$ and $\alpha(\gamma' + \gamma) \leq \delta$. Using assumptions on $\mathcal{C}$ and $\mathcal{M}$, it follows from [18, p. 483] that the operator-type assumptions **(A1)** to **(A4)** are fulfilled with $\mathcal{B} := \mathcal{B}_{\alpha,\gamma}$ and with $\widehat{\mathcal{B}}$ defined as the Banach space of all the $\mathbb{C}$-valued functions on $\mathbb{X}$ satisfying $|f|_{\alpha,\gamma'} = \sup_{x\in\mathbb{X}} |f(x)|/p(x)^{\alpha(\gamma'+1)}$. Note that inclusion $\widehat{\mathcal{B}} \subset \mathbb{L}^1(\pi)$ is continuous since $p(\cdot)^{\alpha(\gamma'+1)}$ is π-integrable (use $\alpha(\gamma'+1) \leq \delta$ and [18, Prop. 11.1]). Then Corollary 6.2 follows from Theorem IV. □

Proof of Application 3 of Section 1. From $\mathcal{C} < 1$ and $\mathbb{E}[\mathcal{M}^{2(b+1)}] < \infty$, $(S_n)_n$ satisfies the standard CLT, see [18, Prop. 11.3]. From non-sublattice condition in $\mathbb{S}$, the covariance matrix Γ of the CLT is positive definite, see Remark 2.2. We conclude thanks to Corollary 6.2. □

A more precise use of results of [18, Sect. 11] enables to obtain the conclusion of Corollary 6.2 under some weaker mean contractive conditions on $\mathcal{C}$ (instead of $\mathcal{C} < 1$ a.s.). Alternative conditions for the CLT can be found in [8, 2, 35] and references therein.

6.4. **Complement on Application 4 of Section 1.** Let $(B_n, A_n)_n$ be a sequence of i.i.d. random variables with values in $\mathbb{R}^2 \rtimes Sim(\mathbb{R}^2)$, independent of X_0, where $Sim(\mathbb{R}^2)$ is the similarity group of $\mathbb{R}^2$. Let us consider the affine iterative model

$$X_n = A_n X_{n-1} + B_n.$$

Now assumptions and statements of Application 4 (Section 1) are specified. Let ν denote the distribution of (B_1, A_1). We write $\bar\nu$ the projection of ν on $Sim(\mathbb{R}^2)$, we denote by $G_{\bar\nu}$ the closed subgroup of $Sim(\mathbb{R}^2)$ generated by the support of $\bar\nu$. We recall that $Sim(\mathbb{R}^2) = \mathbb{R}_+^* \times O(\mathbb{R}^2)$, where $O(\mathbb{R}^2)$ is the orthogonal group. We also denote by $|\cdot|$ the matrix norm associated with the euclidean norm on $\mathbb{R}^2$. We suppose that:

- there exists a unique stationary distribution π and its support is unbounded,
- no affine subspace of $\mathbb{R}^2$ is invariant by the support of ν,
- $\mathbb{E}[|A_1|^2] = 1$, $\mathbb{E}[|A_1|^2 \log|A_1|] < \infty$ and $\mathbb{E}[|B_1|^2] < \infty$,
- the projection of $G_{\bar\nu}$ on $\mathbb{R}_+^*$ is equal to $\mathbb{R}_+^*$.

In this case, measure π admits an expectation in $\mathbb{R}^2$, called m_0, and there exists a gaussian random variable Z such that, for every $x \in \mathbb{R}^2$, under the probability measure $\mathbb{P}_x$, the following sequence of random variables

$$\left(\frac{1}{\sqrt{n\log(n)}}\Big(\sum_{k=1}^n X_k - n m_0\Big)\right)_{n\geq 1}$$

converges in distribution to Z ([5, Th. 1.5]).

Corollary 6.3. *If $x \mapsto x$ is non-sublattice in $\mathbb{R}^2$, then, for every $(x, s) \in \mathbb{R}^2 \times \mathbb{R}^2$, we have*

$$\mathbb{P}_x\left(\forall\varepsilon > 0,\ \left|\sum_{k=1}^{n} X_k - nm_0 - s\right| < \varepsilon,\ i.o.\right) = 1.$$

Proof. Again we apply Theorem IV. Thanks to Lemmas 3.9 and 3.12 of [5], operator-type assumptions **(A1)** to **(A4)** hold true on some Lipschitz weighted spaces similar to those introduced in Subsection 6.3. Probabilistic-type assumptions follow from the above non-standard CLT (see also [5, Prop 3.18]) and from the non-sublattice assumption on $\xi(x) = x$. □

Appendix A. Complement on Assertion (a) in Proposition 5.1

Proposition A.1. *Assume that Q is strongly ergodic on $\mathcal{B}$ (see* **(A1)***) and that*

(A2') *There exists $\alpha > 0$ such that, for every $t \in B(0, \alpha)$, we have $Q(t) \in \mathcal{L}(\mathcal{B}) \cap \mathcal{L}(\widehat{\mathcal{B}})$ and:*

$$\forall t \in B(0, \alpha),\ \lim_{h\to 0} \|Q(t + h) - Q(t)\|_{\mathcal{B},\widehat{\mathcal{B}}} = 0, \tag{32}$$

(A3') *There exist $\kappa_1 \in [0, 1)$ and $C \in (0; +\infty)$ such that*

$$\forall n \geq 1,\ \forall t \in B(0, \alpha),\ \forall f \in \mathcal{B},\quad \|Q(t)^n f\|_{\mathcal{B}} \leq C\,\kappa_1^n\,\|f\|_{\mathcal{B}} + C\,\|f\|_{\widehat{\mathcal{B}}}.$$

Then Property (18) of Proposition 5.1 is fulfilled. Moreover properties (19) and **(A5)** *are equivalent (with the same covariance matrix Γ and the same function $L(\cdot)$).*

Since **(A2)**-**(A3)** imply **(A2')**-**(A3')**, Proposition A.1 completes the proof of Proposition 5.1(a).

Proof. The fact that (18) holds under Hypotheses **(A1)** and **(A2')**-**(A3')** follows from [16, p. 428]. The equivalence between (19) and **(A5)** is proved in [16, lem. 4.2] when $A_n = \sqrt{n}$ in **(A5)**, see also [18, Lem. 5.2]. When the non-standard CLT holds in **(A5)**, the proof is similar, we just outline below the main arguments. Without loss of generality, we suppose that Γ is the identity matrix. First observe that we have by Lemma 5.2 (applied with $f = \mathbf{1}_{\mathbb{X}}$ and $m = 0$):

$$\forall u \in B(0, \alpha),\quad \mathbb{E}_{(\pi,0)}\big[e^{i\langle u,S_n\rangle}\big] = \pi\big(Q(u)^n\mathbf{1}_{\mathbb{X}}\big). \tag{33}$$

The proof of the "if-part" in Proposition A.1 is easy: indeed, assume that (19) holds, and let $(A_n)_n$ be a sequence of positive real numbers such that $A_n^2 \sim nL(A_n)$. From (33) and (18), we obtain for any fixed $t \in \mathbb{R}^2$ and for n sufficiently large

$$\mathbb{E}_{(\pi,0)}\big[e^{i\langle t,S_n/A_n\rangle}\big] = \lambda\big(t/A_n\big)^n\,\pi\big(\Pi(t/A_n)\mathbf{1}_{\mathbb{X}}\big) + O(\kappa^n).$$

Using (19), $A_n^2 \sim nL(A_n)$ and the fact that L is slowly varying, one can easily see that $\lim_n \lambda\big(t/A_n\big)^n = e^{-|t|^2/2}$. Hence the desired CLT in Hypothesis **(A5)** holds true.

Conversely, assume that **(A5)** holds. Let us prove that the function $\lambda(\cdot)$ in (18) satisfies:

$$\psi(u) := \frac{\lambda(u) - 1}{|u|^2\,L\big(|u|^{-1}\big)} + 1/2 \to 0 \quad \text{when } u \to 0.$$

From Levy's theorem, we have: $\forall t \in B(0, \alpha)$, $\lim_n \mathbb{E}_{(\pi,0)}[e^{i\langle t,S_n/A_n\rangle}] = \exp(-|t|^2/2)$. Thus, by using (33), (18) and the complex logarithm function $\log(\cdot)$, this gives $\lim_n n\log\lambda(t/A_n) = -|t|^2/2$, from which we easily deduce: $\lim_n \psi(t/A_n) = 0$ (use $\log(z) \sim (z - 1)$ when $z \to 1$, $n \sim A_n^2/L(A_n)$ and $L(A_n) \sim L(A_n/|t|)$). More precisely, by using the classical refinement of Levy's theorem in terms of uniform convergence on compact sets and the fact that the property $\lim_{x\to+\infty} \frac{L(kx)}{L(x)} = 1$ is uniform in k on each compact subset of $(0; +\infty)$ (according to formula

(25)), one can see that the limit $\lim_n \psi(t/A_n) = 0$ is uniform on $C := \{t \in \mathbb{R}^2 : \alpha/2 \le |t| < \alpha\}$ (see [16, lem. 4.2] for details). So, given $\varepsilon > 0$, one can choose $N_0 = N_0(\varepsilon)$ such that: $n \ge N_0,\ t \in C \ \Rightarrow\ |\psi(t/A_n)| < \varepsilon$. Next, since $\lim_n A_{n+1}/A_n = 1$, one can suppose that N_0 is such that: $\forall n \ge N_0,\ 1/(2A_n) < 1/A_{n+1}$. From that, we easily deduce that $\cup_{n \ge N_0} C/A_n = \{t \in \mathbb{R}^2 : 0 < |u| < \alpha/A_{N_0}\}$. Therefore we have: $0 < |u| < \alpha/A_{N_0} \ \Rightarrow\ |\psi(u)| < \varepsilon$. $\square$

Remark A.2. *Expansion (19) may be adapted to cover the convergence in distribution of S_n (properly normalized) to a stable distribution of index $0 < p < 2$. Then Propositions 4.2 and 4.3 extend (with A_n such that $A_n^p \sim n\,L(A_n)$), but we have $\sum_{n\ge1} A_n^{-2} < \infty$ since for n large enough, $A_n^{p/2-1} \le L(A_n)$, $nL(A_n) \le 2A_n^p$ and thus $A_n^{-2} \le (2/n)^{4/(p+2)}$. Therefore we obtain: $\sum_{n\ge1} \mathbb{P}_{(\mu,0)}\big(|S_n - s| < \varepsilon\big) < \infty$. This gives the expected transience property.*

Appendix B. Complement on Assertion (*c*) in Proposition 5.1

Without loss of generality, we suppose that the MRW $(X_n, S_n)_{n\in\mathbb{N}}$ is the canonical version defined on $\Omega = (\mathbb{X} \times \mathbb{R}^2)^{\mathbb{N}}$. We recall that, in the sense given in (2), S_1 takes its values in a closed two-dimensional subgroup $\mathbb{S}$ of $\mathbb{R}^2$: this corresponds to cases *(H1) (H2) (H3)* described at the beginning of Section 2. In this appendix, we prove that, under Hypotheses **(A1)**-**(A4)**, Property (22) is linked to the non-sublattice condition of Hypothesis **(A6)**. To that effect, introduce the following:

Definition B.1. *Under Hypothesis (2) we shall say that $(S_n)_n$ is* **arithmetic** *in $\mathbb{S}$ w.r.t. $\mathcal{B}$ if there exist $t \in \mathbb{R}^2 \setminus \mathbb{S}^*$, $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, $w \in \mathcal{B}$ such that, for π-almost every $x \in \mathbb{X}$, we have $|w(x)| = 1$ and the following property:*

$$\forall n \ge 1, \quad e^{i\langle t, S_n\rangle}\, w(X_n) = \lambda^n\, w(x) \quad \mathbb{P}_{(x,0)} - a.s.. \tag{34}$$

Under Hypothesis **(A1)**-**(A4)**, we consider the set

$$G := \{t \in \mathbb{R}^2 \ :\ r(Q(t)) = 1\}.$$

Recall that the dual subgroup $\mathbb{S}^*$ of $\mathbb{S}$ is defined in (8). Since $Q(0) = Q$, $r(Q) = 1$ and $Q(\cdot)$ is $\mathbb{S}^*$-periodic, $\mathbb{S}^*$ is contained in G.

Proposition B.2. *Assume that Hypotheses* **(A1)**-**(A4)** *hold true. Then the following assertions hold:*

(i) Property (22) $\Leftrightarrow$ $G = \mathbb{S}^*$ $\Leftrightarrow$ $(S_n)_n$ is not arithmetic in $\mathbb{S}$ w.r.t. $\mathcal{B}$;

(ii) If Hypotheses **(A5)**-**(A6)** *hold, then Property (22) is fulfilled;*

(iii) If $(S_n)_n$ is sublattice in $\mathbb{S}$ and the function $\chi(\cdot)$ in (6) is such that, for every $t \in \mathbb{R}^2$, we have $e^{i\langle t,\chi(\cdot)\rangle} \in \mathcal{B}$, then Property (22) does not hold.

Remark B.3. *In ρ-mixing or V-geometrical ergodicity cases (see Subections 6.1 and 6.2), the condition on $\chi(\cdot)$ in Assertion (iii) is automatically fulfilled, so that the non-sublattice assumption is equivalent to Condition (22). For Lipschitz iterative models, the non-sublattice assumption is just a sufficient condition for Condition (22) to hold true on the weighted-Lipschitz spaces defined in Subsection 6.3 (because the condition on $\chi(\cdot)$ in Assertion (iii) of Proposition B.2 is not automatically fulfilled). The non-arithmeticity condition, which is equivalent to Condition (22), can be simplified in the special case of additional functionals (see* [18, Section 5]*).*

When $\mathbb{S} = \mathbb{R}^2$ and S_n is an additive functional (see (11)), Proposition B.2 is established in [18, Section 12]. Here we give the adaptation to general MRWs and subgroups $\mathbb{S}$.

Proof of Proposition B.2. First, using Hypotheses **(A1)-(A4)** and the same arguments as in [18, Lem. 12.1], we obtain

$$\forall t \in \mathbb{R}^2 \setminus G, \quad r(Q(t)) < 1. \tag{35}$$

Second, an easy adaptation of [18, Lem. 12.3] shows that, for any compact subset K of $\mathbb{R}^2 \setminus G$, there exists $\rho = \rho(K) \in [0,1)$ such that

$$\sup_{t \in K} \|Q(t)^n\|_{\mathcal{B}} = O(\rho^n) \tag{36}$$

(consider compact subsets K of $\mathbb{R}^2 \setminus G$ instead of compact subsets of $\mathbb{R}^2 \setminus \{0\}$ in the proof of [18, Lem. 12.3]). Next, since $\mathbb{S}^* \subset G$, the previous property yields the first equivalence in *(i)*: indeed, if $G = \mathbb{S}^*$, then (36) obviously gives (22) . Conversely, if (22) is true, then for every $t \in \mathbb{R}^2 \setminus \mathbb{S}^*$ we have $r(Q(t)) < 1$, thus $t \in \mathbb{R}^2 \setminus G$. Therefore Condition (22) gives $G \subset \mathbb{S}^*$, hence $G = \mathbb{S}^*$.

In addition [18, Lem. 12.1] gives the following equivalence:

Property (A): we have $t \in G$ if and only if there exist $\lambda \in \mathbb{C}$, $|\lambda| = 1$, and $w \in \mathcal{B}$, $w \neq 0$, such that we have the following equality: $Q(t)w = \lambda w$ in $\mathcal{B}$. Moreover the previous function $w(\cdot)$ is such that $|w| = \pi(|w|)$ π-a.s..

The fact that $Q(t)w = \lambda w$ implies $|w| = \pi(|w|)$ π-a.s. is easy to obtain. Indeed, we have $|w| = |\lambda^n w| = |Q(t)^n w| \leq Q^n |w|$ for every $n \geq 1$, thus we deduce from **(A4)** that $|w| \leq \pi(|w|)$ π-a.s. So $g := \pi(|w|) - |w|$ is nonnegative, and $\pi(g) = 0$, hence $|w| = \pi(|w|)$ π-a.s..

Finally, from the previous property, we can deduce the following.

Property (B): we have $\mathbb{S}^* \neq G$ if and only if there exist $t \in \mathbb{R}^2 \setminus \mathbb{S}^*$, $\lambda \in \mathbb{C}$, $|\lambda| = 1$, and $w \in \mathcal{B}$, $w \neq 0$, such that $|w| = 1$ π-a.s. and $Q(t)w = \lambda w$ in $\mathcal{B}$.

To prove the second equivalence in *(i)*, one needs the following.

Lemma B.4. *We have $\mathbb{S}^* \neq G$ if and only if there exist $t \in \mathbb{R}^2 \setminus \mathbb{S}^*$, $\lambda \in \mathbb{C}$, $|\lambda| = 1$, and $w \in \mathcal{B}$, $w \neq 0$, such that for π-a.e. $x \in \mathbb{X}$ we have $|w(x)| = 1$ and*

$$\forall n \geq 1, \quad \mathbb{E}_{(x,0)}\big[e^{i\langle t, S_n\rangle}\, w(X_n)\big] = \lambda^n\, w(x). \tag{37}$$

Proof. Assume that $\mathbb{S}^* \neq G$, and let (t, λ, w) be as stated in Property (B). Then we have: $\forall n \geq 1,\ Q(t)^n w = \lambda^n w$ in $\mathcal{B}$. Since, by hypothesis, $\mathcal{B} \subset \mathbb{L}^1(\pi)$ with continuous inclusion, it follows that $Q(t)^n w = \lambda^n w$ in $\mathbb{L}^1(\pi)$, hence we have (37) for π-a.e. $x \in \mathbb{X}$ (use Lemma 5.2 with $m = 0$). Conversely, let $t \in \mathbb{R}^2 \setminus \mathbb{S}^*$ and (λ, w) as stated in Lemma B.4. Then we have for π-a.e. $x \in \mathbb{X}$: $\forall n \geq 1,\ Q(t)^n w(x) = \lambda^n w(x)$. This implies that $t \in G$. Indeed, if $t \notin G$, then by (35) we would have $r(Q(t)) < 1$, thus $\lim_n Q(t)^n w = 0$ in $\mathcal{B}$, and so in $\mathbb{L}^1(\pi)$: this would give the property: $w = 0$ π-a.s., which is impossible since by hypothesis $|w| = 1$ π-a.s.. □

Using the facts that $\mathbb{P}_{(x,0)}$ is a probability measure and $|w| = 1$ π-a.s., the property stated in Lemma B.4 is equivalent to the arithmeticity of $(S_n)_n$ in $\mathbb{S}$ w.r.t. $\mathcal{B}$, which proves the second equivalence in *(i)*.

Now we prove Assertion *(ii)* of Proposition B.2. Under Hypothesis **(A1)-(A4)**, G is a closed subgroup of $\mathbb{R}^2$, and under the additional Hypothesis **(A5)**, G is discrete, see [18, Prop. 12.4]. Observe that, since $\mathbb{S}^* \subset G$, we have $G^* \subset \mathbb{S}$. To prove Assertion *(ii)* of Proposition B.2, one needs to use the following statement, which is an easy adaptation of the proof of [18, Prop. 12.4]:

Property (C): there exist a bounded measurable function $\chi : \mathbb{X} \to \mathbb{R}^2$ and a family $(\beta_t)_{t \in G}$ of real numbers such that, for π-almost every $x \in \mathbb{X}$, we have

$$\forall t \in G,\ \forall n \geq 1,\ \big\langle t, S_n + \chi(X_n) - \chi(x)\big\rangle \in n\beta_t + 2\pi\mathbb{Z} \ \ \mathbb{P}_{(x,0)} - a.s.. \tag{38}$$

The fact that G is discrete plays an important role in Property (C) to obtain the existence of the above function χ, which does not depend on t.

Assume that Condition (22) is not fulfilled. Then from Assertion *(i)* of Proposition B.2, $\mathbb{S}^*$ is a proper subgroup of G. Hence G^* is a proper subgroup of $\mathbb{S}$. Consequently, from Property (C), $(S_n)_n$ is sublattice in $\mathbb{S}$. This proves *(ii)*.

Finally we establish Assertion *(iii)* of Proposition B.2. Suppose that $(S_n)_n$ is sublattice in $\mathbb{S}$, with $\mathbb{S}_0$, $\chi(\cdot)$ and $(\beta_t)_{t\in\mathbb{S}_0^*}$ as indicated in (6), and with the additional condition: $\forall t \in \mathbb{R}^2$, $e^{i\langle t,\chi(\cdot)\rangle} \in \mathcal{B}$. Since by hypothesis $\mathbb{S}_0$ is strictly contained in $\mathbb{S}$, there exists $t_0 \in \mathbb{S}_0^* \setminus \mathbb{S}^*$. We deduce from (6) that, for π-almost every $x \in \mathbb{X}$, we have

$$\forall n \geq 1, \quad e^{i\langle t_0,S_n\rangle} e^{i\langle t_0,\chi(X_n)\rangle} = e^{in\beta_{t_0}} e^{i\langle t_0,\chi(x)\rangle} \ \ \mathbb{P}_{(x,0)} - a.s..$$

So we obtain (34) with $\lambda := e^{i\beta_{t_0}}$ and $w := e^{i\langle t_0,\chi(\cdot)\rangle} \in \mathcal{B}$. Hence, from Assertion (i) of Proposition B.2, Condition (22) does not hold. □

Remark B.5. *When $(S_n)_n$ is an AF (see (11)), Conditions (34) and (6) in Definition B.1 may be stated only for $n = 1$ and specified with absorbing sets (instead of properties fulfilled π-a.s.), see* [18] *and Remark 2.1. Similarly, for general MRW, if $\mathcal{B}$ is composed of π-classes of functions (for instance $\mathcal{B} = \mathbb{L}^2(\pi)$), then the equivalence in Lemma B.4 is valid when (37) holds for $n = 1$. Indeed this condition says that for π-a.e. $x \in \mathbb{X}$: $Q(t)w(x) = \lambda\, w(x)$. So $Q(t)w = \lambda\, w$ in $\mathcal{B}$ and the proof of Lemma B.4 can be then repeated. Consequently, under the previous condition on $\mathcal{B}$, Conditions (34) and (6) may be also stated only for $n = 1$ (and then (9) is not relevant).*

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