Additional material on local limit theorem for finite Additive Markov Processes

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Abstract

This paper proposes additional material to the main statements of [HL13] which are are recalled in Section 2. In particular an application of [HL13, Th 2.2] to Renewal Markov Processes is provided in Section 4 and a detailed checking of the assumptions of [HL13, Th 2.2] for the joint distribution of local times of a finite jump process is reported in Section 5. A uniform version of [HL13, Th 2.2] with respect to a compact set of transition matrices is given in Section 6 (see [HL13, Remark 2.4]). The basic material on the semigroup of Fourier matrices and the spectral approach used in [HL13] is recalled in Section 3 in order to obtain a good understanding of the properties involved in this uniform version.

Keywords: Gaussian approximation, Local time, Spectral method, Markov random walk.

Contents

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1 Notations

Any vector $v = (v_k)_{k\in\{1,\dots,N\}} \in \mathbb{C}^N$ is a considered as a row-vector and $v^\top$ is the corresponding column-vector. The vector with all components equal to 1 is denoted by $\mathbf{1}$. The euclidean product scalar and its associated norm on $\mathbb{C}^N$ is denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively. The set of $N\times N$-matrices with complex entries is denoted by $\mathcal{M}_N(\mathbb{C})$. Let $\|\cdot\|_\infty$ denote the supremum norm on $\mathbb{C}^N$: $\forall v\in\mathbb{C}^N$, $\|v\|_\infty = \max_{k\in\{1,\dots,N\}}|v_k)|$. For the sake of simplicity, $\|\cdot\|_\infty$ also stands for the associated matrix norm:

$$\forall A\in\mathcal{M}_N(\mathbb{C}),\quad \|A\|_\infty := \sup_{\|v\|_\infty=1}\|Av^\top\|_\infty.$$

We also use the following norm $\|\cdot\|_0$ on $\mathcal{M}_N(\mathbb{C})$:

$$\forall A = (A_{k,\ell})_{(k,\ell)\in\{1,\dots,N\}^2}\in\mathcal{M}_N(\mathbb{C}),\quad \|A\|_0 := \max_{(k,\ell)\in\{1,\dots,N\}}|A_{k,\ell}|.$$

It is easily seen that

$$\forall A\in\mathcal{M}_N(\mathbb{C}),\quad \|A\|_0\le\|A\|_\infty\le N\|A\|_0. \tag{1}$$

For any bounded positive measure ν on $\mathbb{R}^d$, we define its Fourier transform as:

$$\forall\zeta\in\mathbb{R}^d,\quad \widehat{\nu}(\zeta) := \int_{\mathbb{R}^d} e^{i\langle\zeta,y\rangle}\,d\nu(y).$$

Let $\mathcal{A} = (\mathcal{A}_{k,\ell})_{(k,\ell)\in\{1,\dots,N\}^2}$ be a $N\times N$-matrix with entries in the set of bounded positive measures on $\mathbb{R}^d$. We set

$$\forall B\in B(\mathbb{R}^d),\qquad \mathcal{A}(1_B) := \big(\mathcal{A}_{k,\ell}(1_B)\big)_{(k,\ell)\in\mathbb{X}^2} \tag{2a}$$

$$\forall\zeta\in\mathbb{R}^d,\qquad \widehat{\mathcal{A}}(\zeta) := (\widehat{\mathcal{A}}_{k,\ell}(\zeta))_{(k,\ell)\in\mathbb{X}^2}. \tag{2b}$$

We denote by $\lfloor t\rfloor$ the integer part of any $t\in\mathbb{T}$.

2 The LLT for the density process

Let $\{(X_t,Z_t)\}_{t\in\mathbb{T}}$ be an MAP with state space $\mathbb{X}\times\mathbb{R}^d$, where $\mathbb{X} := \{1,\dots,N\}$ and the driving Markov process $\{X_t\}_{t\in\mathbb{T}}$ has transition semi-group $\{P_t\}_{t\in\mathbb{T}}$. We refer to [Asm03, Chap. XI] for the basic material on such MAPs. The conditional probability to $\{X_0 = k\}$ and its associated expectation are denoted by $\mathbb{P}_k$ and $\mathbb{E}_k$ respectively. Note that if $T:\mathbb{R}^d\to\mathbb{R}^m$ is a linear transformation, then $\{X_t, T(Z_t)\}_{t\in\mathbb{T}}$ is still a MAP on $\mathbb{X}\times R^m$ (see Lemma C.1). We suppose that $\{X_t\}_{t\in\mathbb{T}}$ has a unique invariant probability measure π. Set $m := \mathbb{E}_\pi[Z_1] = \sum_k \pi(k)\mathbb{E}_k[Z_1]\in\mathbb{R}^d$. Consider the centered MAP $\{(X_t,Y_t)\}_{t\in\mathbb{T}}$ where $Y_t := Z_t - t\,m$. The two next assumptions are involved in both CLT and LLT.

(I-A) : *The stochastic $N\times N$-matrix $P := P_1$ is irreducible and aperiodic.*

(M$\boldsymbol{\alpha}$) : *The family of r.v. $\{Y_v\}_{v\in(0,1]\cap\mathbb{T}}$ satisfies the uniform moment condition of order α:*

$$M_\alpha := \max_{k\in\mathbb{X}}\sup_{v\in(0,1]\cap\mathbb{T}}\mathbb{E}_k\big[\|Y_v\|^\alpha\big] < \infty. \tag{3}$$

The next theorem provides a CLT for $t^{-1/2}Y_t$, proved for $d := 1$ in [KW64] for $\mathbb{T}=\mathbb{N}$ and in [FH67] for $\mathbb{T} = [0,\infty)$ (see [FHL12] for ρ-mixing driving Markov processes).

Theorem 2.1 *Under Assumptions* **(I-A)** *and* **(M2)***,* $\{t^{-1/2}Y_t\}_{t\in\mathbb{T}}$ *converges in distribution to a d-dimensional Gaussian law* $\mathcal{N}(0,\Sigma)$ *when* $t\to+\infty$*.*

Now let us specify the notations and assumptions involved in our LLT. First we assume that the MAP $\{(X_t,Y_t)\}_{t\in\mathbb{T}}$ satisfies the following usual non-lattice condition:

(NL) : *there is no* $a\in\mathbb{R}^d$*, no closed subgroup* H *in* $\mathbb{R}^d$*,* $H\neq\mathbb{R}^d$*, and finally no function* $\beta:\mathbb{X}\to\mathbb{R}^d$ *such that:*

$$\forall k\in\mathbb{X},\quad Y_1+\beta(X_1)-\beta(k)\ \in\ a+H\quad \mathbb{P}_k\text{-a.s.},$$

Second we introduce the assumptions on the density process of $t^{-1/2}Y_t$. For any $t\in\mathbb{T}$ and $(k,\ell)\in\mathbb{X}^2$, we define the bounded positive measure $\mathcal{Y}_{k,\ell,t}$ on $\mathbb{R}^d$:

$$\forall B\in B(\mathbb{R}^d),\quad \mathcal{Y}_{k,\ell,t}(1_B):=\mathbb{P}_k\big\{X_t=\ell,Y_t\in B\big\}. \tag{4}$$

Let ℓ_d denote the Lebesgue measure on $\mathbb{R}^d$. From the Lebesgue decomposition of $\mathcal{Y}_{k,\ell,t}$ w.r.t. ℓ_d, there are two bounded positive measures $\mathcal{G}_{k,\ell,t}$ and $\mu_{k,\ell,t}$ on $\mathbb{R}^d$ such that

$$\forall B\in B(\mathbb{R}^d)\qquad \mathcal{Y}_{k,\ell,t}(1_B):=\mathcal{G}_{k,\ell,t}(1_B)+\mu_{k,\ell,t}(1_B) \tag{5a}$$

$$\text{where}\qquad \mathcal{G}_{k,\ell,t}(1_B)=\int_B g_{k,\ell,t}(y)\,dy \tag{5b}$$

for some measurable function $g_{k,\ell,t}:\mathbb{R}^d\to[0,+\infty)$, and such that $\mu_{k,\ell,t}$ and ℓ_d are mutually singular. The measure $\mathcal{G}_{k,\ell,t}$ is called the absolutely continuous (a.c.) part of $\mathcal{Y}_{k,\ell,t}$ with associated density $g_{k,\ell,t}$. For any $t\in\mathbb{T}$, we introduce the following $N\times N$-matrices with entries in the set of bounded positive measures on $\mathbb{R}^d$

$$\mathcal{Y}_t:=(\mathcal{Y}_{k,\ell,t})_{(k,\ell)\in\mathbb{X}^2},\quad \mathcal{G}_t:=(\mathcal{G}_{k,\ell,t})_{(k,\ell)\in\mathbb{X}^2},\quad \mathcal{M}_t:=(\mu_{k,\ell,t})_{(k,\ell)\in\mathbb{X}^2}, \tag{6a}$$

and for every $y\in\mathbb{R}^d$, the following real $N\times N$-matrix:

$$G_t(y):=\big(g_{k,\ell,t}(y)\big)_{(k,\ell)\in\mathbb{X}^2}. \tag{6b}$$

Then the component-wise equalities (5a)-(5b) read as follows in a matrix form: for any $t\in\mathbb{T}$

$$\forall B\in B(\mathbb{R}^d),\quad \mathcal{Y}_t(1_B)=\mathcal{G}_t(1_B)+\mathcal{M}_t(1_B)=\int_B G_t(y)dy+\mathcal{M}_t(1_B). \tag{6c}$$

The assumptions on the a.c. part $\mathcal{G}_t$ and the singular part $\mathcal{M}_t$ of $\mathcal{Y}_t$ are the following ones.

AC1 : *There exist* $c>0$ *and* $\rho\in(0,1)$ *such that*

$$\forall t>0,\quad \|\mathcal{M}_t(1_{\mathbb{R}^d})\|_0\le c\rho^t \tag{7}$$

and there exists $t_0>0$ *such that*

$$\rho^{t_0}\max(2,cN)\le 1/4 \tag{8a}$$

$$\Gamma_{t_0}(\zeta):=\sup_{w\in[t_0,2t_0)}\|\widehat{G}_w(\zeta)\|_0\longrightarrow 0\quad \textit{when } \|\zeta\|\to+\infty. \tag{8b}$$

AC2 : *For any $t > 0$, there exists an open convex subset $\mathcal{D}_t$ of $\mathbb{R}^d$ such that G_t vanishes on $\mathbb{R}^d \setminus \overline{\mathcal{D}}_t$, where $\overline{\mathcal{D}}_t$ denotes the adherence of $\mathcal{D}_t$. Moreover G_t is continuous on $\overline{\mathcal{D}}_t$ and differentiable on $\mathcal{D}_t$, with in addition*

$$\sup_{t>0} \sup_{y\in\overline{\mathcal{D}}_t} \|G_t(y)\|_0 < \infty \tag{9a}$$

$$\sup_{y\in\partial\mathcal{D}_t} \|G_t(y)\|_0 = O\big(t^{-(d+1)/2}\big) \quad \textit{where} \ \ \partial\mathcal{D}_t := \overline{\mathcal{D}}_t \setminus \mathcal{D}_t \tag{9b}$$

$$j = 1, \ldots , d : \quad \sup_{t>0} \sup_{y\in\mathcal{D}_t} \Big\|\frac{\partial G_t}{\partial y_j}(y)\Big\|_0 < \infty. \tag{9c}$$

The next theorem gives a LLT for the density of the a.c. part of the probability distribution of $t^{-1/2}Y_t$. This is the main contribution of [HL13].

Theorem 2.2 *Let $\{(X_t, Y_t)\}_{t\in\mathbb{T}}$ be a centered MAP satisfying Assumptions* **(I-A)**, **(M3)**, **(AC1)-(AC2)**. *Moreover assume that the matrix Σ of Theorem 2.1 is invertible. Then, for every $k \in \mathbb{X}$, the density $f_{k,t}(\cdot)$ of the a.c. part of the probability distribution of $t^{-1/2}Y_t$ under $\mathbb{P}_k$ satisfies the following property:*

$$\sup_{y\in\mathbb{R}^d} \big|f_{k,t}(y) - \eta_{\Sigma}(y)\big| \le O\big(t^{-1/2}\big) + O\big(\sup_{y\notin\mathcal{D}_t} \eta_{\Sigma}(t^{-1/2}y)\big) \tag{10}$$

where $\eta_{\Sigma}(\cdot)$ denotes the density of the non-degenerate d-dimensional Gaussian distribution $\mathcal{N}(0,\Sigma)$ involved in Theorem 2.1.

3 Semigroup of Fourier matrices and basic lemmas

We assume that the conditions of Theorem 2.2 hold, and for the sake of simplicity that Σ is the identity matrix. The proof of Theorem 2.2 involves Fourier analysis as in the i.d.d. case. In our Markov context, this study is based on the semi-group property of the matrix family $\{\widehat{\mathcal{Y}}_t(\zeta)\}_{t\in\mathbb{T}}$ for every $\zeta \in \mathbb{R}^d$ which allows to analyze the characteristic function of Y_t. In this section, we provide a collection of lemmas which highlights the connections between the assumptions of Section 2 and the behavior of this semi-group. This is the basic material for the derivation of Theorem 2.2 in [HL13, Section 3].

Recall that, for every $(k,\ell) \in \mathbb{X}^2$, the measures $\mathcal{G}_{k,\ell,t}$ (with density $g_{k,\ell,t}$) and $\mu_{k,\ell,t}$ denote the a.c. and singular parts of the bounded positive measure $\mathcal{Y}_{k,\ell,t}(1.) = \mathbb{P}_k\{X_t = \ell, Y_t \in \cdot\}$ on $\mathbb{R}^d$. Using the notation of (5a)-(5b), the bounded positive measure $\sum_{\ell=1}^N \mathcal{G}_{k,\ell,t}$ with density

$$g_{k,t} := \sum_{\ell=1}^{N} g_{k,\ell,t}$$

is the a.c. part of the probability distribution of Y_t under $\mathbb{P}_k$, while $\mu_{k,t} := \sum_{\ell=1}^N \mu_{k,\ell,t}$ is its singular part. That is, we have for any $k \in \mathbb{X}$ and $t > 0$:

$$\forall B \in B(\mathbb{R}^d), \quad \mathbb{P}_k\{Y_t \in B\} = \int_B g_{k,t}(y)\, dy + \mu_{k,t}(1_B). \tag{11}$$

The bounded positive measure $\mathcal{Y}_t$ is defined in (6a) and its Fourier transform $\widehat{\mathcal{Y}}_t$ is (see (2b)):

$$\forall (t,\zeta) \in \mathbb{T}\times\mathbb{R}^d,\ \forall (k,\ell) \in \mathbb{X}^2, \quad \big(\widehat{\mathcal{Y}}_t(\zeta)\big)_{k,\ell} = \mathbb{E}_k\big[1_{\{X_t=\ell\}}\, e^{i\langle\zeta,Y_t\rangle}\big]. \tag{12}$$

Note that $\widehat{\mathcal{Y}}_t(0) = P_t$. From the additivity of the second component Y_t, we know that that $\{\widehat{\mathcal{Y}}_t(\zeta)\}_{t\in\mathbb{T},\zeta\in\mathbb{R}^d}$ is a semi-group of matrices (e.g. see [FHL12] for details), that is using the usual product in the space $\mathcal{M}_N(\mathbb{C})$ of complex $N \times N$-matrices:

$$\forall \zeta \in \mathbb{R}^d,\ \forall (s,t) \in \mathbb{T}^2, \quad \widehat{\mathcal{Y}}_{t+s}(\zeta) = \widehat{\mathcal{Y}}_t(\zeta)\,\widehat{\mathcal{Y}}_s(\zeta). \tag{SG}$$

In particular the following property holds true

$$\forall \zeta \in \mathbb{R}^d,\ \forall n \in \mathbb{N}, \qquad \widehat{\mathcal{Y}}_n(\zeta) := \big(\mathbb{E}_k[1_{\{X_n=\ell\}}\, e^{i\langle\zeta,Y_n\rangle}]\big)_{(k,\ell)\in\mathbb{X}^2} = \widehat{\mathcal{Y}}_1(\zeta)^n. \tag{13}$$

For every $k \in \mathbb{X}$ and $t \in \mathbb{T}$, we denote by $\phi_{k,t}$ the characteristic function of Y_t under $\mathbb{P}_k$:

$$\forall \zeta \in \mathbb{R}^d, \quad \phi_{k,t}(\zeta) := \mathbb{E}_k\big[e^{i\langle\zeta,Y_t\rangle}\big] = e_k\widehat{\mathcal{Y}}_t(\zeta)\mathbf{1}^\top, \tag{14}$$

where e_k is the k-th vector of the canonical basis of $\mathbb{R}^d$.

The first lemma below provides the control of $\phi_{k,t}$ on a ball $B(0,\delta) := \{\zeta \in \mathbb{R}^d : \|\zeta\| < \delta\}$ for some $\delta > 0$, using a perturbation approach. The second one is on the control of $\phi_{k,t}$ on the annulus $\{\zeta \in \mathbb{R}^d : \delta \le \|\zeta\| \le A]$ for any $A > \delta$. Finally the third lemma focuses on the Fourier transform of the density $g_{k,\ell,t}$ of the a.c. part $\mathcal{G}_{k,\ell,t}$ of $\mathcal{Y}_{k,\ell,t}$ in (5a) on the domain $\{\zeta \in \mathbb{R}^d : \|\zeta\| \ge A\}$ for some $A > 0$.

Lemma 3.1 ([HL13, Lem. 4.1]) *Under Assumptions* **(I-A)** *and* **(M3)***, there exists a real number $\delta > 0$ such that, for all $\zeta \in B(0,\delta)$, the characteristic function of Y_t satisfies*

$$\forall k \in \mathbb{X},\ \forall t \in \mathbb{T}, \quad \phi_{k,t}(\zeta) = \lambda(\zeta)^{\lfloor t\rfloor}\, L_{k,t}(\zeta) + R_{k,t}(\zeta), \tag{15}$$

with $\mathbb{C}$-valued functions $\lambda(\cdot)$, $L_{k,t}(\cdot)$ and $R_{k,t}(\cdot)$ on $B(0,\delta)$ satisfying the next properties for $t \in \mathbb{T}$, $\zeta \in \mathbb{R}^d$ and $k \in \mathbb{X}$:

$$\|\zeta\| < \delta \ \Rightarrow\ \lambda(\zeta) = 1 - \frac{\|\zeta\|^2}{2} + O(\|\zeta\|^3) \tag{16a}$$

$$t \ge 2,\ \|t^{-1/2}\zeta\| < \delta \ \Rightarrow\ \big|\lambda(t^{-1/2}\zeta)^{\lfloor t\rfloor} - e^{-\|\zeta\|^2/2}\big| \le C\,\frac{(1+\|\zeta\|^3)e^{-\|\zeta\|^2/8}}{\sqrt{t}} \tag{16b}$$

$$\|\zeta\| < \delta \ \Rightarrow\ \big|L_{k,t}(\zeta) - 1\big| \le C\,\|\zeta\| \tag{16c}$$

$$\exists\, r \in (0,1), \quad \sup_{\|\zeta\|\le\delta} |R_{k,t}(\zeta)| \le C\, r^{\lfloor t\rfloor} \tag{16d}$$

where the constant $C > 0$ in (16b)-(16d) *only depends on δ and on M_3 in* **(M3)** *(see* (3)*).*

Proof of (16b). From (16a) we obtain for $v \in \mathbb{R}^d$ such that $|v| \le \delta$ (up to reduce δ)

$$|\lambda(v)| \le 1 - \frac{\|v\|^2}{2} + \frac{\|v\|^2}{4} \le e^{-\|v\|^2/4}.$$

Therefore we have for any $t \ge 2$ and any $\zeta \in \mathbb{R}^d$ such that $t^{-1/2}\|\zeta\| \le \delta$,

$$\big|\lambda\big(t^{-1/2}\zeta\big)\big| \le e^{-\frac{\|\zeta\|^2}{4t}}. \tag{17}$$

Now consider any $t \ge 2$ and any $\zeta \in \mathbb{R}^d$ such that $t^{-1/2}\|\zeta\| \le \delta$. Set $n := \lfloor t\rfloor$, and write

$$\lambda\big(t^{-1/2}\zeta\big)^n - e^{-\frac{\|\zeta\|^2}{2}} = \big(\lambda\big(t^{-1/2}\zeta\big) - e^{-\frac{\|\zeta\|^2}{2n}}\big)\sum_{k=0}^{n-1}\lambda\big(t^{-1/2}\zeta\big)^{n-k-1}e^{\frac{-k\|\zeta\|^2}{2n}}.$$

We have

$$\begin{aligned}
\Big|\sum_{k=0}^{n-1}\lambda\big(t^{-1/2}\zeta\big)^{n-k-1}e^{\frac{-k\|\zeta\|^2}{2n}}\Big| &\le \sum_{k=0}^{n-1}\big|\lambda\big(t^{-1/2}\zeta\big)\big|^{n-k-1}e^{\frac{-k\|\zeta\|^2}{2n}}\\
&\le \sum_{k=0}^{n-1}e^{-\frac{\|\zeta\|^2(n-k-1)}{4t}}e^{\frac{-k\|\zeta\|^2}{4t}}\\
&\le n\,e^{-\frac{n\|\zeta\|^2}{4t}}e^{\frac{\|\zeta\|^2}{4t}}\le b\,t\,e^{-\frac{\|\zeta\|^2}{8}}
\end{aligned}$$

where $b := \sup_{|u|\le\delta}\exp(u^2/4)$, since $n/t \ge (t-1)/t \ge 1/2$ for $t \ge 2$. Moreover, using (16a) and the fact that $|e^a - e^b| \le |a-b|$ for any $a, b \in (-\infty, 0)$, we obtain that there exist positive constants D and D' such that

$$\begin{aligned}
\big|\lambda\big(t^{-1/2}\zeta\big)-e^{-\frac{\|\zeta\|^2}{2n}}\big| &\le \big|\lambda\big(t^{-1/2}\zeta\big)-e^{-\frac{\|\zeta\|^2}{2t}}\big|+\big|e^{-\frac{\|\zeta\|^2}{2t}}-e^{-\frac{\|\zeta\|^2}{2n}}\big|\\
&\le D\,t^{-\frac{3}{2}}\|\zeta\|^3+\frac{\|\zeta\|^2}{2}\Big|\frac{1}{t}-\frac{1}{n}\Big|\\
&\le D\,t^{-\frac{3}{2}}\|\zeta\|^3+\frac{\|\zeta\|^2}{2}\,\frac{1}{t(t-1)}\\
&\le D'\big(t^{-\frac{3}{2}}\|\zeta\|^3+t^{-2}\|\zeta\|^2\big).
\end{aligned}$$

Thus

$$\begin{aligned}
\left|\lambda\big(t^{-1/2}\zeta\big)^n-e^{-\frac{\|\zeta\|^2}{2}}\right| &\le bD'\,t\,e^{-\frac{\|\zeta\|^2}{8}}\big(t^{-\frac{3}{2}}\|\zeta\|^3+t^{-2}\|\zeta\|^2\big)\\
&\le bD'\,e^{-\frac{\|\zeta\|^2}{8}}\left(\frac{\|\zeta\|^3}{\sqrt{t}}+\frac{\|\zeta\|^2}{t}\right)\\
&\le \frac{bD'}{\sqrt{t}}\,e^{-\frac{\|\zeta\|^2}{8}}\big(\|\zeta\|^3+\|\zeta\|^2\big)\\
&\le \frac{D''}{\sqrt{t}}\,e^{-\frac{\|\zeta\|^2}{8}}\big(1+\|\zeta\|^3\big)
\end{aligned}$$

for some positive constant D''. □

Lemma 3.2 ([HL13, Lem. 4.2]) *Assume that Conditions* **(I-A)**, **(AC1)** *and* **(Mα)** *for some $\alpha > 0$ hold. Let δ, A be any real numbers such that $0 < \delta < A$. There exist constants $D \equiv D(\delta, A) > 0$ and $\tau \equiv \tau(\delta, A) \in (0,1)$ such that*

$$\forall k \in \mathbb{X},\ \forall t \in \mathbb{T}, \quad \sup_{\delta\le\|\zeta\|\le A}|\phi_{k,t}(\zeta)| \le D\,\tau^{\lfloor t\rfloor}. \tag{18}$$

Lemma 3.3 ([HL13, Th. 4.3]) *Under Condition* **(AC1)**, *there exist positive constants A and C such that the following property holds:*

$$|\zeta| \ge A \implies \forall (k,\ell) \in \mathbb{X}^2,\ \forall t \in [t_0, +\infty[, \quad \big|\widehat{g}_{k,\ell,t}(\zeta)\big| \le C\,\frac{t}{2^{t/t_0}}.$$

4 Application to the Markov Renewal Processes

Let $\{X_n, Y_n\}_{n\in\mathbb{N}}$ be a discrete-time MAP with state space $\mathbb{X}\times\mathbb{R}$ and $\mathbb{X} := \{1,\dots,N\}$. When $\{\xi_n\}_{n\geq 0}$, with $\xi_0 := Y_0$ and $\xi_n := Y_n - Y_{n-1}$ for $n \geq 1$, is a sequence of non-negative random variables, then $\{X_n, Y_n\}_{n\in\mathbb{N}}$ is also known as a Markov Renewal Process (MRP). The so-called semi-Markov kernel $Q(\cdot;\{\cdot\}\times dy)$ is defined by (e.g. see [Asm03, VII.4])

$$\forall B \in B(\mathbb{R}),\ \forall (k,\ell)\in\mathbb{X}^2,\quad \mathbb{P}_k\{X_1=\ell, Y_1\in B\} = \int_B Q(k;\{\ell\}\times dy) \tag{19}$$

and is nothing else but the measure $\mathcal{Y}_{k,\ell,1}$ in (4). The transition probability matrix P associated with the Markov chain $\{X_n\}_{n\in\mathbb{N}}$ is given by

$$\forall (k,\ell)\in\mathbb{X}^2,\quad P(k,\ell) = Q(k;\{\ell\}\times\mathbb{R}) = \int_{\mathbb{R}} Q(k;\{\ell\}\times dy).$$

For any $n\geq 2$, the bounded positive measure $\mathcal{Y}_{k,\ell,n}$ is defined by the convolution product of the semi-Markov kernel Q, that is

$$\forall B\in B(\mathbb{R}^d),\quad \mathcal{Y}_{k,\ell,n}(1_B) := \mathbb{P}_k\{X_n=\ell, Y_n\in B\} = Q^{\star n}(k,\{\ell\}\times B) \tag{20}$$

Then, the Theorem 2.2 for the density process of $Y_n/\sqrt{n}$ could be specified to this specific class of MAPs. Since we only have to replace time t by n in all the material developed in Section 2, we omit the details. Note that the only simplification in assumptions of Theorem 2.2 is on the uniform moment condition $\mathbf{(M\boldsymbol{\alpha})}$ which reduces to: the r.v. Y_1 satisfies the following moment condition of order α:

$$M_\alpha := \max_{k\in\mathbb{X}} \mathbb{E}_k\big[\|Y_1\|^\alpha\big] < \infty. \tag{21}$$

We will only illustrate our main result on the MRP embedded in a Markovian Arrival Process (e.g. see [Asm03, XI.1]).

Recall that a N-state Markovian Arrival Process is a continuous-time MAP $\{(J_t,N_t)\}_{t\geq 0}$ on the state space $\{1,\dots,N\}\times\mathbb{N}$, where N_t represents the number of arrivals up to time t, while the states of the driving Markov process $\{J_t\}_{t\geq 0}$ are called phases. Let Y_n be the time at the nth arrival ($Y_0 = 0$ a.s.) and let X_n be the state of the driving process just after the nth arrival. Then $\{(X_n,Y_n)\}_{n\in\mathbb{N}}$ is known to be an MRP with the following semi-Markov kernel Q on $\{1,\dots,N\}\times\mathbb{R}$: for any $(k,\ell)\in\mathbb{X}^2$

$$Q(k;\{\ell\}\times dy) := (e^{yD_0}D_1)(k,\ell)\,1_{(0,\infty)}(y)dy = e_k e^{yD_0} D_1 {e_\ell}^\top\,1_{(0,\infty)}(y)\,dy \tag{22}$$

which is parametrized by a pair of $N\times N$-matrices usually denoted by D_0 and D_1. Such a process is also an instance of Markov Random Walk. The matrix D_0+D_1 is the infinitesimal generator of the background Markov process $\{J_t\}_{t\geq 0}$. Matrix D_0 is always assumed to be stable and D_0+D_1 to be irreducible. In this case, $\{J_t\}_{t\geq 0}$ has a unique invariant probability measure denoted by π. The process $\{X_n\}_{n\in\mathbb{N}}$ is a Markov chain with state space $\mathbb{X} := \{1,\dots,N\}$ and transition probability matrix P:

$$\forall (k,\ell)\in\mathbb{X}^2,\quad P(k,\ell) = Q(k;\{\ell\}\times\mathbb{R}) = \left((-D_0)^{-1}D_1\right)(k,\ell). \tag{23}$$

This Markov chain has an invariant probability measure $\boldsymbol{\phi}$ (different of π). From (22), the bounded positive measure $\mathcal{Y}_{k,\ell,1}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ with density $g_{k,\ell,1}(y) = e_k e^{yD_0} D_1 e_\ell{}^\top 1_{(0,\infty)}(y)$. Then matrix G_t defined in (6b) has the form

$$G_1(y) = e^{yD_0} D_1 \, 1_{(0,\infty)}(y).$$

For any $n \geq 2$, the positive measure $\mathcal{Y}_{k,\ell,n}$ in (20) is absolutely continuous with respect to the Lebesgue measure with density given by

$$\forall y \in \mathbb{R}, \quad G_n(y) = (G_{n-1} \star G_1)(y) =: \left(\sum_{j \in \mathbb{X}} \big(G_{k,j,1} \star G_{j,\ell,n-1}\big)(y) \right)_{k,\ell \in \mathbb{X}^2}. \tag{24}$$

Let us check the assumptions of Theorem 2.2. Assumption **(I-A)** on P in (23) is standard in the literature on the Markov Arrival Process. It is well known that Y_1 has a moment of order 3 given by $E_k[(Y_1)^3] = 3!\, e_k(-D_0)^{-3}\, \mathbf{1}^\top$, so that Condition (21) holds with $\alpha = 3$. Let us give some details for checking Conditions **(AC1)-(AC2)**.

(AC2) : The open convex $\mathcal{D}_n$ involved in **(AC2)** is given for any $n \geq 1$ by

$$\mathcal{D}_n :=]0, \infty[, \qquad \text{with } \overline{\mathcal{D}}_n = [0, \infty), \quad \partial\mathcal{D}_n = \{0\}.$$

It is clear that $G_1(y)$ is continuous on $\overline{\mathcal{D}}_1$ and differentiable on $\mathcal{D}_1$ with differential

$$\forall y > 0, \quad \frac{dG_1}{dy}(y) = D_0 e^{yD_0} D_1. \tag{25}$$

Since G_n is obtained from convolution product of G_1, the same properties are also valid for G_n. Next, we prove by induction that

$$\sup_{n>0} \sup_{y \in [0,+\infty)} \|G_n(y)\|_0 \leq N \|D_1\|_0. \tag{26}$$

For $n := 1$, we have from norm equivalence (1) and $\|e^{D_0}\|_\infty \leq 1$ that

$$\begin{aligned} \forall y \in [0, +\infty), \quad \|G_1(y)\|_0 &= \|e^{yD_0} D_1\|_0 \leq \|e^{yD_0} D_1\|_\infty \leq \|e^{yD_0}\|_\infty \|D_1\|_\infty \\ &\leq N\|D_1\|_0. \end{aligned}$$

Using the definition (24) of $G_n(y)$ and the induction hypothesis, we have

$$\begin{aligned} |G_{k,\ell,n}(y)| &= \sum_j \int_{\mathbb{R}} |G_{k,j,1}(u)||G_{j,\ell,n-1}(t-u)| du \leq N\|D_1\|_0 \int_{\mathbb{R}} G_{k,j,1}(u) du \\ &\leq N\|D_1\|_0 \, \mathbb{P}_k\{X_1 = \ell\} \leq N\|D_1\|_0. \end{aligned}$$

The proof of estimate (26) is complete.

Second, we easily obtain from (25) that

$$\sup_{y \in (0,+\infty)} \| \frac{dG_1}{dy}(y) \|_0 \leq \big(N \max(\|D_0\|_0, \|D_1\|_0)\big)^2$$

Next, using an induction, we can obtain from the following well known property of the convolution product

$$\frac{dG_n}{dy}(y) = \left(G_1 \star \frac{dG_{n-1}}{dy} \right)(y)$$

that

$$\sup_{n>0}\sup_{y\in(0,+\infty)}\|\frac{dG_n}{dy}(y)\|_0 \leq \big(N\max(\|D_0\|_0,\|D_1\|_0)\big)^2.$$

Finally, $G_1(0)=D_1$ and $G_n(0)=0$ for any $n\geq 2$.
(AC1) : First, observe that $\mu_{k,\ell,n}$ is 0 for any $(k,\ell)\in\mathbb{X}^2$ and $n\geq 1$. Thus (7) is satisfied for $\rho=0$. In Condition (AC1), we only have to check that

$$\|\widehat{G}_{n_0}(\zeta)\|_0 \longrightarrow 0 \quad \text{when } |\zeta|\to+\infty$$

for n_0 large enough. It follows from the convolution definition (24) of G_n that

$$\forall n\in\mathbb{N},\ \forall\zeta\in\mathbb{R},\qquad \widehat{G}_n(\zeta) := \big(\mathbb{E}_k\big[1_{\{X_n=\ell\}}\, e^{i\langle\zeta,Y_n\rangle}\big]\big)_{(k,\ell)\in\mathbb{X}^2} = \widehat{G}_1(\zeta)^n$$

where

$$\widehat{G}_1(\zeta) = \int_0^{+\infty} e^{i\zeta y}e^{D_0 y}dy\, D_1$$

is integrable. In fact, using an integration by parts, we obtain that

$$\forall\zeta\neq 0,\qquad \widehat{G}_1(\zeta) \;=\; \frac{-1}{i\zeta}\big[D_1+\int_0^{+\infty} e^{i\zeta y}D_0e^{yD_0}D_1\,dy\big]$$

so that

$$\forall\zeta\neq 0,\qquad \|\widehat{G}_1(\zeta)\|_0 \leq \frac{2\|D_1\|_0}{|\zeta|}.$$

Therefore, we deduce from norm equivalence (1) that for any integer $n_0\geq 1$,

$$\|\widehat{G}_{n_0}(\zeta)\|_0 \leq \frac{(2N\|D_1\|_0)^{n_0}}{|\zeta|^{n_0}} \longrightarrow 0 \quad \text{when } |\zeta|\to+\infty.$$

5 Application to the local times of a jump process

5.1 The local limit theorem for the density of local times

Let $\{X_t\}_{t\geq 0}$ be a Markov jump process with finite state space $\mathbb{X}:=\{1,\ldots,N\}$ and generator G. Its transition semi-group is given by

$$\forall t\geq 0,\quad P_t := e^{tG}.$$

The local time $L_t(i)$ at time t associated with state $i\in\mathbb{X}$, or the sojourn time in state i on the interval $[0,t]$, is defined by

$$\forall t\geq 0,\quad L_t(i) := \int_0^t 1_{\{X_s=i\}}\,ds.$$

It is well known that $L_t(i)$ is an additive functional of $\{X_t\}_{t\geq 0}$ and that $\{(X_t,L_t(i))\}_{t\geq 0}$ is an MAP. In this section, we consider the MAP $\{(X_t,L_t)\}_{t\geq 0}$ where L_t is the random vector of the local times

$$L_t := (L_t(1),\ldots,L_t(N)).$$

Note that, for all $t > 0$, we have $\langle L_t, \mathbf{1}\rangle = t$, that is L_t is $\mathcal{S}_t$-valued where

$$\mathcal{S}_t := \big\{y \in [0, +\infty)^N : \langle y, \mathbf{1}\rangle = t\big\}.$$

Let us assume that $\{X_t\}_{t\geq 0}$ has an invariant probability measure π. This happens when the generator G is irreducible. Set $m = (m_1, \dots, m_N) := \mathbb{E}_\pi[L_1]$. We define the $\mathcal{S}_t^{(0)}$-valued centered r.v.

$$Y_t = L_t - tm$$

where $\mathcal{S}_t^{(0)} := T_{-tm}(\mathcal{S}_t)$ with the translation T_{-tm} by vector $-tm$ in $\mathbb{R}^N$. Note that $\mathcal{S}_t^{(0)}$ is a subset of the hyperplane H of $\mathbb{R}^N$ defined by

$$H := \big\{y \in \mathbb{R}^N : \ \langle y, \mathbf{1}\rangle = 0\big\}. \tag{27}$$

Let Λ be the bijective map from H into $\mathbb{R}^{N-1}$ defined by $\Lambda(y) := (y_1, \dots, y_{N-1})$ for $y := (y_1, \dots, y_N) \in H$. We introduce the following $(N-1)$-dimensional random vector

$$Y'_t := \Lambda(Y_t) = (Y_t(1), \dots, Y_t(N-1)).$$

The following lemma follows from Lemma C.1.

Lemma 5.1 *The process $\{(X_t, Y'_t)\}_{t\geq 0}$ is a $\mathbb{X} \times \overline{\mathcal{D}}_t$-valued MAP where $\mathcal{D}_t$ is the open convex of $\mathbb{R}^{N-1}$ defined by*

$$\mathcal{D}_t := \big\{y' \in \mathbb{R}^{N-1} : j = 1, \dots, N-1, \ y'_j \in (-m_j t, (1-m_j)t), \ \langle y', \mathbf{1}\rangle < m_N t\big\}. \tag{28}$$

Proposition 5.1 *If G and the sub-generators $G_{i^c i^c} := (G(k,\ell))_{k,\ell\in\{i\}^c}$, $i = 1, \dots, N$ are irreducible then the MAP $\{(X_t, Y'_t)\}_{t\geq 0}$ satisfies the conditions* **(M3)** *and* **(AC1)-(AC2)***.*

Note that the matrix Σ in the CLT for $\{t^{-1/2}Y'_t\}_{t>0}$ is invertible from [HL13, Remark 2.3] and that $O(\sup_{y\notin\mathcal{D}_t} \eta_\Sigma(t^{-1/2}y)) = O(t^{-1/2})$ from (28). Under the assumptions of Proposition 5.1 on the generator of $\{X_t\}_{t\geq 0}$, Theorem 2.2 gives that

$$\sup_{y'\in\mathbb{R}^{N-1}} \big|f'_{k,t}(y') - (2\pi)^{-(N-1)/2}(\det\Sigma)^{-1/2} e^{-\frac{1}{2}\langle y'^\top, \Sigma y'^\top\rangle}\big| = O\big(t^{-1/2}\big),$$

where $f'_{k,t}$ is the density of the a.c. part of the probability distribution of $t^{-1/2}Y'_t$. An explicit form of Σ is provided in [HL13, Remark 3.1].

In order to obtain a statement in terms of Y_t, we can use the following lemma. Recall that ℓ_{N-1} denotes the Lebesgue measure on $\mathbb{R}^{N-1}$ and $B(H)$ stands for the Borelian σ-algebra on H. Let ν be the measure defined on $(H, B(H))$ as the image measure of ℓ_{N-1} under Λ^{-1}, that is: for any positive and $B(H)$-measurable function $\psi : H \to \mathbb{R}$,

$$\int_H \psi(y)\,\nu(dy) := \int_{\mathbb{R}^{N-1}} \psi(\Lambda^{-1}y')\,dy'.$$

Using the bijection Λ, the following lemma can be easily proved (see Appendix A).

Lemma 5.2 *For any $(k,\ell) \in \mathbb{X}^2$, the a.c. and singular parts of the Lebesgue decomposition of the measure $\mathbb{P}_k\{X_t = \ell, Y_t \in \cdot\}$ on H with respect to ν are obtained from image measure under Λ of the a.c. and singular parts of the Lebesgue decomposition of the measure $\mathbb{P}_k\{X_t = \ell, Y'_t \in \cdot\}$ on $\mathbb{R}^{N-1}$ with respect to ℓ_{N-1} .*

Therefore we know that the density, say $f_{k,t}$, of the a.c. part of the probability distribution of

$$t^{-1/2}Y_t = t^{-1/2}(L_t - tm)$$

with respect to the measure ν on the hyperplane H is given by

$$\forall h \in H, \quad f_{k,t}(h) := f'_{k,t}(\Lambda h).$$

Finally, we have obtained the following local limit theorem for $f_{k,t}$.

Proposition 5.2 ([HL13, Prop. 3.1]) *If G and the sub-generators $G_{i^c i^c} := (G(k,\ell))_{k,\ell\in\{i\}^c}$, $i = 1,\ldots,N$ are irreducible, then the density $f_{k,t}$ of the a.c. part of the probability distribution of $t^{-1/2}(L_t - tm)$ under $\mathbb{P}_k$ satisfies*

$$\sup_{h\in H} \big| f_{k,t}(h) - (2\pi)^{-(N-1)/2}(\det\Sigma)^{-1/2} e^{-\frac{1}{2}\langle \Lambda h^\top, \Sigma\Lambda h^\top\rangle} \big| = O\big(t^{-1/2}\big).$$

It remains to prove that Proposition 5.1 holds true.

5.2 The main lines of the derivation of Proposition 5.1

Let us introduce the randomized Markov chain $\{Z_n\}_{n\in\mathbb{N}}$ with state space $\mathbb{X}$ and transition matrix

$$\widetilde{P} := I + G/a \quad \text{with } a > \max(|G(j,j)|, j \in \mathbb{X}).$$

Since G is assumed to be irreducible, the transition matrix $\widetilde{P}$ is irreducible and aperiodic. Moreover, since $\widetilde{P}(i,i) > 0$ for any $i \in \mathbb{X}$, the sub-stochastic matrix $\widetilde{P}_{i^c i^c} := (\widetilde{P}(k,\ell))_{k,\ell\in\{i\}^c}$ is aperiodic.

Let us define the following convex open set of $\mathbb{R}^{N-1}$

$$\forall t > 0, \quad \mathcal{C}_t := \big\{ y \in]0,t[^{N-1}, \langle y, \mathbf{1}\rangle < t \big\}.$$

For any $t > 0$, the adherence of $\mathcal{C}_t$ is denoted by $\overline{\mathcal{C}}_t$ and is $\overline{\mathcal{C}}_t := \big\{ y \in [0,t]^{N-1}, \langle y, \mathbf{1}\rangle \le t \big\}$ and the boundary of $\mathcal{C}_t$ is $\partial\mathcal{C}_t := \overline{\mathcal{C}}_t \backslash \mathcal{C}_t$ given by

$$\partial\mathcal{C}_t = \bigcup_{i=1}^{N} \big\{ y = (y_1,\ldots,y_{N-1}) \in \overline{\mathcal{C}}_t \mid y_i := 0 \big\} \cup \big\{ y \in \overline{\mathcal{C}}_t \mid \langle y, \mathbf{1}\rangle = t \big\}. \tag{29}$$

The joint conditional distribution of $(X_t, L'_t) := (X_t, L_t(1), \ldots, L_t(N-1))$ under $\mathbb{P}_k$ is 0 on $\mathbb{R}^{N-1}\backslash\overline{\mathcal{C}}_t$ and its a.c. part has the following density density $\psi_{k,\ell,t}$ from [Ser99, Cor. 4.4]

$$\begin{aligned} \forall (k,\ell) \in \mathbb{X}^2, \forall y \in \mathbb{R}^{N-1},\ \psi_{k,\ell,t}(y) := 1_{\mathcal{C}_t}(y)\, a^{N-1} \sum_{n=0}^{\infty} e^{-at}\frac{(at)^n}{n!} \\ \times \sum_{\substack{k_1 \ge 0,\ldots,k_{N-1}\ge 0,\\ \sum_{j=1}^{N-1} k_j \le n}} x^{t;y}_{n;k_1,\ldots,k_{N-1}}\, p_{k,\ell}(n+N, k_1,\ldots,k_{N-1}) \end{aligned} \tag{30}$$

where the non-negative coefficient $p_{k,\ell}(n+N,k_1,\ldots,k_{N-1})$ satisfies $0 \le \sum_{\ell=1}^{N-1} p_{k,\ell}(n+N,k_1,\ldots,k_{N-1}) \le 1$ and we have the following relation

$$\sum_{\substack{k_2 \ge 0,\ldots,k_N\ge 0,\\ \sum_{j=2}^{N} k_j \le n}} n!\, x^{t;y_2,\ldots,y_N}_{n;k_2,\ldots,k_N} = 1 \tag{31}$$

For every $(k,\ell) \in \mathbb{X}^2$, the Lebesgue decomposition of the positive measure $\mathcal{F}_{k,\ell,t}$ defined by $\mathbb{P}_k\{X_t = \ell, L'_t \in \cdot\}$ writes as follows

$$\forall B \in B(\mathbb{R}^{N-1}), \quad \mathcal{F}_{k,\ell,t}(1_B) = \mathbb{P}_k\{X_t = \ell, L'_t \in B\} = \int_B \psi_{k,\ell,t}(y)\, dy + \alpha_{k,\ell,t}(1_B). \tag{32}$$

First note that Condition **(I-A)** is fulfilled since the matrix $P := P_1 = e^G$ is irreducible and aperiodic when the generator G of the driving jump process is irreducible. Second, the random variables $\|L_t\|$ (and so $\|Y_t\|$) are bounded, so that the moment condition **(M$\boldsymbol{\alpha}$)** is satisfied for any $\alpha > 0$. This allows us to apply Theorem 2.1 to derive a CLT for $\{t^{-1/2}Y'_t\}_{t\geq 0}$. Next, the main steps of the proof are to check that Conditions **(AC1)-(AC2)** hold under the assumptions of Proposition 5.1 on the generator G.

5.2.1 Checking Condition (AC2)

For every $y \in \mathbb{R}^{N-1}$, we introduce the following real $N \times N$-matrix

$$\Psi_t(y) := (\psi_{k,\ell,t}(y))_{(k,\ell)\in\mathbb{X}^2}.$$

where $\psi_{k,\ell,t}$ is defined in (30). The useful properties of $\psi_{k,\ell,t}$ are given in the next lemma which is proved in Appendix B.

Lemma 5.3 *For any $t > 0$, Ψ_t vanishes on $\mathbb{R}^d \setminus \overline{\mathcal{C}}_t$, is continuous on $\overline{\mathcal{C}}_t$ and differentiable on $\mathcal{C}_t$. We have*

$$\sup_{t>0} \sup_{y\in\mathbb{R}^{N-1}} \|\Psi_t(y)\|_0 \leq \sup_{t>0} \sup_{y\in\mathbb{R}^{N-1}} \|\Psi_t(y)\mathbf{1}^\top\|_0 \leq a^{N-1}; \tag{33a}$$

$$j = 1, \dots, N-1 \qquad \sup_{t>0} \sup_{y\in\mathcal{C}_t} \Big\|\frac{\partial \Psi_t}{\partial y_j}(y)\Big\|_0 \leq 2a^N. \tag{33b}$$

There exists $\rho \in (0,1)$ such that

$$\forall y \in \partial\mathcal{C}_t, \quad \|\Psi_t(y)\|_0 = O(e^{at(\rho-1)}(1+at)). \tag{33c}$$

Now, let us check that Conditions **(AC2)** hold for the function

$$G_t(y) = \Psi_t(y + m't) \tag{34}$$

associated with the absolutely continuous part of the probability distribution of the centered r.v. $(X_t, Y'_t) = (X_t, L'_t - m't)$ under $\mathbb{P}_k$ and $m' := (\pi_1, \dots, \pi_{N-1})$. It vanishes on the open convex $\mathcal{D}_t := T_{-m't}(\mathcal{C}_t)$ which is the translate of $\mathcal{C}_t$ by the translation $T_{-m't}$ of vector $-m't$. More precisely,

$$\mathcal{D}_t = \big\{y \in \mathbb{R}^{N-1} : j = 1, \dots, N-1,\ y_j \in (-m_j t, (1-m_j)t),\ \langle y, \mathbf{1}\rangle < m_N t\big\}.$$

with adherence

$$\overline{\mathcal{D}}_t = \big\{y \in \mathbb{R}^{N-1} : j = 1, \dots, N-1,\ y_j \in [-m_j t, (1-m_j)t],\ \langle y, \mathbf{1}\rangle \leq m_N t\big\}. \tag{35a}$$

and boundary

$$\partial\mathcal{D}_t = \bigcup_{j=1}^{N-1} \big\{y \in \overline{\mathcal{D}}_t \mid y_j = -m_j t\big\} \cup \big\{y \in \overline{\mathcal{D}}_t : \langle y, \mathbf{1}\rangle = m_N t\big\}. \tag{35b}$$

Then it follows from the basic properties of function Ψ_t on $\mathcal{C}_t$ stated in Lemma 5.3 that G_t is continuous on $\overline{\mathcal{D}}_t$ and differentiable on $\mathcal{D}_t$ with differential

$$\forall j = 1, \dots, N-1,\ \forall y \in \mathcal{D}_t, \quad \frac{\partial G_t}{\partial y_j}(y) = \frac{\partial \Psi_t}{\partial y_j}(y + m't).$$

Moreover, we obtain from (33a)-(33c)

$$\sup_{t>0} \sup_{y \in \mathbb{R}^{N-1}} \|G_t(y)\|_0 \le \sup_{t>0} \sup_{y \in \mathbb{R}^{N-1}} \|G_t(y)\mathbf{1}^\top\|_0 \le \sup_{t>0} \sup_{y \in \mathbb{R}^{N-1}} \|\Psi_t(y)\mathbf{1}^\top\|_0 \le a^{N-1};$$

$$\forall y \in \partial\mathcal{D}_t, \quad \|G_t(y)\|_0 = \|\Psi_t(y + m't)\|_0 = O\big(\frac{1}{t}\big);$$

$$\sup_{t>0} \sup_{y \in \mathcal{D}_t} \big\|\frac{\partial G_t}{\partial y}(y)\big\|_0 \le \sup_{t>0} \sup_{y \in \mathcal{C}_t} \big\|\frac{\partial \Psi_t}{\partial y}(y)\big\|_0 \le 2a^N.$$

5.2.2 Checking Condition (AC1)

For any $i \in \{1, \dots, N\}$, we introduce the following $(N-1) \times (N-1)$-subgenerator of G, $G_{i^c i^c} := (G(k,\ell))_{k,\ell \in \{i\}^c}$. If $G_{i^c i^c}$ is irreducible then $\|e^{tG_{i^c i^c}}\|_0 = O(e^{-r_i t})$ where $-r_i$ is the Perron-Frobenius negative eigenvalue of $G_{i^c i^c}$. Thus, if $G_{i^c i^c}$ is irreducible for any $i \in \{1, \dots, N\}$

$$\max_{i \in \{1,\dots,N\}} \|e^{tG_{i^c i^c}}\|_0 = O(e^{-rt}) \qquad \text{where } r := \min_i (r_i) > 0. \tag{36}$$

In a first step, we study the singular part $\mathcal{A}_t$ of the Lebesgue decomposition (32) where $\mathcal{A}_t$ is the $N \times N$-matrix with entries in the set of bounded positive measures on $\mathbb{R}^{N-1}$

$$\mathcal{A}_t := (\alpha_{k,\ell,t})_{(k,\ell) \in \mathbb{X}^2}.$$

We show that $\|\mathcal{A}_t(1_{\mathbb{R}^{N-1}})\|_0$ goes to 0 at a geometric rate when t grows to infinity. Note that $\mathcal{A}_t(1_B) = 0$ for every $B \in B(\mathbb{R}^{N-1})$ such that $B \cap \partial\mathcal{C}_t = \emptyset$. Next, it remains to show that there exist $c > 0$ and $\rho \in (0,1)$ such that

$$\forall t > 0, \quad \|\mathcal{A}_t(1_{\partial\mathcal{C}_t})\|_0 \le c\rho^t.$$

First, let us consider, for any $i \in \{1, \dots, N-1\}$, the set $d_{i,t} := \big\{(y_1, \dots, y_{N-1}) \in \overline{\mathcal{C}}_t \mid y_i = 0\big\}$:

$$\alpha_{k,\ell,t}(1_{d_{i,t}}) \quad \le \quad \mathbb{P}_k\{L_t(i) = 0\} = \begin{cases} 0 & \text{if } k = i \\ \sum_\ell e^{tG_{i^c i^c}}(k,\ell) & \text{if } k \ne i. \end{cases}$$

Thus, $\max_i \|\mathcal{A}_t(1_{d_{i,t}})\|_0 = O(e^{-rt})$ from (36). Second let us denote $s_t := \big\{(y_1, \dots, y_{N-1}) \in \overline{\mathcal{C}}_t \mid \sum_{j=1}^{N-1} y_j = t\big\}$. We can write for all $(k,\ell) \in \mathbb{X}^2$

$$\begin{aligned} \alpha_{k,\ell,t}(1_{s_t}) \quad &\le \quad \mathbb{P}_k\{L_t(N) = 0, X_t = \ell\} \\ &\le \quad \mathbb{P}_k\{L_t(N) = 0\} = \begin{cases} 0 & \text{if } k = N \\ \sum_\ell e^{tG_{N^c N^c}}(k,\ell) & \text{if } k \ne N. \end{cases} \end{aligned}$$

Therefore, $\|\mathcal{A}_t(1_{s_t})\|_0 = O(e^{-rt})$. Combining the previous estimates, we obtain that there exist $c > 0$ and $\rho \in (0,1)$ such that

$$\forall t > 0, \quad \|\mathcal{A}_t(1_{\mathbb{R}^{N-1}})\|_0 = \|\mathcal{A}_t(1_{\partial\mathcal{C}_t})\|_0 \le c\rho^t.$$

It follows that there exist $c > 0$ and $\rho \in (0,1)$ such that

$$\forall t > 0, \quad \|\mathcal{M}_t(1_{\mathbb{R}^{N-1}})\|_0 = \|\mathcal{A}_t(1_{\mathbb{R}^{N-1}})\|_0 \leq c\rho^t.$$

where $\mathcal{M}_t$ is the matrix associated with the singular part of the probability distribution of (X_t, Y'_t).

In a second step, we prove that for every $t_0 > 0$

$$\sup_{t\in[t_0,2t_0)} \|\widehat{\Psi}_t(\zeta)\|_0 \longrightarrow 0 \quad \text{when } \|\zeta\| \to +\infty.$$

Indeed, for every $t > 0$, for every $(k,\ell) \in \mathbb{X}^2$, the Fourier transform $\widehat{\psi}_{k,\ell,t}$ of $\psi_{k,\ell,t}$ has the following form (if $N = 2$ consider only the first integral)

$$\widehat{\psi}_{k,\ell,t}(\zeta) = \int_0^t \int_0^{t-y_1} \cdots \int_0^{t-\sum_{j\leq N-2} y_j} \psi_{k,\ell,t}(y) e^{i\zeta_{N-1}y_{N-1}} dy_{N-1} \dots dy_2\, dy_1.$$

Using an integration by part, we obtain that for every $\zeta_{N-1} \neq 0$

$$\int_0^{t-\sum_{j\leq N-2} y_j} \psi_{k,\ell,t}(y) e^{i\zeta_{N-1}y_{N-1}}\, dy_{N-1} =$$
$$\frac{1}{i\zeta_{N-1}} \left(\Big[\psi_{k,\ell,t}(y) e^{i\zeta_{N-1}y_{N-1}}\Big]_{y_{N-1}=0}^{y_{N-1}=t-\sum_{j\leq N-2} y_j} - \int_0^{t-\sum_{j\leq N-2} y_j} \frac{\partial \psi_{k,\ell,t}}{\partial y_{N-1}}(y)\, e^{iy_{N-1}\zeta_{N-1}}\, dy_{N-1} \right)$$

so that, we obtain from the bounds (33a)-(33b) that

$$\begin{aligned} |\widehat{\psi}_{k,\ell,t}(\zeta)| &\leq \frac{1}{|\zeta_{N-1}|} \int_0^t \cdots \int_0^{t-\sum_{j\leq N-3} y_j} \big(2a^{N-1} + 2a^N(t - \sum_{j\leq N-2} y_j)\big) dy_{N-2} \dots dy_1 \\ &\leq \frac{1}{|\zeta_{N-1}|} \big(2a^{N-1} \frac{t^{N-2}}{(N-2)!} + 2a^N \frac{t^{N-1}}{(N-1)!}\big). \end{aligned}$$

Finally, for every $t_0 > 0$

$$\sup_{t\in[t_0,2t_0)} \|\widehat{\Psi}_t(\zeta)\|_0 \leq \frac{2a^{N-1}(1+a)\max(1,t_0^N)}{|\zeta_{N-1}|}.$$

For each $j \in \{1,\dots,y_{N-2}\}$, using similar computations from an integration by parts with respect to the variable y_j, the same inequality holds with ζ_j in place of ζ_{N-1}. Therefore, we obtain that

$$\forall \zeta \in \mathbb{R}^{N-1}\backslash\{0\}, \quad \sup_{t\in[t_0,2t_0)} \|\widehat{\Psi}_t(\zeta)\|_0 \leq \frac{2a^{N-1}(1+a)\max(1,t_0^N)}{\|\zeta\|_0}.$$

and this quantity goes to 0 when $\|\zeta\| \to +\infty$. Since $G_t(y) = \Psi_t(y + tm)$ we have $\widehat{G}_t(\zeta) = e^{-i\langle m',\zeta\rangle t}\widehat{\Psi}_t(\zeta)$ for any $\zeta \in \mathbb{R}^{N-1}$ and

$$\forall t > 0,\ \forall \zeta \in \mathbb{R}^{N-1}, \quad \|\widehat{G}_t(\zeta)\|_0 = \|\widehat{\Psi}_t(\zeta)\|_0.$$

It follows that

$$\Gamma(\zeta) \equiv \Gamma_{t_0}(\zeta) := \sup_{t\in[t_0,2t_0)} \|\widehat{G}_t(\zeta)\|_0 \longrightarrow 0 \quad \text{when } \|\zeta\|_0 \to +\infty.$$

6 A uniform LLT with respect to transition matrix P

Let $\mathcal{P}$ denote the set of irreducible and aperiodic stochastic $N \times N$-matrices. The topology in $\mathcal{P}$ is associated (for instance) with the distance $d(P,P') := \|P - P'\|_\infty$ $(P,P' \in \mathcal{P})$. Again $(X_t,Y_t)_{t\in\mathbb{T}}$ is a centered MAP with state space $\mathbb{X}\times\mathbb{R}^d$, where $\mathbb{X} := \{1,\ldots,N\}$, and $\{P_t\}_{t\in\mathbb{T}}$ denotes the transition semi-group of the Markov process $\{X_t\}_{t\in\mathbb{T}}$. In this section, the stochastic matrix $P := P_1$ is assumed to belong to a compact subset $\mathcal{P}_0$ of $\mathcal{P}$, and we give assumptions for the LLT of Theorem 2.2 to hold uniformly in $P \in \mathcal{P}_0$.

For every $k \in \mathbb{X}$ the underlying probability measure $\mathbb{P}_k$ and the associated expectation $\mathbb{E}_k$ depend on P. To keep in mind this dependence, they are denoted by $\mathbb{P}_k^P$ and $\mathbb{E}_k^P$ respectively. Similarly we use the notations Σ^P, $\mathcal{Y}_t^P$, $\mathcal{G}_t^P$, $\mathcal{M}_t^P$ and G_t^P for the covariance matrix of Theorem 2.1 and the matrices in (6a)-(6b) respectively. Let $\mathfrak{M}$ denote the space of the probability measures on $\mathbb{R}^d$ equipped with the total variation distance d_{TV}.

Let $\mathcal{P}_0$ be any compact subset of $\mathcal{P}$. Let us consider the following assumptions:

U1 : *For any $(k,\ell)\in\mathbb{X}^2$, the map $P\mapsto\mathcal{Y}_{k,\ell,1}^P$ is continuous from $(\mathcal{P}_0,d)$ into $(\mathfrak{M},d_{TV})$.*

U2 : *There exist positive constants α and β such that*

$$\forall P\in\mathcal{P}_0,\ \forall\zeta\in\mathbb{R}^d,\quad \alpha\,\|\zeta\|^2 \le \langle\zeta,\Sigma^P\zeta\rangle \le \beta\,\|\zeta\|^2. \tag{37}$$

U3 : *The conditions* **(M3)**, **(AC1)** *and* **(AC2)** *hold uniformly in $P\in\mathcal{P}_0$.*

Theorem 6.1 *Under Assumptions (U1)-(U3), for every $k\in\mathbb{X}$, the density $f_{k,t}^P(\cdot)$ of the a.c. part of the probability distribution of $t^{-1/2}Y_t$ under $\mathbb{P}_k^P$ satisfies the following asymptotic property when $t\to+\infty$:*

$$\sup_{P\in\mathcal{P}_0}\sup_{y\in\mathbb{R}^d}\big|f_{k,t}^P(y)-\eta_{\Sigma^P}(y)\big| = O(t^{-1/2}) + O\big(\sup_{P\in\mathcal{P}_0}\sup_{y\notin\mathcal{D}_t}\eta_{\Sigma^P}(t^{-1/2}y)\big).$$

Assumptions (U3) read as follows:

U3-1 : *The r.v. $\{Y_v\}_{v\in(0,1]\cap\mathbb{T}}$ satisfies the following moment condition:*

$$M := \sup_{P\in\mathcal{P}_0}\max_{k\in\mathbb{X}}\sup_{v\in(0,1]\cap\mathbb{T}}\mathbb{E}_k^P[\|Y_v\|^3] < \infty. \tag{38}$$

U3-2 : *There exist $c>0$ and $\rho\in(0,1)$ such that*

$$\forall t>0,\quad \sup_{P\in\mathcal{P}_0}\|\mathcal{M}_t^P(1_{\mathbb{R}^d})\|_0 \le c\rho^t \tag{39}$$

and there exists $t_0>0$ such that

$$\rho^{t_0}\max(2,cN)\le 1/4 \tag{40a}$$

$$\Gamma_{t_0}(\zeta) := \sup_{P\in\mathcal{P}_0}\sup_{w\in[t_0,2t_0)}\|\widehat{G_w^P}(\zeta)\|_0 \longrightarrow 0 \quad \textit{when } \|\zeta\|\to+\infty. \tag{40b}$$

U3-3 : *For any $t > 0$, there exists an open convex subset $\mathcal{D}_t$ of $\mathbb{R}^d$ such that G_t vanishes on $\mathbb{R}^d \setminus \overline{\mathcal{D}}_t$, where $\overline{\mathcal{D}}_t$ denotes the adherence of $\mathcal{D}_t$. Moreover G_t is continuous on $\overline{\mathcal{D}}_t$ and differentiable on $\mathcal{D}_t$, with in addition*

$$\sup_{P\in\mathcal{P}_0}\sup_{t>0}\sup_{y\in\overline{\mathcal{D}}_t}\|G_t^P(y)\|_0 < \infty \tag{41a}$$

$$\sup_{P\in\mathcal{P}_0}\sup_{y\in\partial\mathcal{D}_t}\|G_t^P(y)\|_0 = O\big(\frac{1}{t}\big) \quad \textit{where} \;\; \partial\mathcal{D}_t := \overline{\mathcal{D}}_t \setminus \mathcal{D}_t \tag{41b}$$

$$j = 1,\ldots,d: \quad \sup_{P\in\mathcal{P}_0}\sup_{t>0}\sup_{y\in\mathcal{D}_t}\big\|\frac{\partial G_t^P}{\partial y_j}(y)\big\|_0 < \infty. \tag{41c}$$

Proof. The proof of Theorem 6.1 borrows the same way as for Theorem 2.2. What we only have to do is to prove that the bounds in Lemmas 3.1-3.3 are uniform in $P \in \mathcal{P}_0$ under Assumptions (U1)-(U2). For Lemma 3.3, this is obvious by using (U3-2). The proof of Theorem 6.1 will be complete if we establish the uniform version of Lemma 3.1 and 3.2. This is done below. □

Lemma 6.1 *Assume that Condition (U1) holds and that, for some $\alpha > 0$,*

$$M_\alpha := \sup_{P\in\mathcal{P}_0}\max_{k\in\mathbb{X}}\sup_{v\in(0,1]\cap\mathbb{T}} \mathbb{E}_k^P[\|Y_v\|^\alpha] < \infty.$$

Then the map $(P,\zeta) \mapsto \widehat{\mathcal{Y}_1^P}(\zeta)$ is continuous from $\mathcal{P}_0 \times \mathbb{R}^d$ into $\mathcal{M}_N(\mathbb{C})$.

Proof. We may suppose that $\alpha \in (0,1]$. Let $(k,\ell) \in \mathbb{X}^2$, $(P,P') \in \mathcal{P}_0^2$ and $(\zeta,\zeta') \in \mathbb{R}^d \times \mathbb{R}^d$. To simplify we write $\mathcal{Y}$ and $\mathcal{Y}'$ for $\mathcal{Y}_{k,\ell,1}^P$ and $\mathcal{Y}_{k,\ell,1}^{P'}$. Then

$$\begin{aligned}
\big|\big(\widehat{\mathcal{Y}_1^P}(\zeta)\big)_{k,\ell} - \big(\widehat{\mathcal{Y}_1^{P'}}(\zeta')\big)_{k,\ell}\big| &= \big|\int_{\mathbb{R}^d} e^{i\langle\zeta,y\rangle}\mathcal{Y}(dy) - \int_{\mathbb{R}^d} e^{i\langle\zeta',y\rangle}\mathcal{Y}'(dy)\big| \\
&\le \int_{\mathbb{R}^d}\big|e^{i\langle\zeta,y\rangle} - e^{i\langle\zeta',y\rangle}\big|\,\mathcal{Y}(dy) \\
&\qquad + \big|\int_{\mathbb{R}^d} e^{i\langle\zeta',y\rangle}\mathcal{Y}(dy) - \int_{\mathbb{R}^d} e^{i\langle\zeta',y\rangle}\mathcal{Y}'(dy)\big| \\
&\le 2\,\|\zeta-\zeta'\|^\alpha \int_{\mathbb{R}^d}\|y\|^\alpha\,\mathcal{Y}(dy) + d_{TV}(\mathcal{Y},\mathcal{Y}') \\
&\le 2M_\alpha\,\|\zeta-\zeta'\|^\alpha + d_{TV}(\mathcal{Y},\mathcal{Y}')
\end{aligned}$$

(use $|e^{iu}-1| \le 2|u|^\alpha$, $u \in \mathbb{R}$, and the Cauchy-Schwarz inequality to obtain the second inequality above). The desired continuity property then follows from (U1). □

From now on, we sometimes omit the notational exponent P. The uniformity in Lemma 3.1 is obtained as follows. Recall that, for any $P \in \mathcal{P}_0$ fixed, Formula (15) with $t = n \in \mathbb{N}$ follows from the standard perturbation theory. Similarly, using Lemma 6.1, Formula (15) with $t = n \in \mathbb{N}$ can be obtained for every $P \in \mathcal{U}_0$ and for all $\zeta \in \mathbb{R}^d$ such that $\|\zeta\| \le \delta$, with $\delta > 0$ independent from $P_0 \in \mathcal{P}_0$ since $\mathcal{P}_0$ is compact. Moreover the associated functions $\lambda(\cdot)$, $L_{k,t}(\cdot)$ and $R_{k,t}(\cdot)$ in (15) (depending on P) satisfy the properties (16a)-(16d) in a uniform way in $P \in \mathcal{P}_0$ from (U3-1). Note that Condition (U2) is useful to obtain (16b) uniformly in $P \in \mathcal{P}_0$. The passage from the discrete-time case to the continuous-time again follows from [FHL12, Prop. 4.4] since the derivatives of $\zeta \mapsto \widehat{\mathcal{Y}_v}(\zeta)$ are uniformly bounded in $(P,v) \in \mathcal{P}_0 \times (0,1]$ from (U3-1).

From $\phi_{k,t}(\zeta) = e_k \widehat{\mathcal{Y}}_1(\zeta)^{\lfloor t \rfloor} \widehat{\mathcal{Y}}_v(\zeta) \mathbf{1}^\top$ with $v := t - \lfloor t \rfloor \in [0,1)$ and $\|\widehat{\mathcal{Y}_v}(\zeta)\mathbf{1}\|_\infty \le \|\widehat{\mathcal{Y}_v}(0)\|_\infty \le 1$, the uniformity in Lemma 3.2 follows from the following lemma.

Lemma 6.2 *Let δ, A be such that $0 < \delta < A$. Then there exist $D \equiv D(\delta, A) > 0$ and $\tau \equiv \tau(\delta, A) \in (0,1)$ such that*

$$\forall n \in \mathbb{N}, \quad \sup_{P \in \mathcal{P}_0} \sup_{\delta \le \|\zeta\| \le A} \|\widehat{\mathcal{Y}_1^P}(\zeta)^n\|_\infty \le D\,\tau^n. \tag{42}$$

Proof. The spectral radius of any matrix $T \in \mathcal{M}_N(\mathbb{C})$ is denoted by $r(T)$. Suppose that

$$\rho_0 := \sup_{P \in \mathcal{P}_0} \sup_{\delta \le \|\zeta\| \le A} r\big(\widehat{\mathcal{Y}_1^P}(\zeta)\big) < 1. \tag{43}$$

Then (42) holds. Indeed, consider any $\tau \in (\rho_0, 1)$ and denote by Γ_τ the oriented circle in $\mathbb{C}$ centered at 0 with radius τ. Let I denote the identity $N \times N$-matrix. Property (43) and standard spectral calculus then give

$$\begin{aligned}
\sup_{P \in \mathcal{P}_0} \sup_{\delta \le \|\zeta\| \le A} \|\widehat{\mathcal{Y}_1^P}(\zeta)^n\|_\infty &\le \sup_{P \in \mathcal{P}_0} \sup_{\delta \le \|\zeta\| \le A} \Big\| \frac{1}{2i\pi} \oint_{\Gamma_\tau} z^n \big(zI - \widehat{\mathcal{Y}_1^P}(\zeta)\big)^{-1} dz \Big\|_\infty \\
&\le \tau^{n+1} \sup_{P \in \mathcal{P}_0} \sup_{\delta \le \|\zeta\| \le A} \|(zI - \widehat{\mathcal{Y}_1^P}(\zeta))^{-1}\|_\infty.
\end{aligned}$$

This proves (42) since the last bound is finite from the continuity of $(P, \zeta) \mapsto \widehat{\mathcal{Y}_1^P}(\zeta)$ on the compact set $\mathcal{P}_0 \times \{\zeta \in \mathbb{R}^d : \delta \le \|\zeta\| \le A\}$ (Lemma 6.1).

It remains to prove (43). Assume that $\rho_0 \ge 1$. Then $\rho_0 = 1$ since $r(\widehat{\mathcal{Y}_1}(\zeta)) \le \|\widehat{\mathcal{Y}_1}(\zeta)\|_\infty \le \|\widehat{\mathcal{Y}_1}(0)\|_\infty \le 1$. Thus there exists some sequences $(P_n)_n \in \mathcal{P}_0^{\mathbb{N}}$ and $(\zeta_n)_n \in (\mathbb{R}^d)^{\mathbb{N}}$ satisfying $\delta \le \|\zeta_n\| \le A$ such that

$$\lim_n r\big(\widehat{\mathcal{Y}_1^{P_n}}(\zeta_n)\big) = 1$$

By compactness one may suppose that $(P_n)_n$ and $(\zeta_n)_n$ respectively converge to some $P_\infty \in \mathcal{P}_0$ and some $\zeta_\infty \in \mathbb{R}^d$ such that $\delta \le \|\zeta_\infty\| \le A$. From Lemma 6.1 we obtain that

$$\lim_n \widehat{\mathcal{Y}_1^{P_n}}(\zeta_n) = \widehat{\mathcal{Y}_1^{P_\infty}}(\zeta_\infty).$$

Below, $\mathcal{Y}_1^{P_\infty}(\zeta_\infty)$ is simply denoted by $\mathcal{Y}_1(\zeta_\infty)$. From the upper semi-continuity of the map $T \mapsto r(T)$ on $\mathcal{M}_N(\mathbb{C})$ and from $r(\widehat{\mathcal{Y}_1}(\zeta_\infty)) \le 1$, it follows that $r(\widehat{\mathcal{Y}_1}(\zeta_\infty)) = 1$. Write $S := \widehat{\mathcal{Y}_1}(\zeta_\infty)$ to simplify. Then S admits an eigenvalue λ of modulus one. Let $f = (f(k))_{k \in \mathbb{X}} \in \mathbb{C}^N$ be an associated nonzero eigenvector. We have

$$\forall k \in \mathbb{X}, \quad |\lambda f(k)| = |f(k)| \le |(Sf)(k)| \le (\widehat{\mathcal{Y}_1}(0)|f|)(k) = (P|f|)(k),$$

where $|f| := (|f(k)|)_{k \in \mathbb{X}}$. Recall that the P_∞-invariant probability measure $\pi = (\pi(\ell))_{\ell \in \mathbb{X}}$ is such that $\forall \ell \in \mathbb{X},\ \pi(\ell) > 0$ since P_∞ is irreducible. From $\pi(P_\infty|f|) = \pi(|f|)$ and the positivity of $P_\infty|f| - |f|$, it follows that $P_\infty|f| = |f|$. Thus $|f| = c\mathbf{1}$ with some constant c. We may assume that $|f| = \mathbf{1}$. Equality $\lambda f = Sf$ rewrites as

$$\forall k \in \mathbb{X}, \quad \lambda f(k) = \sum_{\ell=1}^N f(\ell)\, \widehat{\mathcal{Y}_1}(\zeta_\infty)_{k,\ell} = \sum_{\ell=1}^N f(\ell)\, \mathbb{E}_k^{P_\infty}\big[1_{\{X_1 = \ell\}}\, e^{i\langle \zeta_\infty, Y_1\rangle}\big] = \mathbb{E}_k^{P_\infty}\big[f(X_1)\, e^{i\langle \zeta_\infty, Y_1\rangle}\big].$$

From $|f(k)| = 1$ for every $k \in \mathbb{X}$ and from standard convexity arguments, we obtain

$$\forall k \in \mathbb{X}, \quad \lambda f(k) = f(X_1)\, e^{i\langle \zeta_\infty, Y_1\rangle} \quad \mathbb{P}_k^{P_\infty} - \text{a.s..}$$

Now writing $\lambda = e^{ib}$ with $b \in \mathbb{R}$ and $f(\ell) = e^{ig(\ell)}$ for every $\ell \in \mathbb{X}$, one can deduce from the previous equality that

$$\forall k \in \mathbb{X}, \quad \langle \zeta_\infty, Y_1\rangle + g(X_1) - g(k) \in b + 2\pi\mathbb{Z} \quad \mathbb{P}_k^{P_\infty} - \text{a.s..}$$

Define $a := \frac{b}{\|\zeta_\infty\|^2}\zeta_\infty \in \mathbb{R}^d$ and $\theta : \mathbb{X} \to \mathbb{R}^d$ by $\theta(\ell) := \frac{g(\ell)}{\|\zeta_\infty\|^2}\zeta_\infty$. Consider the following closed subgroup $H := \frac{2\pi\mathbb{Z}}{\|\zeta_\infty\|^2}\,\zeta_\infty \oplus (\mathbb{R}\cdot\zeta_\infty)^\perp$ in $\mathbb{R}^d$. Then the previous property is equivalent to

$$\forall k \in \mathbb{X}, \quad Y_1 + \theta(X_1) - \theta(k) \in a + H \quad \mathbb{P}_k^{P_\infty} - \text{a.s..}$$

But Assumption (U3-2) obviously implies that P_∞ satisfies **(AC1)**, so that the last property is impossible as seen in the proof of Lemma 3.2. Property (43) is proved. $\square$

A Proof of Lemma 5.2

Assume that

$$\forall B' \in B(\mathbb{R}^{N-1}), \quad \mathbb{P}_k\big\{X_t = \ell, Y'_t \in B'\big\} = \int_{\mathbb{R}^{N-1}} g'(y)\, dy + \mu'(A)$$

avec $\mu' \perp \ell_{N-1}$. Let $B \in B(H)$. Then

$$\begin{aligned}
\mathbb{P}_k\big\{X_t = \ell, Y_t \in B\big\} &= \mathbb{P}_k\big\{X_t = \ell, \Lambda(Y_t) \in \Lambda(B)\big\} \\
&= \int_{\Lambda(B)} g'(y)\, dy + \mu'(\Lambda(B)) \\
&= \int_{\mathbb{R}^{N-1}} 1_B(\Lambda^{-1}(y))\, g'\big(\Lambda(\Lambda^{-1}y)\big)\, dy + \mu'(\Lambda(B)) \\
&= \int_B (g' \circ \Lambda)(x)\, d\eta(x) + \mu'(\Lambda(B)).
\end{aligned}$$

Let $d\mu$ be the image measure of $d\mu'$ under Λ^{-1}, that is : $\forall B \in B(H),\ \mu(B) := \mu'(\Lambda(B))$. Then we have $\mu \perp \eta$. Indeed, we know that there exist two disjoint sets $E', F' \in B(\mathbb{R}^{N-1})$ such that

$$\forall B' \in B(\mathbb{R}^{N-1}), \quad \ell_{N-1}(B') = \ell_{N-1}(B' \cap E') \quad \text{and} \quad \mu'(B') = \mu'(B' \cap F').$$

Then, introducing $E := \Lambda^{-1}(E')$ and $F := \Lambda^{-1}(F')$, we clearly have for any $B \in B(H)$

$$\eta(B) = \ell_{N-1}(\Lambda(B)) = \ell_{N-1}\big(\Lambda(B) \cap E'\big) = \ell_{N-1}\big(\Lambda(B \cap E)\big) = \eta(B \cap E)$$

and

$$\mu(B) = \mu'(\Lambda(B)) = \mu'\big(\Lambda(B) \cap F'\big) = \mu'\big(\Lambda(B \cap F)\big) = \mu(B \cap F),$$

so that η and μ are supported by the two disjoint sets E et F.

B Proof of Lemma 5.3

Let us recall that the density $\psi_{k,\ell,t}$ is given by (30). We can be a little bit more precise on the properties of the coefficients invoked in (30). Indeed, we know from [Ser99] that

$$p_{k,\ell}(n+N,k_1,\dots,k_{N-1}) := \mathbb{P}_k\{V^1_{n+N-1}=k_1+1,\dots,V^{N-1}_{n+N-1}=k_{N-1}+1, Z_{n+N-1}=\ell\}$$

with V^i_n the number of visits to state i at time n

$$\forall i=1,\dots,N-1 \quad V^i_n=\sum_{j=0}^{n} 1_{\{Z_j=i\}}. \tag{44}$$

Moreover, if $y=(y_1,\dots,y_{N-1})$

$$x^{t;y}_{n;k_1,\dots,k_{N-1}} := \frac{n!}{k_1!\cdots k_{N-1}!(n-\sum_{j=1}^{N-1}k_j)!}\prod_{j=1}^{N-1}\left(\frac{y_j}{t}\right)^{k_j}\left(1-\frac{\sum_{j=1}^{N-1}y_j}{t}\right)^{n-\sum_{j=1}^{N-1}k_j} \tag{45}$$

Recall that these coefficients satisfy the property (31).

We deduce (33a) from $\sum_{\ell\in\mathbb{X}} p_{k,\ell}(n+N,k_1,\dots,k_{N-1})\le 1$ and (31):

$$\forall y\in\mathbb{R}^{N-1},\quad |\psi_{k,\ell,t}(y)|\le\sum_{j\in\mathbb{X}}|\psi_{k,j,t}(y)|\le 1_{\mathcal{C}_t}(y)\,a^{N-1}\sum_{n=0}^{\infty}e^{-at}\frac{(at)^n}{n!}\le a^{N-1}.$$

Let us check (33b) for $j:=1$. We obtain

$$\begin{aligned}
&\frac{\partial\psi_{k,\ell,t}}{\partial y_1}\\
&= a^{N-1}\sum_{n=1}^{\infty}e^{-at}\frac{(at)^n}{n!}\frac{1}{t}\\
&\quad\times\Bigg[\sum_{\substack{k_1\ge 1,\dots,k_{N-1}\ge 0,\\ \sum_{j=1}^{N-1}k_j\le n}} n\,x^{t;y}_{n-1;k_1-1,\dots,k_{N-1}}\,p_{k,\ell}(n+N,k_1,\dots,k_{N-1})\\
&\qquad-\sum_{\substack{k_1\ge 0,\dots,k_{N-1}\ge 0,\\ \sum_{j=1}^{N-1}k_j\le n-1}} n\,x^{t;y}_{n-1;k_1,\dots,k_{N-1}}\,p_{k,\ell}(n+N,k_1,\dots,k_{N-1})\Bigg]
\end{aligned}$$

so that

$$\begin{aligned}
\frac{\partial\psi_{k,\ell,t}}{\partial y_1} &= a^{N}\sum_{n=1}^{\infty}e^{-at}\frac{(at)^{n-1}}{(n-1)!}\\
&\times\Bigg[\sum_{\substack{k_1\ge 0,\dots,k_{N-1}\ge 0,\\ \sum_{j=1}^{N-1}k_j\le n-1}} x^{t;y}_{n-1;k_1,\dots,k_{N-1}}\,p_{k,\ell}(n+N,k_1+1,k_2,\dots,k_{N-1})\\
&\qquad-\sum_{\substack{k_1\ge 0,\dots,k_{N-1}\ge 0,\\ \sum_{j=1}^{N-1}k_j\le n-1}} x^{t;y}_{n-1;k_1,\dots,k_{N-1}}\,p_{k,\ell}(n+N,k_1,\dots,k_{N-1})\Bigg].
\end{aligned}$$

Finally,

$$\frac{\partial \psi_{k,\ell,t}}{\partial y_1}(y) = a^N \sum_{n=0}^{\infty} e^{-at} \frac{(at)^n}{n!} \sum_{\substack{k_1 \geq 0, \dots, k_{N-1} \geq 0, \\ \sum_{j=1}^{N-1} k_j \leq n}} x^{t;y}_{n;k_1,\dots,k_{N-1}}$$
$$\times \big[p_{k,\ell}(n+N+1, k_1+1, k_2, \dots, k_{N-1}) - p_{k,\ell}(n+N+1, k_1, \dots, k_{N-1})\big].$$

It follows that

$$\forall y \in \mathcal{C}_t, \quad \big|\frac{\partial \psi_{k,\ell,t}}{\partial y_1}(y)\big| \leq \sum_{\ell \in \mathbb{X}} \big|\frac{\partial \psi_{k,\ell,t}}{\partial y_1}(y)\big| \leq a^N 2 \sum_{n=0}^{\infty} e^{-at} \frac{(at)^n}{n!} \leq 2a^N.$$

The same computation for each variable y_j $j = 2, \dots, N-1$ gives the bound in (33b).

For any $t > 0$, the definition of function Ψ_t can be extended on the boundary $\partial\mathcal{C}_t$ (see (29)) since the coefficients in (45) are well defined on $\overline{\mathcal{C}}_t$. Moreover, since these coefficients as function of y are continuous on $\overline{\mathcal{C}}_t$, it is easily seen that the extended version of Ψ_t is continuous on $\overline{\mathcal{C}}_t$ and is also denoted by Ψ_t in the sequel. Next, we study the behaviour of this function on the boundary $\partial\mathcal{C}_t$.

We have for any $y \in \big\{(y_1, \dots, y_{N-1}) \in \overline{\mathcal{C}}_t : y_i := 0\big\}$ that $x^{t;y}_{n;k_1,\dots,k_{N-1}} = 0$ if $k_i \geq 1$, so that

$$\psi_{k,\ell,t}(y) := a^{N-1} \sum_{n=0}^{\infty} e^{-at} \frac{(at)^n}{n!}$$
$$\times \sum_{\substack{k_1 \geq 0, \dots, k_i = 0, \dots, k_{N-1} \geq 0, \\ \sum_{j=1}^{N-1} k_j \leq n}} x^{t;y}_{n;k_1,\dots,0,\dots,k_{N-1}}\, p_{k,\ell}(n+N, k_1, \dots, 0, \dots, k_{N-1}).$$

Moreover, using $p_{k,\ell}(n+N, k_1, \dots, 0, \dots, k_{N-1}) \leq \mathbb{P}_k\{V^i_{n+N-1} = 1\}$ and (31), we obtain

$$\forall y \in \big\{(y_1, \dots, y_{N-1}) \in \overline{\mathcal{C}}_t \mid y_i := 0\big\}, \quad |\psi_{k,\ell,t}(y)| \leq a^{N-1} \sum_{n=0}^{\infty} e^{-at} \frac{(at)^n}{n!} \mathbb{P}_k\{V^i_{n+N-1} = 1\}. \tag{46a}$$

Finally, for any $y \in \overline{\mathcal{C}}_t$ such that $\langle y, \mathbf{1}\rangle = t$, we have $x^{t;y}_{n;k_1,\dots,k_{N-1}} = 0$ if $\sum_{j=1}^{N-1} k_j < n$, so that

$$\psi_{k,\ell,t}(y) := a^{N-1} \sum_{n=0}^{\infty} e^{-at} \frac{(at)^n}{n!}$$
$$\times \sum_{\substack{k_1 \geq 0, \dots, k_i \geq 0, \dots, k_{N-1} \geq 0, \\ \sum_{j=1}^{N-1} k_j = n}} x^{t;y}_{n;k_1,\dots,k_{N-1}}\, p_{k,\ell}(n+N, k_1, \dots, k_{N-1}).$$

Since $n = \sum_{j=1}^{N-1} k_j$, we have $\sum_{j=1}^{N-1}(k_j + 1) = n + N - 1$ and for any $(k, \ell) \in \mathbb{X}^2$,

$$p_{k,\ell}(n+N, k_1, \dots, k_{N-1}) \leq \mathbb{P}_k\{V^N_{n+N-1} = 0\}.$$

It allows us to write that

$$y \in \left\{(y_1, \dots, y_{N-1}) \in \overline{\mathcal{C}}_t \mid \sum_{j=1}^{N-1} y_j = t\right\}, \quad |\psi_{k,\ell,t}(y)| \leq a^{N-1} \sum_{n=0}^{\infty} e^{-at} \frac{(at)^n}{n!} \mathbb{P}_k\{V^N_{n+N-1} = 0\}. \tag{46b}$$

We deduce from the next Lemma B.1 and (46a)-(46b) that there is $\rho \in (0,1)$ such that

$$\forall y \in \partial\mathcal{C}_t, \quad \|\Psi_t(y)\|_0 = O\big(e^{at(\rho-1)}(1+at)\big).$$

Lemma B.1 *For a discrete time finite Markov chain $\{Z_n\}_{n\in\mathbb{N}}$ with transition matrix $\widetilde{P}$, let V_n^i be the number of visits to state i with respect to n transitions of the Markov chain (see* (44)*). Let us introduce the $(N-1)\times(N-1)$-matrix $\widehat{P}_{i^c i^c} := (\widetilde{P}(k,\ell))_{k,\ell\in\mathbb{X}\setminus\{i\}}$ and assume that this matrix is irreducible and aperiodic. Let $0<\rho_i<1$ be the Perron-Frobenius eigenvalue of $\widehat{P}_{i^c i^c}$. We have the following geometric estimates of the probabilities in* (46a)-(46b) *:*

1. $\mathbb{P}_i\{V_n^i=0\}=0$ *and* $\mathbb{P}_k\{V_n^i=0\}=O({\rho_i}^n)$
2. $\mathbb{P}_k\{V_n^i=1\}=O({\rho_i}^n+n{\rho_i}^{n-1})$ *for* $k\neq i$ *and* $\mathbb{P}_i\{V_n^i=1\}=O({\rho_i}^n)$

Remark B.1 *Note that the irreducibility of the sub-generators $G_{i^c i^c}$ allows us to derive exponential rate of convergence of $\|\Psi_t(y)\|_0$ on ∂C_t when t growths to infinity, given that only a rate in $1/t$ is required in Condition* **(AC2)**.

Proof. Since $\widehat{P}_{\{i\}^c\{i\}^c}$ is irreducible and aperiodic, $\|\widehat{P}_{\{i\}^c\{i\}^c}\mathbf{1}^\top\|_\infty\| \le \|\widehat{P}_{\{i\}^c\{i\}^c}\|_\infty \le C\rho_i$ for some constant C and $\rho_i\in(0,1)$ is the Perron-Frobenius eigenvalue of $\widehat{P}_{\{i\}^c\{i\}^c}$.

For the first assertion, note that $\mathbb{P}_i\{V_n^i=0\}=0$ since $V_n^i\ge 1$ and for $k\neq i$

$$\begin{aligned}\forall n\ge 0, \quad \mathbb{P}_k\{V_n=0\} &= \mathbb{P}_k\{\sum_{\ell=1}^n 1_{\{Z_\ell=i\}}=0\} = \mathbb{P}_k\{Z_1\neq i, Z_2\neq i,\dots,Z_n\neq i\} \\ &= \sum_{\ell\in\mathbb{X}}(\widehat{P}_{\{i\}^c\{i\}^c})^n(k,\ell) = e_k(\widehat{P}_{\{i\}^c\{i\}^c})^n\mathbf{1}^\top. \end{aligned} \tag{47}$$

Then, it follows that $\mathbb{P}_k\{V_n^i=0\}=O({\rho_i}^n)$.

For the second assertion, note that $\mathbb{P}_i\{V_n^i=1\}=\mathbb{P}_i\{\sum_{j=1}^n 1_{\{Z_j=i\}}=0\}=e_i(\widehat{P}_{\{i\}^c\{i\}^c})^n\mathbf{1}^\top$ so that, we deduce that $\mathbb{P}_i\{V_n^i=1\}=O({\rho_i}^n)$. For $k\neq i$, we can write the following renewal equation

$$\begin{aligned}\forall n\ge 0, \quad \mathbb{P}_k\{V_{n+1}=1\} &= \sum_{j\in\mathbb{X}} P(k,j)\mathbb{P}_j\{V_n=1\} \\ &= \sum_{j\neq i} P(k,j)\mathbb{P}_j\{V_n=1\} + P(k,i)\mathbb{P}_i\{V_n=1\}\end{aligned}$$

with $\mathbb{P}_j\{V_0=1\}=0$ if $j\neq i$ and $\mathbb{P}_i\{V_0=1\}=1$. Let us introduce the following column vector $v_i(n+1)=(\mathbb{P}_k\{V_{n+1}=1\})_{k\neq i}$. Then we have from the previous equation

$$\forall n\ge 0, \quad v_i(n+1) = P_{\{i\}^c\{i\}^c}v_i(n)+\mathbb{P}_i\{V_n=1\}P_{\{i\}^c\{i\}}, \qquad v_i(0)=0, \mathbb{P}_i\{V_0=1\}=1.$$

Then, we can obtain the following representation of vector $v_i(n+1)$

$$v_i(n+1)=\sum_{k=0}^n \mathbb{P}_i\{V_k=1\}{P_{\{i\}^c\{i\}^c}}^{n-k}P_{\{i\}^c\{i\}}. \tag{48}$$

Note that $\mathbb{P}_i\{V_k=1\}$ is known from (47). We obtain

$$v_i(n+1) = {P_{\{i\}^c\{i\}^c}}^n P_{\{i\}^c\{i\}} + \sum_{k=1}^n \Big(P_{\{i\}\{i\}^c}{P_{\{i\}^c\{i\}^c}}^{k-1}\,1^\top\Big)\times {P_{\{i\}^c\{i\}^c}}^{n-k}P_{\{i\}^c\{i\}}.$$

The kth component of the vector is given by

$$\begin{aligned}
\mathbb{P}_k\{V_{n+1}=1\} &= {e_k}^\top v_i(n+1) \\
&= {e_k}^\top {P_{\{i\}^c\{i\}^c}}^n P_{\{i\}^c\{i\}} \\
&\quad + \sum_{k=1}^{n} \left(P_{\{i\}\{i\}^c} {P_{\{i\}^c\{i\}^c}}^{k-1} 1^\top \right) \times {e_k}^\top {P_{\{i\}^c\{i\}^c}}^{n-k} P_{\{i\}^c\{i\}} \\
&\leq {e_k}^\top {P_{\{i\}^c\{i\}^c}}^n 1^\top + \sum_{k=1}^{n} \left(P_{\{i\}\{i\}^c} {P_{\{i\}^c\{i\}^c}}^{k-1} 1^\top \right) \times {e_k}^\top {P_{\{i\}^c\{i\}^c}}^{n-k} 1^\top.
\end{aligned}$$

Since $\|P_{\{i\}^c\{i\}^c}\| \leq C\rho_i$, there exists K such that

$$\mathbb{P}_k\{V_{n+1}=1\} \leq K({\rho_i}^n + \sum_{k=1}^{n} {\rho_i}^{k-1}\, {\rho_i}^{n-k}) = K({\rho_i}^n + n{\rho_i}^{n-1}).$$

□

C Linear transformation of MAPs

Let $\{(X_t,Y_t)\}_{t\in\mathbb{T}}$ be an MAP with state space $\mathbb{X}\times\mathbb{R}^d$, where $\mathbb{X} := \{1,\ldots,N\}$ and $\mathbb{T} := \mathbb{N}$ or $\mathbb{T} := [0,\infty)$. Recall from [Asm03] that $\{(X_t,Y_t)\}_{t\in\mathbb{T}}$ is a Markov process with a transition semi-group, denoted by $\{Q_t\}_{t\in\mathbb{T}}$, which satisfies

$$\forall (k,y)\in\mathbb{X}\times\mathbb{R}^d,\ \forall(\ell,B)\in\mathbb{X}\times B(\mathbb{R}^d),\quad Q_t(k,y;\{\ell\}\times B) = Q_t(k,0;\{\ell\}\times B-y). \tag{49}$$

Note that $Q_t(k,y;\{\ell\}\times B) := \mathbb{P}_k\{X_t=\ell, Y_t\in B\} = \mathcal{Y}_{k,\ell,t}(1_B)$. Let us consider any linear transformation $T:\mathbb{R}^d\to\mathbb{R}^m$ and introduce the $\mathbb{X}\times\mathbb{R}^m$-valued process $\{(X_t,TY_t)\}_{t\in\mathbb{T}}$.

Lemma C.1 *The process $\{(X_t,TY_t)\}_{t\in\mathbb{T}}$ is an MAP with state space $\mathbb{X}\times\mathbb{R}^m$ and transition semi-group $Q_t^{(T)}$ defined by: $\forall(k,z)\in\mathbb{X}\times\mathbb{R}^m,\ \forall(\ell,B)\in\mathbb{X}\times B(\mathbb{R}^m)$,*

$$Q_t^{(T)}(k,z;\{\ell\}\times B) = Q_t(k,0;\{\ell\}\times T^{-1}(B-z)). \tag{50}$$

Note that we only have to prove that $\{(X_t,TY_t)\}_{t\in\mathbb{T}}$ is a Markov process with transition semi-group defined by (50), since the additivity property for $\{(X_t,TY_t)\}_{t\in\mathbb{T}}$ is clearly satisfied from (50):

$$\begin{aligned}
Q_t^{(T)}(k,z;\{\ell\}\times B) &= Q_t(k,0;\{\ell\}\times T^{-1}(B-z)) \\
&=: Q_t^{(T)}(k,0;\{\ell\}\times B-z).
\end{aligned}$$

Proof. Let $\mathbb{F}_t^{(X,Y)} := \sigma(X_u,Y_u,\ u\leq t)$ and $\mathbb{F}_t^{(X,TY)} := \sigma(X_u,TY_u,\ u\leq t)$ be the filtration generated by the processes $\{(X_t,Y_t)\}_{t\in\mathbb{T}}$ and $\{(X_t,TY_t)\}_{t\in\mathbb{T}}$ respectively. Since $\{(X_t,Y_t)\}_{t\in\mathbb{T}}$ is an MAP with transition semi-group $\{Q_t\}_{t\in\mathbb{T}}$, we have by definition for any bounded function g on $\mathbb{X}\times\mathbb{R}^d$

$$\begin{aligned}
\mathbb{E}\left[g(X_{t+s},Y_{t+s}) \mid \mathbb{F}_s^{(X,Y)}\right] &= \sum_{\ell\in\mathbb{X}} \int_{\mathbb{R}^d} g(\ell,y_1) Q_t(X_s,Y_s;\{\ell\}\times dy_1) \\
&= \sum_{\ell\in\mathbb{X}} \int_{\mathbb{R}^d} g(\ell,y_1+Y_s) Q_t(X_s,0;\{\ell\}\times dy_1).
\end{aligned}$$

Using the tower rule and the last representation of the condition expectation, we obtain for any $\ell \in \mathbb{X}$ and $B \in B(\mathbb{R}^m)$

$$\begin{aligned}
\mathbb{E}\left[1_{\{\ell\}\times B}(X_{t+s}, TY_{t+s}) \mid \mathbb{F}_s^{(X,TY)}\right] &= \mathbb{E}\left[\mathbb{E}\left[1_{\{\ell\}\times B}(X_{t+s}, TY_{t+s}) \mid \mathbb{F}_s^{(X,Y)}\right] \mid \mathbb{F}_s^{(X,TY)}\right] \\
&= \mathbb{E}\left[\int_{\mathbb{R}^d} 1_B(T(y_1 + Y_s))Q_t(X_s, 0; \{\ell\} \times dy_1) \mid \mathbb{F}_s^{(X,TY)}\right] \\
&= \mathbb{E}\left[\int_{\mathbb{R}^d} 1_B(Ty_1 + TY_s)Q_t(X_s, 0; \{\ell\} \times dy_1) \mid \mathbb{F}_s^{(X,TY)}\right] \\
&= \int_{\mathbb{R}^d} 1_B(Ty_1 + TY_s)Q_t(X_s, 0; \{\ell\} \times dy_1) \\
&= \int_{\mathbb{R}^d} 1_{T^{-1}(B-TY_s)}(y_1)Q_t(X_s, 0; \{\ell\} \times dy_1) \\
&= Q_t(X_s, 0; \{\ell\} \times T^{-1}(B - TY_s)).
\end{aligned}$$

Then $\{(X_t, TY_t)\}_{t\in\mathbb{T}}$ is a Markov process with transition semi-group given by (50). □

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