MULTIPLICATIVE ERGODICITY OF LAPLACE TRANSFORMS FOR ADDITIVE FUNCTIONAL OF MARKOV CHAINS

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Abstract. This article is motivated by the quantitative study of the exponential growth of Markov-driven bifurcating processes (see [13] for applications of our results). In this respect, a key property is the multiplicative ergodicity, which deals with the asymptotic behaviour of some Laplace-type transform of nonnegative additive functional of a Markov chain. We establish a spectral version of this multiplicative ergodicity property in a general framework. Our approach is based on the use of the operator perturbation method. We apply our general results to two examples of Markov chains, including linear autoregressive models. In these two examples the operator-type assumptions reduce to some expected finite moment conditions on the functional (no exponential moment conditions are assumed in this work).

Contents

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1. Introduction

Let $(\mathbb{X}, \mathcal{X})$ be a measurable space, let $(X_n)_{n\geq 0}$ be a sequence of random variables taking their values in $\mathbb{X}$, and finally let ξ and κ be two measurable functions on $\mathbb{X}$ with values in $[0, +\infty)$ and in $\mathbb{N}^*$ (the set of positive integers) respectively. Define

$$\forall n \geq 0, \quad S_n := \sum_{k=0}^{n} \xi(X_k) \tag{1}$$

The authors in [13] investigated the exponential growth of some branching processes, for which the process $(X_n)_{n\geq 0}$ is used to describe the model, while $\xi(\cdot)$ and $\kappa(\cdot)$ are seen respectively as lifetimes of cells and as numbers of new cells (see also [20]).

This exponential growth is defined in [13] provided that the two following quantities are well defined and finite

$$\nu := \inf\{\gamma > 0,\ G(\gamma) < \infty\} \quad \text{and} \quad C_\nu := \lim_{\gamma\to 0^+} \frac{\gamma}{\gamma+\nu} G(\nu+\gamma) \tag{2}$$

with $G : [0, +\infty) \to [0, +\infty]$ defined by $G(\gamma) := \sum_{n\geq 0} g_n(\gamma)$, where $g_n(\gamma)$ is expressed in terms of the following Laplace-type transform of S_n

$$g_n(\gamma) := \mathbb{E}\left[\left(\prod_{j=0}^{n-1} \kappa(X_j)\right)(\kappa(X_n) - 1)e^{-\gamma S_n}\right].$$

The following notion of multiplicative ergodicity, introduced in [16] and [17], has proved to be efficient in [13] to study the finiteness of ν and the existence of C_ν.

Definition 1.1. *Let $\gamma_1 > 0$. We say that $(S_n, \kappa(X_n))_n$ is* **multiplicatively ergodic** *on $J = [0, \gamma_1)$ if there exist two continuous maps A and ρ from J to $(0, +\infty)$ such that, for every compact subset K of $(0, \gamma_1)$, there exist $M_K > 0$ and $\theta_K \in (0, 1)$ such that, for every $n \geq 1$,*

$$\forall \gamma \in K, \quad |g_n(\gamma) - A(\gamma)(\rho(\gamma))^n| \leq M_K(\rho(\gamma)\theta_K)^n. \tag{3}$$

When $\kappa(\cdot)$ is constant, we simply say that $(S_n)_n$ is multiplicatively ergodic on J.

Let us briefly explain why this definition is of great interest for obtaining the finiteness of ν and C_ν (see in [13] for details). Assume that $(S_n, \kappa(X_n))_n$ is multiplicatively ergodic on $J = [0, \gamma_1)$. Then

- For every $\gamma \in J$ we have: $G(\gamma) = \sum_{n\geq 0} g_n(\gamma) < \infty \iff \rho(\gamma) < 1$.
- For every compact subset K of J, we obtain from the definition of ν in (2) that
$$\forall \gamma \in K \cap (\nu, +\infty), \quad \left| G(\gamma) - \frac{A(\gamma)}{1-\rho(\gamma)} \right| \leq \frac{M_K}{1-\rho(\gamma)\theta_K}.$$
- $\nu < \gamma_1$ means that $\nu = \inf\{\gamma \in J \ : \ \rho(\gamma) < 1\} < \gamma_1$.
- If moreover ρ is differentiable at ν with $\rho(\nu) = 1$ and $\rho'(\nu) \neq 0$, then
$$C_\nu := \lim_{\gamma\to 0} \frac{\gamma}{\gamma+\nu} G(\nu+\gamma) \ = \ -\frac{A(\nu)}{\nu\rho'(\nu)}, \tag{4}$$

Throughout this paper $(X_n)_n$ is a Markov chain on $(\mathbb{X}, \mathcal{X})$ with Markov kernel $P(x, dy)$, invariant probability π, and initial probability μ (i.e. μ is the distribution of X_0). In this context, the multiplicative ergodicity can be investigated via the spectral behavior of Laplace-type operators associated with (P, κ, ξ). Indeed an easy induction gives (see [13, Section 2.2] for details)
$$\forall n \geq 1, \quad g_n(\gamma) = \mu\Big(\kappa\, e^{-\gamma\xi}\, P_\gamma^{n-1}\left(Ph_{\kappa,\gamma}\right)\Big) \tag{5}$$
where $h_{\kappa,\gamma} := \big(\kappa - 1\big)\, e^{-\gamma\xi}$ and where P_γ is the nonnegative Laplace kernel on $\mathbb{X}$ defined by
$$\forall x \in \mathbb{X}, \quad P_\gamma(x, dy) := \kappa(y)e^{-\gamma\xi(y)}P(x, dy). \tag{6}$$
In others words P_γ acts on functions $h : \mathbb{X} \to \mathbb{C}$ as : $P_\gamma h := P(h\kappa e^{-\gamma\xi})$.

The purpose of this work is, first to present operator-type assumptions on the Laplace kernels P_γ ensuring that the multiplicative ergodicity holds, second to apply this approach to two Markov models. Actually the kernels P_γ are assumed to continuously act on some suitable Banach space $\mathcal{B}$ and to have on $\mathcal{B}$ some nice spectral properties involving the spectral radius $r(\gamma)$ of P_γ and the associated eigen-projector (see Hypothesis 2.1). Those assumptions, together with perturbation-type hypotheses (see Hypotheses 2.3 and 2.3*), are needed in order to study the behavior of the iterated operator P_γ^n and to deduce from (5) that ν is finite and is given by (see Theorem 2.4)
$$\nu = \inf\{\gamma > 0, \ r(\gamma) < 1\}.$$
The existence and the finiteness of C_ν are discussed in Theorem 2.5.

Our spectral approach is based on the quasi-compactness property and on the method of perturbation of operators. This method, introduced by Nagaev [22, 23] and by Le Page and Guivarc'h [18, 8] to prove a wide class of limit theorems (central limit theorem, local limit theorem, large and moderate deviations principles), has known an impressive development in the past decades (e.g. see [3, 11] and the references therein). Unfortunately, since no exponential moment condition on ξ is assumed in this work, the classical perturbation method does not apply in general in our context to the family of Laplace operators (see Remark 2.2 for details). Instead, through our Hypotheses 2.3 or 2.3*, we use the Keller and Liverani perturbation theorem [15, 1] which involves several Banach spaces instead of a single one (e.g. see [14] and the references therein). The fact that we work with several spaces complicates our study compared to the classical approach. This is the price to pay for obtaining the multiplicative ergodicity property in Markovian models under weak moment assumptions on ξ, as illustrated in our instances presented in Theorems 5.1 and 6.1.

Further complementary discussions and reductions of our operator-type hypotheses are addressed in Section 3, especially when the spaces involved in those hypotheses are assumed to be Banach lattices. The proofs of the main Theorems 2.4 and 2.5 are given in Section 4. Finally we present in Sections 5 and 6 two applications, namely two Markovian examples satisfying our Hypotheses 2.1 and 2.3 (or 2.3*), thus the multiplicative ergodicity property of Definition 1.1: the first one is derived from Knudsen gas (Theorem 5.1); the second one concerns the linear autoregressive processes (Theorem 6.1). These two theorems, already used in [13], are obtained under weak integrability assumptions on the observable ξ (the lifetime).

2. Notations and main results

For any normed complex vector spaces $(\mathcal{B}_0, \|\cdot\|_{\mathcal{B}_0})$ and $(\mathcal{B}_1, \|\cdot\|_{\mathcal{B}_1})$, the space of continuous $\mathbb{C}$-linear operators from $\mathcal{B}_0$ to $\mathcal{B}_1$ is written $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_1)$. We endow the space $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_1)$ with the operator norm $\|\cdot\|_{\mathcal{B}_0,\mathcal{B}_1}$ given by

$$\forall Q \in \mathcal{L}(\mathcal{B}_0, \mathcal{B}_1), \quad \|Q\|_{\mathcal{B}_0,\mathcal{B}_1} = \sup_{f\in\mathcal{B}_0,\ \|f\|_{\mathcal{B}_0}=1} \|Qf\|_{\mathcal{B}_1}.$$

If $\mathcal{B}_0$ and $\mathcal{B}_1$ are the same Banach space (say $\mathcal{B}$ to simplify), we simply write $(\mathcal{L}(\mathcal{B}), \|\cdot\|_{\mathcal{B}})$ for $(\mathcal{L}(\mathcal{B},\mathcal{B}), \|\cdot\|_{\mathcal{B},\mathcal{B}})$. If $Q \in \mathcal{L}(\mathcal{B})$, then we write $\sigma(Q)$ for the spectrum of Q:

$$\sigma(Q) := \{\lambda \in \mathbb{C} \ :\ (Q - \lambda I) \text{ is non invertible}\},$$

where I denotes the identity operator on $\mathcal{B}$. The spectral radius of Q (resp. its essential spectral radius) is denoted by $r(Q)$ (resp. $r_{ess}(Q)$). Recall that

$$r(Q) := \lim_{n\to+\infty} \|Q^n\|_{\mathcal{B}}^{1/n}$$

$$\begin{aligned} r_{ess}(Q) &= \sup\{|\lambda| \ :\ \lambda \in \mathbb{C} \text{ and } (Q - \lambda I) \text{ is non Fredholm}\} \\ &= \lim_{n\to+\infty} \inf_{F\in\mathcal{K}(\mathcal{B})} \|Q^n - F\|_{\mathcal{B}}^{1/n} \end{aligned}$$

where $\mathcal{K}(\mathcal{B})$ denotes the set of compact operators on $\mathcal{B}$. Finally the topological dual space of $\mathcal{B}$ is denoted by $(\mathcal{B}^*, \|\cdot\|_{\mathcal{B}^*})$, and the adjoint operator of Q is denoted by Q^*. Recall that Q and Q^* have the same norm in $\mathcal{L}(\mathcal{B})$ and $\mathcal{L}(\mathcal{B}^*)$ respectively. They also have the same spectrum (thus the same spectral radius), as well as the same essential spectral radius.

With the notations introduced in Introduction, the Laplace kernels $(P_\gamma)_\gamma$ are given by

$$\forall \gamma \in [0, +\infty),\ P_\gamma f = P(\kappa e^{-\gamma\xi} f) \ \text{ and } \ P_\infty f = P(\kappa \mathbf{1}_{\{\xi=0\}} f). \tag{7}$$

Since the action of P_γ is considered latter on several Banach spaces, the notation $P_{\gamma|\mathcal{B}}$ will sometimes be used in order to indicate that the action of P_γ is considered on $\mathcal{B}$. Finally $\mathbb{L}^1(\pi) = \mathbb{L}^1(\mathbb{X}, \mathcal{X}, \pi)$ denotes the usual Lebesgue space on $(\mathbb{X}, \mathcal{X})$ associated with the stationary distribution π of P.

Hypothesis 2.1. *Let $\mathcal{B}$ be a Banach space composed of functions on $\mathbb{X}$ (or of classes of such functions modulo the $\pi-$almost sure equality) such that $\mathcal{B} \subset \mathbb{L}^1(\pi)$. Let J be a subinterval of $[0, +\infty]$. We will say that Hypothesis 2.1 holds on $\mathcal{B}$ and J if, for every $\gamma \in J$, P_γ continuously acts on $\mathcal{B}$ and if*

(i) $r(\gamma) = r(P_{\gamma|\mathcal{B}}) > 0$, *and P_γ is quasi-compact on $\mathcal{B}$ (i.e. $r_{ess}(P_{\gamma|\mathcal{B}}) < r(\gamma)$)*

(ii) $r(\gamma)$ *is the only eigenvalue of modulus $r(\gamma)$ for P_γ, and $r(\gamma)$ is a first order pole of P_γ with moreover* $\dim \operatorname{Ker}(P_\gamma - r(\gamma)I) = \dim \operatorname{Ker}(P_\gamma - r(\gamma)I)^2 = 1$.

Under Hypothesis 2.1, we know from the spectral theory (e.g. see [11, Prop. XIV.2]) that, for every $\gamma \in J$, P_γ has the following spectral gap property: there exists a rank-one projector $\Pi_\gamma \in \mathcal{L}(\mathcal{B})$ (i.e. the eigenprojector associated with the eigenvalue $r(\gamma)$), and some constants $\theta_\gamma \in (0,1)$ and $M_\gamma \in (0,+\infty)$ such that

$$\forall \gamma \in J, \quad \forall f \in \mathcal{B}, \quad \big\|P_\gamma^n f - r(\gamma)^n \Pi_\gamma f\big\|_{\mathcal{B}} \le M_\gamma \left(\theta_\gamma\, r(\gamma)\right)^n \|f\|_{\mathcal{B}}. \tag{8}$$

If moreover $\mu(\kappa e^{-\gamma\xi}\cdot) \in \mathcal{B}^*$, $P(h_{\kappa,\gamma}) \in \mathcal{B}$, and $B(\gamma) := \mu(\kappa\, e^{-\gamma\xi}\Pi_\gamma(Ph_{\kappa,\gamma}))$ is positive, then we deduce from (5) that (3) holds with $A = B/r$ and $\rho = r$ in the specific case $K = \{\gamma\}$.

Remark 2.2. *To establish the multiplicative ergodicity of Definition 1.1, further regularity properties are needed. Due to (8), a natural way is to apply the perturbation theory of linear operators. Unfortunately, the classical operator perturbation method* [22, 23, 8, 9] *does not apply to our context. Indeed, because we do not assume any exponential moment condition on ξ (contrarily to the above mentioned papers), the map $\gamma \mapsto P_\gamma$ is (in general)* **not continuous** *from $(0,+\infty)$ to $\mathcal{L}(\mathcal{B})$. For instance, for linear autoregressive models (Theorem 6.1), we will work with Banach spaces $\mathcal{B}_a = \mathcal{C}_{V^a}$ linked to some weighted-supremum Banach spaces. For the Knudsen gas (Theorem 5.1), we will work with $\mathcal{B}_a = \mathbb{L}^a(\pi)$. In these two cases, the map $\gamma \mapsto P_\gamma$ is not continuous in general from $(0,+\infty)$ to $\mathcal{L}(\mathcal{B}_a)$, but only from $(0,+\infty)$ to $\mathcal{L}(\mathcal{B}_a,\mathcal{B}_b)$ for $a < b$ for the linear autoregressive models (and for $b < a$ for the Knudsen gas). This is the reason why we use below the Keller-Liverani perturbation theorem* [15, 1]. *The price to pay is to consider a chain of Banach spaces instead of a single one.*

In view of the previous remark we introduce two sets of hypotheses which are nothing else but the assumptions of the Keller-Liverani perturbation theorem when applied, first to the family $(P_\gamma)_\gamma$ (Hypothesis 2.3), second to the family $(P_\gamma^*)_\gamma$ (Hypothesis 2.3*): both of them will be relevant for our examples in Sections 5 and 6 (Knudsen gas and linear autoregressive model). Below the notation $\mathcal{B}_0 \hookrightarrow \mathcal{B}_1$ means that $\mathcal{B}_0$ is continuously injected in $\mathcal{B}_1$.

Hypothesis 2.3. *Let $\mathcal{B}_0$ and $\mathcal{B}_1$ be two Banach spaces, let J be a subinterval of $[0,+\infty]$. We will say that $((P_\gamma)_{\gamma\in J}, \mathcal{B}_0, \mathcal{B}_1)$ satisfies Hypothesis 2.3 if*

- $\mathcal{B}_0 \hookrightarrow \mathcal{B}_1$,
- *for every $\gamma \in J$, $P_\gamma \in \mathcal{L}(\mathcal{B}_0) \cap \mathcal{L}(\mathcal{B}_1)$,*
- *the map $\gamma \mapsto P_\gamma$ is continuous from J to $\mathcal{L}(\mathcal{B}_0,\mathcal{B}_1)$,*
- *there exist $c_0 > 0$, $\delta_0 > 0$, $M > 0$ such that*

$$\forall \gamma \in J, \quad r_{ess}\big(P_{\gamma|\mathcal{B}_0}\big) \le \delta_0 \tag{9a}$$

$$\forall \gamma \in J,\ \forall n \ge 1,\ \forall f \in \mathcal{B}_0, \quad \|P_\gamma^n f\|_{\mathcal{B}_0} \le c_0\big(\delta_0^n \|f\|_{\mathcal{B}_0} + M^n \|f\|_{\mathcal{B}_1}\big) \tag{9b}$$

Hypothesis 2.3*. *$((P_\gamma)_{\gamma\in J}, \mathcal{B}_0, \mathcal{B}_1)$ satisfies all the conditions of Hypothesis 2.3, except for (9a) and (9b) which are replaced by the following ones:*

$$\forall \gamma \in J, \quad r_{ess}\big((P_\gamma^*)_{|\mathcal{B}_1^*}\big) \le \delta_0 \tag{10a}$$

$$\forall \gamma \in J,\ \forall n \ge 1,\ \forall f^* \in \mathcal{B}_1^*, \quad \|(P_\gamma^*)^n f^*\|_{\mathcal{B}_1^*} \le c_0(\delta_0^n \|f^*\|_{\mathcal{B}_1^*} + M^n \|f^*\|_{\mathcal{B}_0^*}) \tag{10b}$$

Hypothesis 2.3* can be seen as a dual version of Hypothesis 2.3, but it is worth noticing that the conditions (10a)-(10b) cannot be deduced from (9a)-(9b) (and conversely). Under Hypothesis 2.3 or 2.3* we define the following set:

$$J_0 := \{\gamma \in J : r(\gamma) > \delta_0\}. \tag{11}$$

Finally recall that we have set $h_{\kappa,\gamma} := \big(\kappa - 1\big)\, e^{-\gamma\xi}$.

Theorem 2.4 (Existence of ν)**.**
Let $\mathcal{B}_0 \hookrightarrow \mathcal{B}_3 \hookrightarrow \mathbb{L}^1(\pi)$ be two Banach spaces such that $\mathbf{1}_{\mathbb{X}} \in \mathcal{B}_0$, let J be a subinterval of $[0,+\infty]$. Assume that $((P_\gamma)_{\gamma\in J}, \mathcal{B}_0, \mathcal{B}_3)$ satisfies either Hypothesis 2.3 or Hypothesis 2.3, and that*

- *Hypothesis 2.1 holds on J_0 and $\mathcal{B} := \mathcal{B}_0$ under Hypothesis 2.3*
- *Hypothesis 2.1 holds on J_0 and $\mathcal{B} := \mathcal{B}_3$ under Hypothesis 2.3*.*

Then, setting $r(\gamma) := r(P_{\gamma|\mathcal{B}})$ for every $\gamma \in J$, we have

$$\forall \gamma_0 \in J, \quad \limsup_{\gamma\to\gamma_0} r(\gamma) \le \max(\delta_0, r(\gamma_0))\,, \tag{12}$$

and the function $\gamma \mapsto r(\gamma)$ is continuous on J_0. Moreover there exists a map $\gamma \mapsto \Pi_\gamma$ from J_0 to $\mathcal{L}(\mathcal{B})$ which is continuous from J_0 to $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_3)$ such that, for every compact subset K of J_0, there exist $\theta_K \in (0,1)$ and $M_K \in (0,+\infty)$ such that

$$\forall \gamma \in K,\ \forall f \in \mathcal{B}, \quad \big\| P_\gamma^n f - r(\gamma)^n \Pi_\gamma f \big\|_{\mathcal{B}} \le M_K \, \big(\theta_K\, r(\gamma)\big)^n \|f\|_{\mathcal{B}}. \tag{13}$$

Consequently, under the previous assumptions, the following assertions hold:

(i) If the maps $\gamma \mapsto Ph_{\kappa,\gamma}$ and $\gamma \mapsto \mu(\kappa e^{-\gamma\xi}\cdot)$ are continuous from J_0 to $\mathcal{B}_0$ and to $\mathcal{B}_3^*$ respectively, and if

$$\forall \gamma \in J_0, \quad \pi(\Pi_\gamma \mathbf{1}_{\mathbb{X}}) > 0 \quad \text{ and } \quad B(\gamma) := \mu\left(\kappa\, e^{-\gamma\xi}\Pi_\gamma(Ph_{\kappa,\gamma})\right) > 0, \tag{14}$$

then, under $\mathbb{P}_\mu$, $(S_n, \kappa(X_n))_n$ is multiplicatively ergodic on J_0 with $A(\gamma) := \frac{B(\gamma)}{r(\gamma)}$ and $\rho(\gamma) = r(\gamma)$.

(ii) If moreover $\inf_{\gamma\in J_0} r(\gamma) < 1 < \sup_{\gamma\in J_0} r(\gamma)$, then, under $\mathbb{P}_\mu$, ν is finite and

$$\nu = \inf\{\gamma > 0 \, : \, r(\gamma) < 1\}. \tag{15}$$

Formula (13) can be interpreted as a spectral multiplicative ergodicity property.

To prove the existence of C_ν, we reinforce our assumptions by considering a longer chain of Banach spaces.

Theorem 2.5 (Existence of C_ν)**.**
Assume $\pi(\xi > 0) > 0$. Let $\mathcal{B}_0 \hookrightarrow \mathcal{B}_1 \hookrightarrow \mathcal{B}_2 \hookrightarrow \mathcal{B}_3 \hookrightarrow \mathbb{L}^1(\pi)$ be Banach spaces containing $\mathbf{1}_{\mathbb{X}}$ and let J be a subinterval of $[0,+\infty]$. Assume that one of the two following conditions holds

(a) Either: for $i = 0,1,2$, $((P_\gamma)_{\gamma\in J}, \mathcal{B}_i, \mathcal{B}_{i+1})$ satisfies Hypothesis 2.3, and Hypothesis 2.1 holds with $(J_0, \mathcal{B}_i)$; in this case we set $\mathcal{B} := \mathcal{B}_0$.

(b) Or: for $i = 0,1,2$, $((P_\gamma)_{\gamma\in J}, \mathcal{B}_i, \mathcal{B}_{i+1})$ satisfies Hypothesis 2.3, and Hypothesis 2.1 holds with $(J_0, \mathcal{B}_{i+1})$; in this case we set $\mathcal{B} := \mathcal{B}_3$.*

Assume moreover that the map $\gamma \mapsto P_\gamma$ is continuous from J to $\mathcal{L}(\mathcal{B}_i, \mathcal{B}_{i+1})$ for $i \in \{0,2\}$ and C^1 from J to $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ with derivative $P'_\gamma f = P_\gamma(-\xi f)$ ($f \mapsto \xi f$ being in $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$). Then (13) holds with C^1-smooth maps $\gamma \mapsto r(\gamma) := r(P_{\gamma|\mathcal{B}})$ and $\gamma \mapsto \Pi_\gamma$ from J_0 into $\mathbb{R}$ and into $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_3)$ respectively. Consequently, under the previous assumptions, the assertions (i)-(ii) in Theorem 2.4 can be specified and completed as follows:

(i') If the additional assumptions in Assertion (i) of Theorem 2.4 hold with the present spaces $\mathcal{B}_0$ and $\mathcal{B}_3$, then the functions $A(\cdot)$ and $\rho(\cdot) := r(\cdot)$ are C^1-smooth on J_0.

(ii') If moreover $\inf_{\gamma\in J_0} r(\gamma) < 1 < \sup_{\gamma\in J_0} r(\gamma)$ and if $r'(\nu) \neq 0$, then the constant C_ν of (4) is well defined and finite.

Assertions (i) and (ii) in Theorem 2.4 and 2.5 follow from Formula (5) (see the remarks after Definition 1.1) and from the respective first part of these two theorems. The proof of the first part in Theorems 2.4 and 2.5 is presented in Section 4.

Note that the two above results require, not only to check the spectral property (9a) (or (10a)) and the Doeblin-Fortet inequalities (9b) (or (10b)), but also to prove (14) and moreover $r'(\nu) > 0$ in Theorem 2.5. This is discussed in the next section.

3. Complementary results about the previous hypotheses

Recall that the spaces linked to Hypothesis 2.1 in the assumptions of Theorems 2.4 and 2.5 are contained in $\mathbb{L}^1(\pi)$. In this section we investigate Hypothesis 2.1 and Condition (14) by using some assumptions involving the notion of positivity and non-negativity on such a space (or on its dual space), as defined below. Moreover we give complementary results on the spectral radius $r(\gamma)$ of $(P_\gamma)_{|\mathcal{B}}$ for some Banach space $\mathcal{B}$.

Definition 3.1. *Let $\mathcal{B}$ be a Banach space composed of functions $f : \mathbb{X} \to \mathbb{C}$ (or of classes of such functions modulo π). If $f \in \mathcal{B}$ is a class of functions, we say that it is* **non-negative** *(resp.* **positive***) if one of its representant is so. We say that it is* **non-null** *if the null function is not one of its representant. An element $\psi \in \mathcal{B}^*$ is said to be* **non-negative** *if for every non-negative $f \in \mathcal{B}$, we have $\psi(f) \geq 0$. An element $\psi \in \mathcal{B}^*$ is said to be* **positive** *if for every non-negative non-null $f \in \mathcal{B}$, we have $\psi(f) > 0$.*

Hypothesis 3.2. *For every $\phi \in \mathcal{B}$, $\phi \geq 0$, $\phi \neq 0$, there exists $\psi \in \mathcal{B}^*$, $\psi \geq 0$, such that $\psi(\phi) > 0$. For every $\psi \in \mathcal{B}^*$, $\psi \geq 0$, $\psi \neq 0$, there exists $\phi \in \mathcal{B}$, $\phi \geq 0$ such that $\psi(\phi) > 0$.*

Hypothesis 3.3. *Let $J \subset [0, +\infty]$ be such that: $\forall \gamma \in J,\ P_\gamma \in \mathcal{L}(\mathcal{B})$. For every $\gamma \in J$ such that $r(\gamma) > 0$, the following properties hold: if $\phi \in \mathcal{B}$ is non-null and non-negative, then $P_\gamma \phi > 0$ (modulo π) and every non-null non-negative $\psi \in \mathcal{B}^* \cap \operatorname{Ker}(P_\gamma^* - r(\gamma)I)$ is positive.*

Note that Hypothesis 3.2 is quite general. Let us recall the definition of a Banach lattice.

Definition 3.4. *A complex Banach space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ of functions $f : \mathbb{X} \to \mathbb{C}$ (or of classes of such functions modulo π) is said to be a* **complex Banach lattice** *if it is stable by $|\cdot|$, by real part and if*

$$\forall f, g \in \mathcal{B}, \quad f(\mathbb{X}) \cup g(\mathbb{X}) \subset \mathbb{R} \quad \Rightarrow \quad \min(f, g), \max(f, g) \in \mathcal{B},$$

$$\forall f, g \in \mathcal{B}, \quad |f| \leq |g| \quad \Rightarrow \quad \|\,|f|\,\|_{\mathcal{B}} = \|f\|_{\mathcal{B}} \leq \|g\|_{\mathcal{B}} = \|\,|g|\,\|_{\mathcal{B}}.$$

Such a space satisfies Hypothesis 3.2. Classical instances of Banach lattices of functions are the spaces $(\mathbb{L}^p(\pi), \|\cdot\|_p)$ and $(\mathcal{B}_V, \|\cdot\|_V)$ (see (28) and (35)), as well as the space $(\mathcal{L}^\infty(\mathbb{X}), \|\cdot\|_\infty)$ composed of all the bounded measurable $\mathbb{C}$-valued functions on $\mathbb{X}$, and equipped with its usual norm $\|f\|_\infty := \sup_{x\in\mathbb{X}} |f(x)|$.

3.1. **About Hypothesis 2.1.**

Proposition 3.5. *Let J be a subinterval of $[0,+\infty]$ and let $\mathcal{B}$ be a Banach lattice (as described in Definition 3.4). We assume that Hypothesis 3.3 holds, that $\mathcal{B} \subset \mathbb{L}^1(\pi)$, and that for every $\gamma \in J$*

(i) $P_\gamma \in \mathcal{L}(\mathcal{B})$, $r(\gamma) := r(P_{\gamma|\mathcal{B}}) > 0$, and P_γ is quasi-compact on $\mathcal{B}$,
(ii) for every $f, g \in \mathcal{B}$ with $f > 0$, $P_\gamma f = r(\gamma) f$ and $P_\gamma g = r(\gamma) g$, we have $g \in \mathbb{C} \cdot f$,

Assume moreover that the Markov kernel P satisfies the following condition: 1 is the only complex number λ of modulus 1 such that $P(h/|h|) = \lambda h/|h|$ in $\mathbb{L}^1(\pi)$ for some $h \in \mathcal{B}$, $|h| > 0$ (modulo π). Then Hypothesis 2.1 is fulfilled with J and $\mathcal{B}$.

Proposition 3.5 follows from standard arguments derived from the spectral theory of positive operators. For convenience the proof of Proposition 3.5 is postponed to Appendix A.

Proposition 3.6. *Assume that Hypothesis 2.1 holds for some Banach space $\mathcal{B}$ as described in Definition 3.1 and for some subinterval J of $[0,+\infty]$. Let $\gamma \in J$. Then*

$$\Pi_\gamma = \lim_{n\to+\infty} r(\gamma)^{-n} P_\gamma^n \quad \text{is well defined in } \mathcal{L}(\mathcal{B}), \tag{16}$$

and there exist some nonzero elements $\widehat{\pi}_\gamma \in \mathcal{B}^* \cap \mathrm{Ker}(P_\gamma^* - r(\gamma)I)$ and $\widehat{\phi}_\gamma \in \mathcal{B} \cap \mathrm{Ker}(P_\gamma - r(\gamma)I)$ such that $\widehat{\pi}_\gamma(\widehat{\phi}_\gamma) = 1$ and

$$\forall f \in \mathcal{B}, \quad \Pi_\gamma f = \widehat{\pi}_\gamma(f)\, \widehat{\phi}_\gamma \qquad \text{and} \qquad \forall f^* \in \mathcal{B}^*, \quad \Pi_\gamma^* f^* = f^*(\widehat{\phi}_\gamma)\, \widehat{\pi}_\gamma. \tag{17}$$

If $\mathcal{B}$ satisfies Hypothesis 3.2, then $\widehat{\phi}_\gamma$ and $\widehat{\pi}_\gamma$ are non-negative in $\mathcal{B}$ and $\mathcal{B}^*$ respectively. Under the additional Hypothesis 3.3, $\widehat{\phi}_\gamma > 0$ π-a.s. and $\widehat{\pi}_\gamma > 0$ and, for every non-null and non-negative $f \in \mathcal{B}$, we have $\Pi_\gamma f > 0$ $\pi-$a.s..

Proof. Properties (16) and the existence of $\widehat{\phi}_\gamma$ and $\widehat{\pi}_\gamma$ in (17) follow from Hypothesis 2.1. Now assume that Hypothesis 3.2 holds. Then (16) and the first assertion in Hypothesis 3.2, applied with $\phi = \widehat{\phi}_\gamma$ and the associated $\psi_\gamma \in \mathcal{B}^*$, $\psi_\gamma \geq 0$, imply that, for every $g \in \mathcal{B}$, $g \geq 0$, we have $0 \leq \lim_{n\to+\infty} r(\gamma)^{-n} \psi_\gamma(P_\gamma^n g) = \widehat{\pi}_\gamma(g)\, \psi_\gamma(\widehat{\phi}_\gamma)$, hence $\widehat{\pi}_\gamma \geq 0$ since $\psi_\gamma(\widehat{\phi}_\gamma) > 0$. Next the second assertion in Hypothesis 3.2, applied with $\psi = \widehat{\pi}_\gamma$ and the associated $\phi_\gamma \in \mathcal{B}$, $\phi_\gamma \geq 0$, gives $0 \leq \lim_{n\to+\infty} r(\gamma)^{-n} P_\gamma^n \phi_\gamma = \widehat{\pi}_\gamma(\phi_\gamma)\widehat{\phi}_\gamma$, hence $\widehat{\phi}_\gamma \geq 0$ since $\widehat{\pi}_\gamma(\phi_\gamma) > 0$. Finally, assume that Hypotheses 3.2 and 3.3 hold. Let $\gamma \in J$. Then Equation $P_\gamma \widehat{\phi}_\gamma = r(\gamma)\widehat{\phi}_\gamma$ and Hypothesis 3.3 implies that $\widehat{\phi}_\gamma > 0$ $\pi-$a.s. Moreover, if $\gamma \in J$ and if $f \in \mathcal{B}$, $f \neq 0$, $f \geq 0$, then $\widehat{\pi}_\gamma(f) > 0$ and so $\Pi_\gamma f = \widehat{\pi}_\gamma(f)\, \widehat{\phi}_\gamma$ is positive modulo π. □

Corollary 3.7. *Assume that, for some subinterval $J \subset [0,+\infty)$, Hypothesis 2.1 holds on two Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$ (as described in Definition 3.1) both containing $\mathbf{1}_{\mathbb{X}}$ and satisfying Hypotheses 3.2 and 3.3. Then*

$$\forall \gamma \in J, \quad r(P_{\gamma|\mathcal{B}_1}) = r(P_{\gamma|\mathcal{B}_2}) = \lim_{n\to+\infty} (\pi(P_\gamma^n \mathbf{1}_{\mathbb{X}}))^{1/n}.$$

If moreover $\mathcal{B}_1 \hookrightarrow \mathcal{B}_2$ and if, for $i = 1, 2$, $\Pi_{\gamma,i}$ denotes the rank-one eigen-projector associated with $P_{\gamma|\mathcal{B}_i}$ in (8), then the restriction of $\Pi_{\gamma,2}$ to $\mathcal{B}_1$ equals to $\Pi_{\gamma,1}$.

Proof. For $i = 1, 2$, Property (8) and Proposition 3.6 applied to $P_{\gamma|\mathcal{B}_i}$ (with the notations $\widehat{\phi}_{\gamma,i}$ and $\widehat{\pi}_{\gamma,i}$ related to Proposition 3.6) gives

$$\pi(P_\gamma^n \mathbf{1}_{\mathbb{X}}) = (r(P_{\gamma|\mathcal{B}_i}))^n \widehat{\pi}_{\gamma,i}(\mathbf{1}_{\mathbb{X}})\pi(\widehat{\phi}_{\gamma,i}) + \mathrm{o}\big(r(P_{\gamma|\mathcal{B}_i})^n\big)$$

with $\pi(\widehat{\phi}_{\gamma,i})\, \widehat{\pi}_{\gamma,i}(\mathbf{1}_{\mathbb{X}}) > 0$ from Proposition 3.6. Hence the first assertion holds. Now let $f \in \mathcal{B}_1$. Then $\Pi_{\gamma,1} f = \lim_{n\to+\infty} r(P_{\gamma|\mathcal{B}_2})^{-n}(P_{\gamma|\mathcal{B}_1})^n f$ in $\mathcal{B}_1$ from Proposition 3.6 applied to $P_{\gamma|\mathcal{B}_1}$ and from the previous fact. It follows from $\mathcal{B}_1 \hookrightarrow \mathcal{B}_2$ that this convergence holds in $\mathcal{B}_2$ too. Now Proposition 3.6 applied to $P_{\gamma|\mathcal{B}_2}$ gives $\Pi_{\gamma,1} f = \Pi_{\gamma,2} f$. □

3.2. **About Condition** (14).

Proposition 3.8. *Assume that $(P_\gamma)_\gamma$ and μ satisfy the assumptions of Theorem 2.4(i), excepted (14). Then the real number $B(\gamma)$ given in (14) is well-defined for every $\gamma \in J_0$, and*

$$\forall \gamma \in J_0, \quad B(\gamma) = \widehat{\pi}_\gamma(Ph_{\kappa,\gamma})\, \mu\left(\kappa\, e^{-\gamma\xi}\widehat{\phi}_\gamma\right), \tag{18}$$

where $\widehat{\phi}_\gamma$ and $\widehat{\pi}_\gamma$ are given in (17). Assume moreover that the space $\mathcal{B}$ involved in the assumptions of Theorem 2.4 satisfies Hypotheses 3.2 and 3.3 on J_0 and that one of the following assumptions holds true

- *(i) μ is absolutely continuous with respect to π,*
- *(ii) the first part in Hypothesis 3.3 is reinforced as follows: for every $\gamma \in J$, if $\phi \in \mathcal{B}$ is non-null and non-negative, then $P_\gamma \phi > 0$ everywhere on $\mathbb{X}$.*

Then (14) *holds.*

Proof. Let $\gamma \in J_0$ (thus $r(\gamma) > 0$). First, by assumption $Ph_{\kappa,\gamma} \in \mathcal{B}_0$. Thus $\Pi_\gamma(Ph_{\kappa,\gamma}) \in \mathcal{B}_3$ since $\Pi_\gamma \in \mathcal{L}(\mathcal{B}_0, \mathcal{B}_3)$ (see Theorem 2.4), and $B(\gamma)$ in (14) is then well-defined from the assumptions on μ in Theorem 2.4(i). Formula (18) follows from (17) since $\Pi_\gamma h = \widehat{\pi}_\gamma(h)\widehat{\phi}_\gamma$. Also note that $\widehat{\phi}_\gamma > 0$ (modulo π) and that $\widehat{\pi}_\gamma > 0$ from Proposition 3.6. Thus the first condition of (14) holds since $\pi(\Pi_\gamma \mathbf{1}_{\mathbb{X}}) = \pi(\widehat{\phi}_\gamma)\widehat{\pi}_\gamma(\mathbf{1}_{\mathbb{X}}) > 0$. Moreover $Ph_{\kappa,\gamma} \geq 0$ and $Ph_{\kappa,\gamma} \neq 0$ in $\mathbb{L}^1(\pi)$ since $\pi(Ph_{\kappa,\gamma}) = \pi(h_{\kappa,\gamma}) > 0$. Since $\mathcal{B}_0 \hookrightarrow \mathcal{B}_3 \hookrightarrow \mathbb{L}^1(\pi)$ by hypothesis, it follows that $Ph_{\kappa,\gamma} \neq 0$ in $\mathcal{B}_1$ and in $\mathcal{B}_3$. Thus $\widehat{\pi}_\gamma(Ph_{\kappa,\gamma}) > 0$ from Proposition 3.6. Finally we have $\widehat{\pi}_\gamma(Ph_{\kappa,\gamma})\widehat{\phi}_\gamma > 0$ (π-almost surely in Case (i) and everywhere in Case (ii) since $\widehat{\phi}_\gamma = P_\gamma\widehat{\phi}_\gamma / r(\gamma) > 0$), which ensures the second condition of (14) due to (18). □

3.3. **About the monotonicity of the spectral radius.**

Proposition 3.9. *Let J be a subinterval of $[0, +\infty]$. If $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a complex Banach lattice (as described in Definition 3.4), and if $P_\gamma \in \mathcal{L}(\mathcal{B})$ for every $\gamma \in J$, then the map $\gamma \mapsto r(\gamma)$ is non-increasing on J.*

Proof. For any $0 \leq \gamma < \gamma' \leq \infty$ and for any $f, g \in \mathcal{B}$ such that $|f| \leq |g|$, we have $e^{-\gamma'\xi}|f| \leq e^{-\gamma\xi}|g|$ and so $P_{\gamma'}|f| \leq P_\gamma|g|$, which implies by induction that $P_{\gamma'}^n|f| \leq P_\gamma^n|f|$ for every integer $n \geq 1$. We conclude that $\|P_{\gamma'}^n\|_{\mathcal{B}} \leq \|P_\gamma^n\|_{\mathcal{B}}$ since $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a Banach lattice. This implies that $r(\gamma') \leq r(\gamma)$ and so the desired statement. □

If $\mathcal{B}$ is not a Banach lattice, but only a Banach space as described in Definition 3.1, then the non-increasingness of $r(\cdot)$ can be obtained under Hypotheses 3.2 and 3.3. More precisely:

Proposition 3.10. *Assume that Hypothesis 2.1 is fulfilled for some subinterval J of $[0,+\infty]$ and for some Banach space $\mathcal{B}$ (as described in Definition 3.1) satisfying Hypotheses 3.2 and 3.3. Then $\gamma \to r(\gamma)$ is non-increasing on J.*

Proof. We use the notations of Proposition 3.6. Let $\gamma_1, \gamma_2 \in J$ such that $\gamma_2 < \gamma_1$. Then $\widehat{\pi}_{\gamma_1}(P_{\gamma_1}^n \widehat{\phi}_{\gamma_1}) \leq \widehat{\pi}_{\gamma_1}(P_{\gamma_2}^n \widehat{\phi}_{\gamma_1})$ since $\widehat{\pi}_{\gamma_1}$ and $\widehat{\phi}_{\gamma_1}$ are non-negative (see the previous proof). Moreover $\widehat{\pi}_{\gamma_1}(P_{\gamma_1}^n \widehat{\phi}_{\gamma_1}) = (r(\gamma_1))^n$ and

$$\widehat{\pi}_{\gamma_1}(P_{\gamma_2}^n \widehat{\phi}_{\gamma_1}) = (r(\gamma_2))^n \widehat{\pi}_{\gamma_2}(\widehat{\phi}_{\gamma_1})\widehat{\pi}_{\gamma_1}(\widehat{\phi}_{\gamma_2}) + o\left((r(\gamma_2))^n\right).$$

Since $\widehat{\pi}_{\gamma_1}$ and $\widehat{\pi}_{\gamma_2}$ are positive and since $\widehat{\phi}_{\gamma_1}$ and $\widehat{\phi}_{\gamma_2}$ are non-null non-negative, we have $\widehat{\pi}_{\gamma_2}(\widehat{\phi}_{\gamma_1})\widehat{\pi}_{\gamma_1}(\widehat{\phi}_{\gamma_2}) > 0$. Thus $(r(\gamma_1))^n = O\left((r(\gamma_2))^n\right)$, so $r(\gamma_1) \leq r(\gamma_2)$. □

Remark 3.11. *The non-increasingness of r on J implies that the set J_0 in (11) is an interval. Let us also indicate that it can happen that $r(\cdot)$ is constant on $[0,+\infty)$ (see Appendix B).*

The next result is relevant to check the strict decreasingness of $r(\cdot)$.

Proposition 3.12. *Assume that the assumptions of Theorem 2.5 hold.*

(i) Let $\gamma \in J_0$ be such that $\pi\left(\Pi_\gamma \mathbf{1}_{\mathbb{X}}\right) > 0$ and $\pi\big(\Pi_\gamma(\xi\, \Pi_\gamma \mathbf{1}_{\mathbb{X}})\big) > 0$. Then $r'(\gamma) < 0$.

(ii) Assume moreover that the space $\mathcal{B}_2$ involved in the assumptions of Theorem 2.5 satisfies Hypotheses 3.2 and 3.3 on J_0, and that $\pi(\{\xi = 0\}) < 1$. Then $r'(\cdot) < 0$ on J_0, thus $\gamma \mapsto r(\gamma)$ is strictly decreasing on J_0.

Proof. Assertion (i) follows from Proposition 4.5. Let us derive (ii) from (i). Let $\gamma \in J_0$. First note that $\xi\Pi_\gamma \mathbf{1}_{\mathbb{X}} \in \mathcal{B}_2$. Indeed it follows from the assumptions of Theorem 2.5 that $\Pi_\gamma \mathbf{1}_{\mathbb{X}} \in \mathcal{B}_1$ (use also Corollary 3.7) and that the map $f \mapsto \xi f$ is in $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$. Hence $\xi\Pi_\gamma \mathbf{1}_{\mathbb{X}} \in \mathcal{B}_2$. Moreover, under the assumptions in (ii), we know from the last assertion of Proposition 3.6 (applied on $\mathcal{B}_2$) that $\Pi_\gamma \mathbf{1}_{\mathbb{X}} > 0$ modulo π, thus $\xi\, \Pi_\gamma \mathbf{1}_{\mathbb{X}} \neq 0$ in $\mathbb{L}^1(\pi)$ (since $\pi(\{\xi = 0\}) < 1$), and so $\xi\, \Pi_\gamma \mathbf{1}_{\mathbb{X}} \neq 0$ in $\mathcal{B}_2$ from $\mathcal{B}_2 \hookrightarrow \mathbb{L}^1(\pi)$. Then it follows again from the last assertion of Proposition 3.6 (applied on $\mathcal{B}_2$) that $\Pi_\gamma(\xi\, \Pi_\gamma \mathbf{1}_{\mathbb{X}}) > 0$ modulo π, thus $\pi\big(\Pi_\gamma(\xi\, \Pi_\gamma \mathbf{1}_{\mathbb{X}})\big) > 0$. Hence $r'(\gamma) < 0$ from (i). □

Recall that the parameter ν has been defined in (2). Proposition 3.12 is of interest to check the condition $r'(\nu) \neq 0$ in Theorem 2.5. Moreover a consequence of the monotonicity of $\gamma \mapsto r(\gamma)$ is the following characterisation of $\nu < \infty$.

Proposition 3.13. *Assume that the assumptions of Theorem 2.4(i)-(ii) hold (with J and δ_0 given in Hypothesis 2.3 or 2.3*); in particular for every $\gamma \in J$ the Laplace kernel P_γ is assumed to continuously act on the Banach space $\mathcal{B}$ chosen in the assumptions of Theorem 2.4. Assume moreover that $\delta_0 < 1$. Then, under $\mathbb{P}_\mu$,*

(i) For every $\gamma \in J$ we have: $G(\gamma) < \infty \Leftrightarrow r(\gamma) < 1$.

(ii) If r is non-increasing and if $J = (a, +\infty]$ for some $a \geq 0$, then $\nu < \infty \Leftrightarrow r(\infty) = r(P_{\infty|\mathcal{B}}) < 1$.

Proof. First, if $\gamma \in J_0$, then $G(\gamma) < \infty \Leftrightarrow r(\gamma) < 1$ due to Theorem 2.4(i)-(ii) (use (13) with $K = \{\gamma\}$) and to the first remark after Definition 1.1. Second, if $\gamma \in J \setminus J_0$, then $r(\gamma) \leq \delta_0 < 1$, so that, for some fixed $\delta \in (\delta_0, 1)$, we can deduce from the definition of $g_n(\gamma)$ and $r(\gamma)$, and from assumptions of Theorem 2.4(i)-(ii), that there exists $\tilde{C}_\delta > 0$ such that $G(\gamma) \leq \sum_{n=0}^{+\infty} \tilde{C}_\delta \delta^n < \infty$. Hence (i) is fulfilled. Now, under the assumptions of (ii), it follows from (i) that: $\nu < \infty \Leftrightarrow \limsup_{\gamma \to +\infty} r(\gamma) < 1$. Moreover, due to Theorem 2.4, we know that $\limsup_{\gamma \to +\infty} r(\gamma) \leq \max(\delta_0, r(\infty))$, and even that $\lim_{\gamma \to +\infty} r(\gamma) = r(\infty)$ if $r(\infty) > \delta_0$. Considering the cases $r(\infty) \leq \delta_0$ and $r(\infty) > \delta_0$ then gives the desired equivalence in (ii). $\square$

3.4. **About the positivity of the spectral radius.** Recall that we have set $J_0 := \{\gamma \in J : r(\gamma) > \delta_0\}$ under the assumptions of Theorem 2.4. Another consequence of the monotonicity of $\gamma \mapsto r(\gamma)$ is the following lemma.

Lemma 3.14. *Let $\gamma_1, \gamma_2, \gamma_3$ be such that $0 \leq \gamma_1 < \gamma_2 < \gamma_3$. Assume that the assumptions of Theorem 2.4 hold with $J = (\gamma_1, \gamma_2)$ and that r is non-increasing on J. Moreover suppose that, for every $\gamma \in (\gamma_1, \gamma_3)$, P_γ continuously acts on $\mathcal{B}$ and that the map $f \mapsto \pi(\kappa e^{-\gamma\xi} f)$ is in $\mathcal{B}^*$, where $\mathcal{B}$ is the space given in Theorem 2.4. Assume moreover that $\mathcal{B}$ satisfies Hypotheses 3.2 and 3.3 and that*

$$\Delta_0 := \limsup_{n \to +\infty} \Big(\pi\big(\kappa\, P_0^n \mathbf{1}_{\mathbb{X}}\big)\Big)^{\frac{1}{n}} < \infty. \tag{19}$$

If $J_0 \neq \emptyset$, then we have $r(\gamma) > 0$ for every $\gamma \in (\gamma_1, \gamma_3)$.

Proof. Let $\gamma_0 \in J_0$, $\gamma_0 \neq 0$. Then $r(\gamma) \geq r(\gamma_0) > 0$ for every $\gamma \in (\gamma_1, \gamma_0]$ from Proposition 3.10. Next let $\gamma \in (\gamma_0, \gamma_3)$ and set $p := \gamma/\gamma_0 > 1$. Due to Proposition 3.6, $\widehat{\pi}_{\gamma_0}$ is positive and $\widehat{\phi}_{\gamma_0} > 0$ (modulo π), so that

$$0 < r(\gamma_0) = r\left(\frac{\gamma}{p}\right) = \lim_{n \to +\infty} \Big(\pi\big(\kappa\, e^{-\frac{\gamma}{p}\xi}\, P_{\frac{\gamma}{p}}^n \mathbf{1}_{\mathbb{X}}\big)\Big)^{\frac{1}{n}} = \lim_{n \to +\infty} \left(\mathbb{E}_\pi\left[\left(\prod_{j=0}^n \kappa(X_j)\right) e^{-\frac{\gamma}{p} S_n}\right]\right)^{\frac{1}{n}}$$

due to (13) since $\mathbf{1}_{\mathbb{X}} \in \mathcal{B}$ and $f \mapsto \pi(\kappa e^{-\frac{\gamma}{p}\xi} f)$ is in $\mathcal{B}^*$, and due to a formula similar to (5) (in which we replace $h_{\kappa,\gamma}$ by $\kappa e^{-\gamma\xi}$). Let $q = p/(p-1)$. Writing $\kappa(X_j) = \kappa(X_j)^{1/q}\kappa(X_j)^{1/p}$, it follows from the Hölder inequality that

$$\begin{aligned} r(\gamma_0) &\leq \limsup_{n \to +\infty} \left(\mathbb{E}_\pi\left[\prod_{j=0}^n \kappa(X_j)\right]\right)^{\frac{1}{nq}} \times \limsup_{n \to +\infty} \left(\mathbb{E}_\pi\left[\left(\prod_{j=0}^n \kappa(X_j)\right) e^{-\gamma S_n}\right]\right)^{\frac{1}{np}} \\ &\leq \limsup_{n \to +\infty} \Big(\pi\big(\kappa\, P_0^n \mathbf{1}_{\mathbb{X}}\big)\Big)^{\frac{1}{nq}} \times \limsup_{n \to +\infty} \Big(\pi\big(\kappa\, e^{-\gamma\xi}\, P_\gamma^n \mathbf{1}_{\mathbb{X}}\big)\Big)^{\frac{1}{np}}. \end{aligned}$$

The above first limit superior equals to $\Delta_0^{1/q}$ by hypothesis, and the second limit superior is less than $(r(\gamma))^{\frac{1}{p}}$ from the definition of $r(\gamma)$ and from the fact that $\mathbf{1}_{\mathbb{X}} \in \mathcal{B}$ and $f \mapsto \pi(\kappa e^{-\gamma\xi} f) = \pi(P_\gamma f)$ is in $\mathcal{B}^*$. Thus $0 < r(\gamma_0) \leq \Delta_0^{1/q}\,(r(\gamma))^{\frac{1}{p}}$. $\square$

Remark 3.15. *Condition (19) holds if $P_0 \in \mathcal{L}(\mathcal{B})$ and if the map $f \mapsto \pi(\kappa f)$ is in $\mathcal{B}^*$ since $\Delta_0 \leq r(0)$ from the definition of the spectral radius $r(0)$ of P_0. Moreover note that (19) holds also if κ is bounded by some constant $d > 0$ since $P_0 \leq d\,P$.*

4. Proof of Theorems 2.4 and 2.5

Let us state the Keller-Liverani perturbation theorem. We use the operator-norm notations of the beginning of Section 2.

Theorem 4.1 (Keller-Liverani Perturbation Theorem [15, 1, 6])**.** *Let $(\mathcal{X}_0, \|\cdot\|_{\mathcal{X}_0})$ be a Banach space and $(\mathcal{X}_1, \|\cdot\|_{\mathcal{X}_1})$ be a normed space such that $\mathcal{X}_0 \hookrightarrow \mathcal{X}_1$. Let $J \subset [-\infty, +\infty]$ be an interval and let $(Q(t))_{t\in J}$ be a family of operators. We assume that*

- *For every $t \in J$, $Q(t) \in \mathcal{L}(\mathcal{X}_0) \cap \mathcal{L}(\mathcal{X}_1)$,*
- *$t \mapsto Q(t)$ is a continuous map from J in $\mathcal{L}(\mathcal{X}_0, \mathcal{X}_1)$,*
- *There exist $\delta_0 > 0$, $c_0, M_0 > 0$ such that for every $t \in J$*

$$\forall f \in \mathcal{X}_0,\ \forall n \in \mathbb{Z}_+, \quad \|(Q(t))^n f\|_{\mathcal{X}_0} \leq c_0\big(\delta_0^n \|f\|_{\mathcal{X}_0} + M_0^n \|f\|_{\mathcal{X}_1}\big).$$

Let $t_0 \in J$. Then, for every $\varepsilon > 0$ and every $\delta > \delta_0$, there exists $I_0 \subset J$ containing t_0 such that

$$\sup_{t\in I_0,\, z\in\mathcal{D}(\delta,\varepsilon)} \|(zI - Q(t))^{-1}\|_{\mathcal{X}_0} < \infty,$$

with $\mathcal{D}(\delta,\varepsilon) := \{z \in \mathbb{C},\ d(z, \sigma(Q(t_0)_{|\mathcal{X}_0})) > \varepsilon,\ |z| > \delta\}$.

Furthermore the map $t \mapsto (zI - Q(t))^{-1}$ from J to $\mathcal{L}(\mathcal{X}_0, \mathcal{X}_1)$ is continuous at t_0 in a uniform way with respect to $z \in \mathcal{D}(\delta,\varepsilon)$, i.e.

$$\lim_{t\to t_0,\, t\in J} \sup\big\{\|(zI - Q(t))^{-1} - (zI - Q(t_0))^{-1}\|_{\mathcal{X}_0,\mathcal{X}_1}\ :\ z \in \mathcal{D}(\delta,\varepsilon)\big\} = 0.$$

In particular,

$$\limsup_{t\to t_0} r((Q(t))_{|\mathcal{X}_0}) \leq \max(\delta_0, r((Q(t_0))_{|\mathcal{X}_0}))\,. \tag{20}$$

Finally the map $t \mapsto r((Q(t))_{|\mathcal{X}_0})$ is continuous on $\{t \in J : r((Q(t))_{|\mathcal{X}_0}) > \delta_0 \geq r_{ess}((Q(t))_{|\mathcal{X}_0})\}$.

The next subsections are concerned with the proofs of the first part of Theorem 2.4 and of Theorem 2.5 under Hypothesis 2.3 or under Hypothesis 2.3*. Recall that the assertions (i) and (ii) of these two theorems are then provided by Formula (5).

4.1. Proof of Theorem 2.4 under Hypothesis 2.3.

Here we assume that $((P_\gamma)_{\gamma\in J}, \mathcal{B}_0, \mathcal{B}_3)$ satisfies Hypothesis 2.3 and that Hypothesis 2.1 is fulfilled on J_0 with $\mathcal{B} := \mathcal{B}_0$. From now on, to simplify notations, we write $R_z(\gamma) := (zI - P_\gamma)^{-1}$ for the resolvent when it is well defined. Recall that $J_0 := \{\gamma \in J : r(\gamma) > \delta_0\}$, where $r(\gamma) := r((P_\gamma)_{|\mathcal{B}_0})$. The property (12) and the continuity on J_0 of the function $\gamma \mapsto r(\gamma)$ follow from Theorem 4.1. Moreover we know from Hypothesis 2.1 on J_0 with $\mathcal{B} := \mathcal{B}_0$ that (8) holds. It remains to prove that, if K is a compact subset of J_0, then the constants θ_γ and M_γ in (8) are uniformly bounded by some $\theta_K \in (0,1)$ and $M_K \in (0, +\infty)$, and that $\gamma \mapsto \Pi_\gamma$ is continuous from J_0 to $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_3)$. To that effect we use below the spectral definition of Π_γ.

Let $\chi : J_0 \to (0, +\infty)$ be defined by $\chi(\gamma) := \max\big(\delta_0, \lambda(\gamma)\big)$, where we have set $\lambda(\gamma) := \max\{|\lambda| : \lambda \in \sigma(P_{\gamma|\mathcal{B}_0}) \setminus \{r(\gamma)\}\}$. Due to Theorem 4.1, χ is continuous on J_0. Let K be a compact subset of J_0. We set $\theta := \max_{\gamma\in K} \frac{\chi(\gamma)}{r(\gamma)}$. Since $\chi(\gamma) < r(\gamma)$ for every $\gamma \in K$ and since $r(\cdot)$ and $\chi(\cdot)$ are continuous, we conclude that $\theta \in (0,1)$. Next we consider any $\eta > 0$ such that $\theta + 2\eta < 1$. Let us construct the map $\gamma \mapsto \Pi_\gamma$ from K to $\mathcal{L}(\mathcal{B}_0)$. Let $\gamma_0 \in K$. Since r

is continuous on K, there exists $\varepsilon > 0$ such that, for every $\gamma \in K$ such that $|\gamma - \gamma_0| \leq \varepsilon$, we have $|r(\gamma) - r(\gamma_0)| < \eta r(\gamma_0)$. Let us write $K(\gamma_0)$ for the set of $\gamma \in K$ such that $|\gamma - \gamma_0| \leq \varepsilon$. Observe that, for any $\gamma \in K(\gamma_0)$,

$$\chi(\gamma) \leq \theta r(\gamma) < \theta(1+\eta) r(\gamma_0) < (\theta+\eta) r(\gamma_0) < (1-\eta) r(\gamma_0)$$

and so the eigenprojector Π_γ on $\mathrm{Ker}(P_\gamma - r(\gamma)I)$ can be defined by

$$\Pi_\gamma = \frac{1}{2i\pi} \oint_{\Gamma_1(\gamma_0)} R_z(\gamma)\, dz, \tag{21}$$

where $\Gamma_1(\gamma_0)$ is the oriented circle centered on $r(\gamma_0)$ with radius $\eta\, r(\gamma_0)$. Due to Theorem 4.1, $\gamma \mapsto \Pi_\gamma$ is well defined from $K(\gamma_0)$ to $\mathcal{L}(\mathcal{B}_0)$ and is continuous from $K(\gamma_0)$ to $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_3)$.

Now, for every $\gamma \in K$, we define the oriented circle $\Gamma_0(\gamma) := \big\{z \in \mathbb{C} : |z| = (\theta+\eta)\, r(\gamma)\big\}$. By definition of θ, for every $\gamma \in K$, we have $\chi(\gamma) \leq \theta\, r(\gamma)$ and so $\chi(\gamma) < (\theta+\eta)\, r(\gamma) < r(\gamma)$. Hence, by definition of $\chi(\gamma)$, $R_z(\gamma)$ is well-defined in $\mathcal{L}(\mathcal{B}_0)$ for every $\gamma \in K$ and $z \in \Gamma_0(\gamma)$. From spectral theory, it comes that

$$N_\gamma^n := P_\gamma^n - r(\gamma)^n \Pi_\gamma = \frac{1}{2i\pi} \oint_{\Gamma_0(\gamma)} z^n\, R_z(\gamma)\, dz \tag{22}$$

and so

$$\|P_\gamma^n - r(\gamma)^n \Pi_\gamma\|_{\mathcal{B}_0} \leq M_\gamma \left((\theta+\eta)\, r(\gamma)\right)^{n+1} \quad \text{with} \quad M_\gamma := \sup_{|z|=(\theta+\eta)\, r(\gamma)} \|R_z(\gamma)\|_{\mathcal{B}_0}. \tag{23}$$

We have to prove that

$$M_K := \sup_{\gamma \in K} M_\gamma < \infty. \tag{24}$$

Let $\gamma_0 \in K$. Since $\gamma \mapsto r(\gamma)$ is continuous at γ_0, there exists $\alpha \equiv \alpha(\gamma_0) > 0$ such that, for every $\gamma \in K$ satisfying $|\gamma - \gamma_0| < \alpha$, we have

$$\frac{\theta + \frac{\eta}{2}}{\theta+\eta}\, r(\gamma_0) < r(\gamma) < \frac{\theta + \frac{3\eta}{2}}{\theta+\eta}\, r(\gamma_0).$$

Set $\delta := \frac{\eta}{2}\, r(\gamma_0)$. If $|\gamma - \gamma_0| < \alpha$ and if $|z| = (\theta+\eta)\, r(\gamma)$, we obtain since $\delta_0 \leq \chi(\gamma_0) \leq \theta\, r(\gamma_0)$ and $\theta + 2\eta < 1$:

$$\delta_0 + \delta \leq \chi(\gamma_0) + \delta \leq \big(\theta + \frac{\eta}{2}\big)\, r(\gamma_0) < |z| < \big(\theta + \frac{3\eta}{2}\big)\, r(\gamma_0) < r(\gamma_0) - \delta.$$

From the previous inequalities, let us just keep in mind that $\chi(\gamma_0) + \delta < |z| < r(\gamma_0) - \delta$. Then, by definition of $\chi(\gamma_0)$, we conclude that every complex number z such that $|z| = (\theta+\eta)\, r(\gamma)$ satisfies

$$|z| > \delta_0 + \delta \ \text{ and } \ d\big(z, \sigma(P_{\gamma_0})\big) > \delta.$$

Hence, up to a change of α, due to Theorem 4.1, we obtain that

$$\sup_{\gamma > 0\,:\,|\gamma - \gamma_0| < \alpha} M_\gamma = \sup\left\{\|R_z(\gamma)\|_{\mathcal{B}_0} \ : \ |\gamma - \gamma_0| < \alpha,\ |z| = (\theta+\eta)\, r(\gamma)\right\} < \infty.$$

By a standard compacity argument, we have proved (24). Consequently, with $\theta_K := \theta + \eta$, we deduce from (23) that

$$\|P_\gamma^n - r(\gamma)^n \Pi_\gamma\|_{\mathcal{B}_0} \leq M_K \left(\theta_K\, r(\gamma)\right)^n$$

from which we derive (13) with $\mathcal{B} = \mathcal{B}_0$.

4.2. **Proof of Theorem 2.5 under Hypothesis 2.3.** First we prove the following lemma.

Lemma 4.2. *For every $\gamma \in J_0$ and for $i = 1, 2$, the spectral radius of $P_{\gamma|\mathcal{B}_i}$ is equal to $r(\gamma) := r\big(P_{\gamma|\mathcal{B}_0}\big)$.*

Proof. For $i = 0, 1, 2$ set $r_i(\gamma) := r((P_\gamma)_{|\mathcal{B}_i})$. Due to Theorem 2.4 applied to $((P_\gamma)_\gamma, J, \mathcal{B}_i, \mathcal{B}_{i+1})$, $\pi(P_\gamma^n \mathbf{1}_{\mathbb{X}}) \sim c_i\, r_i(\gamma)^n$ as n goes to infinity, with $c_i := \pi(\Pi_\gamma \mathbf{1}_{\mathbb{X}}) > 0$ from (14). This proves the equality of the spectral radius. □

Proof of Theorem 2.5 under Hypothesis 2.3. We define χ_i as χ in the proof of Theorem 2.4 for each $\mathcal{B}_i$ ($i = 0, 1, 2$). We define now $\chi := \max(\chi_0, \chi_1, \chi_2)$. Let us prove the differentiability of r and Π on J_0. Let $\gamma_0 \in J_0$. Let $\eta > 0$ be such that $r(\gamma_0) > \chi(\gamma_0) + 2\eta$ and let $\varepsilon > 0$ be such that for every $\gamma \in J_0$ satisfying $|\gamma - \gamma_0| < \varepsilon$, we have $r(\gamma) > r(\gamma_0) - \eta > \chi(\gamma_0) + \eta > \chi(\gamma)$. We set $I_0 := J_0 \cap (\gamma_0 - \varepsilon, \gamma_0 + \varepsilon)$ and

$$\mathcal{D}_0 := \{z \in \mathbb{C} \,:\, \chi(\gamma_0) + \eta < |z| < r(\gamma_0) - \eta\} \cup \{z \in \mathbb{C} : |z - r(\gamma_0)| = \eta\}. \tag{25}$$

Due to the hypotheses of Theorem 2.5 and to an easy adaptation of [14, Lemma A.2] (see Remark 4.3), we obtain that, for every $z \in \mathcal{D}_0$, the map $\gamma \mapsto R_z(\gamma)$ is C^1 from I_0 to $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_3)$ with $R_z'(\gamma) = R_z(\gamma) P_\gamma' R_z(\gamma)$ and

$$\lim_{h \to 0} \sup_{z \in \mathcal{D}_0} \frac{\|R_z(\gamma_0 + h) - R_z(\gamma_0) - hR_z'(\gamma_0)\|_{\mathcal{B}_0, \mathcal{B}_3}}{|h|} = 0. \tag{26}$$

Moreover, for every $\gamma \in I_0$, we deduce from spectral theory that Π_γ and N_γ (already defined in the proof of Theorem 2.4) are given by

$$\Pi_\gamma = \frac{1}{2i\pi} \oint_{\Gamma_1} R_z(\gamma)\, dz \quad \text{and} \quad N_\gamma = \frac{1}{2i\pi} \oint_{\Gamma_0} z R_z(\gamma)\, dz,$$

where Γ_1 is the oriented circle centered at $r(\gamma_0)$ with radius η and Γ_0 is the oriented circle centered at 0 with some radius ϑ_0 satisfying $\chi(\gamma_0) + \eta < \vartheta_0 < r(\gamma_0) - \eta$. Thus $\gamma \mapsto \Pi_\gamma$ and $\gamma \mapsto N_\gamma$ are C^1-smooth from J_0 to $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_3)$. Since $\mathbf{1}_{\mathbb{X}} \in \mathcal{B}_0$ by hypothesis this implies the continuous differentiability of $\gamma \mapsto N_\gamma \mathbf{1}_{\mathbb{X}}$ and of $\gamma \mapsto \Pi_\gamma \mathbf{1}_{\mathbb{X}}$ from J_0 to $\mathcal{B}_3$. Since $r(\gamma) = \frac{\pi((P_\gamma - N_\gamma)(\mathbf{1}_{\mathbb{X}}))}{\pi(\Pi_\gamma(\mathbf{1}_{\mathbb{X}}))}$ and $\gamma \mapsto P_\gamma \mathbf{1}_{\mathbb{X}}$ is C^1 from I_0 to $\mathcal{B}_3$ and since $\pi \in \mathcal{B}_3^*$ by hypothesis, we obtain the continuous differentiability of r on I_0. □

Remark 4.3 (Proof of the differentiability of $\gamma \mapsto R_z(\gamma)$)**.** *We adapt the arguments of* [14, Lemma A.2]*, writing*

$$R_z(\gamma) = R_z(\gamma_0) \;+\; R_z(\gamma_0)\,[P_\gamma - P_{\gamma_0}]\,R_z(\gamma_0) \;+\; \vartheta_z(\gamma),$$
$$\text{with} \qquad \vartheta_z(\gamma) := R_z(\gamma_0)\,[P_\gamma - P_{\gamma_0}]\,R_z(\gamma_0)\,[P_\gamma - P_{\gamma_0}]\,R_z(\gamma).$$

Then

$$\frac{\|\vartheta_z(\gamma)\|_{\mathcal{B}_0, \mathcal{B}_2}}{|\gamma - \gamma_0|} \le \|R_z(\gamma_0)\|_{\mathcal{B}_2} \left\| \frac{P_\gamma - P_{\gamma_0}}{\gamma - \gamma_0} \right\|_{\mathcal{B}_1, \mathcal{B}_2} \|R_z(\gamma_0)\|_{\mathcal{B}_1} \|P_\gamma - P_{\gamma_0}\|_{\mathcal{B}_0, \mathcal{B}_1} \|R_z(\gamma)\|_{\mathcal{B}_0}. \tag{27}$$

From the hypotheses of Theorem 2.5 and from the resolvent bounds derived from Theorem 4.1, the last term goes to 0, uniformly in $z \in \mathcal{D}$, when γ goes to γ_0. Similarly we have:

$$\begin{aligned}\big\|R_z(\gamma_0)(P_\gamma - P_{\gamma_0})R_z(\gamma_0) - (\gamma - \gamma_0)R_z(\gamma_0)P_{\gamma_0}' R_z(\gamma_0)\big\|_{\mathcal{B}_0, \mathcal{B}_3} \\ \le\; M\|P_\gamma - P_{\gamma_0} - (\gamma - \gamma_0)P_{\gamma_0}'\|_{\mathcal{B}_1, \mathcal{B}_2} = o(\gamma - \gamma_0)\end{aligned}$$

when again the finite positive constant M is derived from the resolvent bounds of Theorem 4.1. This shows that $R'_z(\gamma_0) = R_z(\gamma_0)P'_{\gamma_0}R_z(\gamma_0)$ in $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_3)$. To prove that $\gamma \mapsto R'_z(\gamma)$ is continuous from J_0 to $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_3)$ in a uniform way with respect to $z \in \mathcal{D}$, observe that $\gamma \mapsto R_z(\gamma)$ is $\mathcal{C}^0$ from J_0 to $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_1)$ (use Theorem 4.1), that $\gamma \mapsto P'_\gamma$ is $\mathcal{C}^0$ (uniformly in $z \in \mathcal{D}$) from J_0 to $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ by hypothesis, and finally that $\gamma \mapsto R_z(\gamma)$ is $\mathcal{C}^0$ (uniformly in $z \in \mathcal{D}$) from J_0 to $\mathcal{L}(\mathcal{B}_2, \mathcal{B}_3)$ (again use Theorem 4.1). Observe that (27) gives the differentiability at γ_0 of the map $\gamma \mapsto R_z(\gamma)$ considered from J to $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_2)$. The additional space $\mathcal{B}_3$ is only required to obtain the continuous differentiability.

4.3. **Proof of Theorems 2.4 and 2.5 under Hypothesis 2.3*.** Here the Keller-Liverani perturbation theorem must be applied to the dual family $(P^*_\gamma)_\gamma$. Actually the hypotheses of Theorem 2.4 are:

- $\mathcal{B}^*_3 \hookrightarrow \mathcal{B}^*_0$,
- For every $\gamma \in J$, $P^*_\gamma \in \mathcal{L}(\mathcal{B}^*_0) \cap \mathcal{L}(\mathcal{B}^*_3)$,
- $\gamma \mapsto P^*_\gamma$ is a continuous map from J in $\mathcal{L}(\mathcal{B}^*_3, \mathcal{B}^*_0)$,
- There exist $\delta_0, c_0, M_0 > 0$ such that, for all $\gamma \in J$, $r_{ess}\big((P_\gamma)^*_{|\mathcal{B}^*_3}\big) \le \delta_0$ and
$$\forall n \ge 1,\ \forall f^* \in \mathcal{B}^*_3, \quad \|(P^*_\gamma)^n f^*\|_{\mathcal{B}^*_3} \le c_0(\delta_0^n \|f^*\|_{\mathcal{B}^*_3} + M^n \|f^*\|_{\mathcal{B}^*_0}).$$
- Hypothesis 2.1 holds on $(J_0, \mathcal{B}_3)$.

Proof of Theorem 2.4 under Hypothesis 2.3.* Under these assumptions it follows from Theorem 4.1 applied to $(P^*_\gamma)_{\gamma\in J}$ with respect to $(\mathcal{B}^*_3, \mathcal{B}^*_0)$ that, for every $\varepsilon > 0$ and every $\delta > \delta_0$, the map $\gamma \mapsto (zI - P^*_\gamma)^{-1}$ is well defined from J_0 to $\mathcal{L}(\mathcal{B}^*_3)$, provided that $z \in \mathcal{D}(\delta, \varepsilon)$ with
$$\mathcal{D}(\delta,\varepsilon) := \{z \in \mathbb{C},\ d(z, \sigma\big((P^*_{\gamma_0})_{|\mathcal{B}^*_2})\big) > \varepsilon,\ |z| > \delta\} = \{z \in \mathbb{C},\ d(z, \sigma\big((P_{\gamma_0})_{|\mathcal{B}_2})\big) > \varepsilon,\ |z| > \delta\}.$$
In addition, the map $\gamma \mapsto (zI - P^*_\gamma)^{-1}$, considered from J_0 to $\mathcal{L}(\mathcal{B}^*_3, \mathcal{B}^*_0)$, is continuous at every $\gamma_0 \in J_0$ in a uniform way with respect to $z \in \mathcal{D}(\delta, \varepsilon)$. By duality this implies that $\gamma \mapsto (zI - P_\gamma)^{-1}$ is well defined from J_0 to $\mathcal{L}(\mathcal{B}_3)$ for every $z \in \mathcal{D}(\delta, \varepsilon)$. Moreover, when this map is considered from J_0 to $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_3)$, it is continuous at γ_0 in a uniform way with respect to $z \in \mathcal{D}(\delta, \varepsilon)$. Finally Hypothesis 2.1 on $(J_0, \mathcal{B}_3)$ enables us to identify the spectral elements associated with $r(\gamma) := r\big((P_\gamma)_{|\mathcal{B}_3}\big)$. Consequently one can prove as in Subsection 4.1 that there exists a map $\gamma \mapsto \Pi_\gamma$ from J_0 to $\mathcal{L}(\mathcal{B}_3)$, which is continuous from J_0 to $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_3)$, such that (13) holds with $\mathcal{B} := \mathcal{B}_3$. □

Proof of Theorem 2.5 under Hypothesis 2.3.* When Theorem 2.5 is stated with Hypothesis 2.3*, then Theorem 2.4 applies on $(\mathcal{B}_0, \mathcal{B}_1)$, $(\mathcal{B}_1, \mathcal{B}_2)$ and $(\mathcal{B}_2, \mathcal{B}_3)$ (with Hypothesis 2.3* in each case). Thus, for every $\gamma \in J_0$, the spectral radius $r_i(\gamma) := r((P_\gamma)_{|\mathcal{B}_i})$ are equal for $i = 1, 2, 3$ (See the proof of Lemma 4.2). Observe that, from our hypotheses, Hypothesis 2.1 holds on $(J_0, \mathcal{B}_i)$ for $i = 1, 2, 3$. Since P^*_γ on $\mathcal{B}^*_i$ inherits the spectral properties of P_γ on $\mathcal{B}_i$, we can prove as above that, for every $\gamma_0 \in J_0$ and for every $\varepsilon > 0$ and $\delta > \delta_0$, the map $\gamma \mapsto (zI - P^*_\gamma)^{-1}$ is well defined from some subinterval I_0 of J_0 containing γ_0 into $\mathcal{L}(\mathcal{B}^*_3)$, provided that $z \in \mathcal{D}_0$ where the set $\mathcal{D}_0$ is defined in (25). In addition, by applying Remark 4.3 with the adjoint operators $(P^*_\gamma)_\gamma$ and the spaces $\mathcal{B}^*_3 \hookrightarrow \mathcal{B}^*_2 \hookrightarrow \mathcal{B}^*_1 \hookrightarrow \mathcal{B}^*_0$, we can prove that the map $\gamma \mapsto (zI - P^*_\gamma)^{-1}$, considered from J_0 to $\mathcal{L}(\mathcal{B}^*_3, \mathcal{B}^*_0)$, is $\mathcal{C}^1$ in a uniform way with respect to $z \in \mathcal{D}_0$. By duality, this gives (26). We conclude the differentiability of $\gamma \mapsto \Pi^*_\gamma$ from J_0 to $\mathcal{L}(\mathcal{B}^*_3, \mathcal{B}^*_1)$ and so the differentiability of $\gamma \mapsto \Pi_\gamma$ from J_0 to $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_3)$. □

4.4. **Complements on the derivative of** $r(\cdot)$**.** Let us first prove the following.

Lemma 4.4. *Let* $J_1 = (a,b) \subset [0,+\infty)$ *and let* $\mathcal{B}_1 \hookrightarrow \mathcal{B}_2$ *be two Banach spaces such that* $f \mapsto \xi f \in \mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)$*. Assume that, for every* $\gamma \in J_1$*,* $P_\gamma \in \mathcal{L}(\mathcal{B}_1)\cap\mathcal{L}(\mathcal{B}_2)$ *and that there exist elements* $\phi_\gamma \in \mathcal{B}_1$ *and* $\pi_\gamma \in \mathcal{B}_2^*$ *such that* $P_\gamma\phi_\gamma = r(\gamma)\phi_\gamma$ *and* $P_\gamma^*\pi_\gamma = r(\gamma)\pi_\gamma$*. Moreover assume that* $\gamma \mapsto P_\gamma$ *and* $\gamma \mapsto r(\gamma)$ *are differentiable from* J_1 *to* $\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)$ *and to* $\mathbb{C}$ *respectively, with respective derivatives at* $\gamma \in J_1$ *given by* $P'_\gamma : f \mapsto P_\gamma(-\xi f)$ *and* $r'(\gamma)$*. Finally assume that* $\gamma \mapsto \phi_\gamma$ *is continuous from* J_1 *to* $\mathcal{B}_1$ *and differentiable from* J_1 *to* $\mathcal{B}_2$ *with derivative* $\gamma \mapsto \phi'_\gamma$*.*

Then we have for every $\gamma \in J_1$*:* $r'(\gamma)\pi_\gamma(\phi_\gamma) = -r(\gamma)\pi_\gamma\big(\xi\phi_\gamma\big)$*. In particular, if* $r(\gamma) > 0$*,* $\pi_\gamma(\phi_\gamma) \geq 0$ *and* $\pi_\gamma\big(\xi\phi_\gamma\big) > 0$*, then* $r'(\gamma) < 0$*.*

Proof. Let $\gamma,\gamma_0 \in J_1$. We have $P_\gamma\phi_\gamma = r(\gamma)\phi_\gamma$ in $\mathcal{B}_2$. From $P_\gamma\phi_\gamma - P_{\gamma_0}\phi_{\gamma_0} = P_{\gamma_0}(\phi_\gamma - \phi_{\gamma_0}) + (P_\gamma - P_{\gamma_0})(\phi_\gamma)$, we obtain that

$$r(\gamma_0)\phi'_{\gamma_0} + r'(\gamma_0)\phi_{\gamma_0} = P_{\gamma_0}(\phi'_{\gamma_0}) + P_{\gamma_0}(-\xi\phi_{\gamma_0}) \quad \text{in } \mathcal{B}_2.$$

We conclude by composing by π_{γ_0} and using the fact that $\pi_{\gamma_0} \circ P_{\gamma_0} = r(\gamma_0)\pi_{\gamma_0}$. □

Proposition 4.5. *Assume that the assumptions of Theorem 2.5 hold. For every* $\gamma \in J_0$*, set* $\phi_\gamma := \Pi_\gamma \mathbf{1}_{\mathbb{X}}$ *and* $\pi_\gamma := \Pi_\gamma^*\pi$*. Then the assumptions of Lemma 4.4 hold with* $J_1 = J_0$ *and with respect to the spaces* $\mathcal{B}_1 \hookrightarrow \mathcal{B}_2$ *(resp.* $\mathcal{B}_1 \hookrightarrow \mathcal{B}_3$*) when the assumptions of Theorem 2.5 hold with Hypothesis 2.3 (resp. with Hypothesis 2.3*). Consequently, for every* $\gamma \in J_0$*,*

$$\pi\left(\Pi_\gamma \mathbf{1}_{\mathbb{X}}\right) > 0 \quad \textit{and} \quad \pi(\Pi_\gamma(\xi\, \Pi_\gamma \mathbf{1}_{\mathbb{X}})) > 0 \implies r'(\gamma) < 0.$$

Proof of Proposition 4.5 under Hypothesis 2.3. We have $\pi_\gamma \in \mathcal{B}_2^*$ since $\pi \in \mathcal{B}_2^*$ and Π_γ^* is well defined in $\mathcal{L}(\mathcal{B}_2^*)$. Moreover $\phi_\gamma \in \mathcal{B}_1$ since $\mathbf{1}_{\mathbb{X}} \in \mathcal{B}_1$ and $\Pi_\gamma \in \mathcal{L}(\mathcal{B}_1)$, and $\gamma \mapsto \phi_\gamma$ is continuous from J to $\mathcal{B}_1$ by Theorem 2.4. Finally $\gamma \mapsto \phi_\gamma$ is differentiable from J to $\mathcal{B}_2$ (see the end of Remark 4.3). We have proved that the assumptions of Lemma 4.4 hold as stated under Hypothesis 2.3. Finally, since $r(\gamma) > 0$ when $\gamma \in J_0$, the desired implication in Proposition 4.5 follows from the conclusion of Lemma 4.4 because $\pi_\gamma(\phi_\gamma) = \pi(\phi_\gamma)$ and $\pi_\gamma(\xi\, \phi_\gamma) = \pi(\Pi_\gamma(\xi\, \Pi_\gamma \mathbf{1}_{\mathbb{X}}))$. □

Proof of Proposition 4.5 under Hypothesis 2.3.* Note that $\pi_\gamma := \Pi_\gamma^*\pi \in \mathcal{B}_3^*$ since $\pi \in \mathcal{B}_3^*$ and Π_γ^* is well defined in $\mathcal{L}(\mathcal{B}_3^*)$. The function $\gamma \mapsto P_\gamma$ is differentiable from J_0 to $\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)$, thus from J_0 to $\mathcal{L}(\mathcal{B}_1,\mathcal{B}_3)$. We have $\phi_\gamma := \Pi_\gamma \mathbf{1}_{\mathbb{X}} \in \mathcal{B}_1$ since $\mathbf{1}_{\mathbb{X}} \in \mathcal{B}_1$ and Π_γ is well defined in $\mathcal{L}(\mathcal{B}_1)$. Moreover $\gamma \mapsto \phi_\gamma$ is continuous from J to $\mathcal{B}_1$ since Π_γ is well defined in $\mathcal{L}(\mathcal{B}_1)$, continuous from J_0 to $\mathcal{L}(\mathcal{B}_0,\mathcal{B}_1)$, and $\mathbf{1}_{\mathbb{X}} \in \mathcal{B}_0$. Finally $\gamma \mapsto \phi_\gamma$ is differentiable from J to $\mathcal{B}_3$ since Π_γ is well defined in $\mathcal{L}(\mathcal{B}_3)$ and differentiable from J_0 to $\mathcal{L}(\mathcal{B}_1,\mathcal{B}_3)$ and $\mathbf{1}_{\mathbb{X}} \in \mathcal{B}_1$. □

5. Application to the Knudsen gas

In this section, we apply our general results for the Knudsen gas. In this model, at each step, either we follow a Markov chain $Z = (Z_n)_n$ (with probability $(1-\alpha)$) or we generate an independent random variable with distribution the invariant probability measure of Z (with probability α). See [2] for more about this model.

Knudsen gas. *Let $\mathbb{X} := \mathbb{R}^d$, let π be some Borel probability measure on $\mathbb{X}$, and let U be a Markov operator with stationary probability π. We fix $\alpha \in (1/2, 1)$. Let $X = (X_n)_n$ be a Markov chain with transition kernel $P := \alpha\pi + (1-\alpha)U$.*

Here we apply the assertions (i)-(ii) of Theorems 2.4 and 2.5 to Knudsen gas by considering the action of the Laplace-type kernels P_γ on the usual Lebesgue space $(\mathbb{L}^a(\pi), \|\cdot\|_a)$ for some suitable $a \in [1, +\infty)$, where

$$\|f\|_a := \left(\int_{\mathbb{X}} |f(x)|^a \, d\pi(x)\right)^{\frac{1}{a}}. \tag{28}$$

Mention that Theorem 2.6 of [13] directly follows from the next theorem.

Theorem 5.1 (Knudsen gas)**.** *Let $p > 1$. Assume that $X = (X_n)_n$ is a Knudsen gas as above described, and that its initial distribution μ on $\mathbb{X}$ is absolutely continuous with respect to π, with density in $\mathbb{L}^p(\pi)$. Assume moreover that $\kappa \equiv 2$. Then, under $\mathbb{P}_\mu$, $(S_n)_n$ is multiplicatively ergodic on the interval $J_0 = \{\gamma > 0 \, : \, r(\gamma) > 2(1-\alpha)\}$, where $r(\gamma)$ denotes le spectral radius of P_γ on $\mathbb{L}^b$ with $b := \frac{p}{p-1}$. If moreover $\alpha > 1/2$ and if*

$$2\alpha \sum_{n \geq 0} (2(1-\alpha))^n \mathbb{P}_\pi\left(\sum_{k=0}^{n} \xi(Z_k) = 0\right) < 1, \tag{29}$$

where $(Z_n)_n$ is a Markov process with transition U, then ν defined in (2) is finite. Finally, if $\pi(\xi^\tau) < \infty$ for some $\tau > 1$ and if $p > \frac{\tau}{\tau-1}$, then the constant C_ν in (4) is well defined and finite.

Theorem 5.1 straightforwardly extends to the case $\kappa(\cdot) \equiv m$, where $m \geq 2$ is any integer. To prove Theorem 5.1, we will check that the hypothesis of Theorems 2.4 and 2.5 are fulfilled via Hypothesis 2.3 on $J = [0, +\infty)$ and on some suitable spaces $\mathbb{L}^a(\pi)$. Before we prove the following.

Lemma 5.2. *Let $1 \leq b < a$.*

- *(i) For every $\gamma \geq 0$, $r_{ess}(P_{\gamma|\mathbb{L}^a(\pi)}) \leq 2(1-\alpha)$.*
- *(ii) The function $\gamma \mapsto P_\gamma$ is continuous from $(0, +\infty]$ to $\mathcal{L}(\mathbb{L}^a(\pi), \mathbb{L}^b(\pi))$.*
- *(iii) For any $\gamma \in [0, +\infty]$ and any $f \in \mathbb{L}^a(\pi)$, $\|P_\gamma f\|_a \leq 2((1-\alpha)\|f\|_a + \alpha\|f\|_1)$.*
- *(iv) For any $\gamma > 0$, for any non-null non-negative $f \in \mathbb{L}^a(\pi)$ and every non-null non-negative $g \in \mathbb{L}^{a'}(\pi)$ with $a' = \frac{a}{a-1}$, we have $\pi(gP_\gamma f) > 0$ and $P_\gamma f > 0$.*
- *(v) If $r(\gamma) > 2(1-\alpha)$, for every $f, g \in \mathbb{L}^a(\pi)$ with $f > 0$, $P_\gamma f = r(\gamma) f$ and $P_\gamma g = r(\gamma) g$, then we have $g \in \mathbb{C} \cdot f$.*
- *(vi) 1 is the only complex number λ of modulus 1 such that $P(h/|h|) = \lambda h/|h|$ in $\mathbb{L}^1(\pi)$ for some $h \in \mathcal{B}$, $|h| > 0$ (modulo π).*
- *(vii) Let $\tau > 1$. If $\pi(\xi^\tau) < \infty$ and if $b < \frac{\tau a}{\tau a + 1}$, then $f \mapsto \xi f$ is in $\mathcal{L}(\mathbb{L}^a(\pi), \mathbb{L}^b(\pi))$ and $\gamma \mapsto P_\gamma$ is C^1 from $[0, +\infty)$ to $\mathcal{L}(\mathbb{L}^a(\pi), \mathbb{L}^b(\pi))$, with $P'_\gamma f = -P_\gamma(\xi f)$.*

Proof.
(i) Observe that $P_\gamma = 2(\alpha\pi(e^{-\gamma\xi}\cdot) + (1-\alpha)U_\gamma)$ with $U_\gamma := U(e^{-\gamma\xi}\cdot)$. Since the sum of a Fredholm operator with a compact operator is Fredholm, we directly obtain $r_{ess}(P_\gamma) = 2(1-\alpha) r_{ess}(U_\gamma) \leq 2(1-\alpha)$.

(ii) For every $0 \le \gamma, \gamma' < \infty$ and every $f \in \mathcal{B}$ such that $\|f\|_a = 1$, we have

$$\begin{aligned}\|P_\gamma f - P_{\gamma'} f\|_b &= 2\|P((e^{-\gamma\xi} - e^{-\gamma'\xi})f)\|_b \\ &\le 2\|(e^{-\gamma\xi} - e^{-\gamma'\xi})f\|_b \le 2\|e^{-\gamma\xi} - e^{-\gamma'\xi}\|_c,\end{aligned}$$

where c is such that $\frac{1}{a} + \frac{1}{c} = \frac{1}{b}$. Hence $\|P_\gamma - P_{\gamma'}\|_{\mathbb{L}^a(\pi),\mathbb{L}^b(\pi)} \le 2\|e^{-\gamma\xi} - e^{-\gamma'\xi}\|_c$, which converges to 0 as γ' goes to γ, by the dominated convergence theorem. In the same way, we prove that $\|P_\gamma - P_\infty\|_{\mathbb{L}^a(\pi),\mathbb{L}^b(\pi)} \le 2\|e^{-\gamma\xi}\|_c$ and hence the continuity of $\gamma \mapsto P_\gamma$ at infinity.

(iii) For every $\gamma \in [0, +\infty]$ and every $f \in \mathbb{L}^a(\pi)$, $\|P_\gamma f\|_a \le 2\|Pf\|_a \le 2((1-\alpha)\|f\|_a + \alpha\|f\|_1)$ since $\|Uf\|_a \le \|f\|_a$. This gives the Doeblin-Fortet inequality.

(iv) For any non-null non-negative $f \in \mathbb{L}^a(\pi)$, we have $P_\gamma f \ge 2\alpha\pi(e^{-\gamma\xi} f)\mathbf{1}_{\mathbb{X}} > 0$. The other assertion of (iv) is then obvious.

(v) Let $f, g \in \mathbb{L}^a(\pi)$ such that $f > 0$, $P_\gamma f = r(\gamma) f$ and $P_\gamma g = r(\gamma) g$ in $\mathbb{L}^a(\pi)$. Set $\beta := \frac{\pi(e^{-\gamma\xi} g)}{\pi(e^{-\gamma\xi} f)}$ and $h := g - \beta f$. Then $\pi(e^{-\gamma\xi} h) = 0$ and $P_\gamma h = r(\gamma)\, h$, which gives $r(\gamma)\, h = 2(1-\alpha)\, U(e^{-\gamma\xi} h)$, so that $r(\gamma)\, |h| \le 2(1-\alpha)\, U(|h|)$. Since $\pi U = \pi$, we obtain: $r(\gamma)\, \pi(|h|) \le 2(1-\alpha)\, \pi(|h|)$. Finally we conclude that $\pi(|h|) = 0$ because $r(\gamma) > 2(1-\alpha)$ and so $g = \beta f$ in $\mathbb{L}^a(\pi)$.

(vi) Let $k \in \mathbb{L}^1(\pi)$ and $\lambda \in \mathbb{C}$ be such that $|\lambda| = 1$, $|k| \equiv \mathbf{1}_{\mathbb{X}}$ and $P(k) = \lambda k$. Then $\lambda k = \alpha\pi(k) + (1-\alpha)U(k)$. Taking the modulus, we obtain $1 \le \alpha|\pi(k)| + (1-\alpha)U(\mathbf{1}_{\mathbb{X}}) \le 1$. By convexity we conclude that $|\pi(k)| = 1$ and that k is constant modulo π, so that $\lambda = 1$.

(vii) Let $f \in \mathbb{L}^a(\pi)$. Observe that $\|\xi f\|_b \le \|\xi f\|_{\frac{\tau a}{\tau a+1}} \le \|\xi\|_\tau \|f\|_a$, thus

$$\forall \gamma \ge 0, \quad \|P_\gamma(\xi f)\|_b = \|P(e^{-\gamma\xi}\xi f)\|_b \le \|e^{-\gamma\xi}\xi f\|_b \le \|\xi f\|_b \le \|\xi\|_\tau \|f\|_a.$$

This proves that, for every $\gamma \ge 0$, the map $f \mapsto P_\gamma(\xi f)$ is in $\mathcal{L}(\mathbb{L}^a(\pi), \mathbb{L}^b(\pi))$. Next, for any $f \in \mathbb{L}^a(\pi)$ and $\gamma, \gamma' > 0$, we have

$$\begin{aligned}\|P_{\gamma'} f - P_\gamma f + (\gamma' - \gamma)P_\gamma(\xi f)\|_b &= \|P\big((e^{-\gamma'\xi} - e^{-\gamma\xi} + (\gamma' - \gamma)e^{-\gamma\xi}\xi)f\big)\|_b \\ &\le \left\|(e^{-\gamma'\xi} - e^{-\gamma\xi} + (\gamma' - \gamma)e^{-\gamma\xi}\xi)f\right\|_b \\ &\le \left\|e^{-\gamma'\xi} - e^{-\gamma\xi} + (\gamma' - \gamma)e^{-\gamma\xi}\xi\right\|_\tau \|f\|_a,\end{aligned}$$

and $\|e^{-\gamma'\xi} - e^{-\gamma\xi} + (\gamma' - \gamma)e^{-\gamma\xi}\xi\|_\tau$ converges to 0 as γ' goes to γ by the dominated convergence theorem. We have proved that $\gamma \mapsto P_\gamma$ is differentiable from $[0, +\infty)$ to $\mathcal{L}(\mathbb{L}^a(\pi), \mathbb{L}^b(\pi))$, with $P'_\gamma f = -P_\gamma(\xi f)$. Finally, since $b < \frac{\tau a}{\tau a+1}$, we can choose $d > 1$ such that $\frac{1}{b} = \frac{1}{d} + \frac{\tau a+1}{\tau a}$, so that we obtain for any $f \in \mathbb{L}^a(\pi)$ and $\gamma, \gamma' > 0$

$$\begin{aligned}\|P_\gamma(\xi f) - P_{\gamma'}(\xi f)\|_b = \|P\big((e^{-\gamma\xi} - e^{-\gamma'\xi})\xi f\big)\|_b &\le \|(e^{-\gamma\xi} - e^{-\gamma'\xi})\xi f\|_b \\ &\le \|e^{-\gamma\xi} - e^{-\gamma'\xi}\|_d \, \|\xi f\|_{\frac{\tau a}{\tau a+1}} \\ &\le \|e^{-\gamma\xi} - e^{-\gamma'\xi}\|_d \, \|\xi\|_\tau \| f\|_a.\end{aligned}$$

from which we deduce that $\gamma \mapsto P'_\gamma$ is continuous from $[0, +\infty)$ to $\mathcal{L}(\mathbb{L}^a(\pi), \mathbb{L}^b(\pi))$ since $\|e^{-\gamma\xi} - e^{-\gamma'\xi}\|_d$ converges to 0 as γ' goes to γ by the dominated convergence theorem. □

Proof of Theorem 5.1. Let us first apply Theorem 2.4 to obtain the existence of ν. Let $b := \frac{p}{p-1}$ and $a > b$. From Assertion (i)-(iii) of Lemma 5.2, $(P_\gamma)_\gamma$ satisfies Hypothesis 2.3 with $J = [0, +\infty)$, $\mathcal{B}_0 = \mathbb{L}^a(\pi)$ and $\mathcal{B}_1 = \mathbb{L}^b(\pi)$ (to obtain (9b), iterate Inequality (iii) of

Lemma 5.2 and use $\|\cdot\|_1 \le \|\cdot\|_b$). Next, if $\gamma \in J_0$ (i.e. $r(\gamma) > 2(1-\alpha)$), then P_γ is quasi-compact from Assertion (i) of Lemma 5.2. Moreover note that Hypothesis 3.2 holds with $\mathcal{B} = \mathbb{L}^a(\pi)$ since $\mathbb{L}^a(\pi)$ is a Banach lattice, and that Hypothesis 3.3 is fulfilled with $\mathcal{B} = \mathbb{L}^a(\pi)$ by using the last property in Assertion (iv) of Lemma 5.2. Then P_γ satisfies the hypotheses of Proposition 3.5 on $\mathcal{B}_0 = \mathbb{L}^a(\pi)$ from Assertions (v) and (vi) of Lemma 5.2. Thus Hypothesis 2.1 holds on J_0 with $\mathcal{B}_0 = \mathbb{L}^a(\pi)$. Theorem 2.4 ensures that $\gamma \mapsto r(\gamma)$ is continuous on J_0 and that Properties (12)-(13) hold.

Consequently, from the assertion (i) of Theorems 2.4, $(S_n)_n$ is multiplicatively ergodic on J_0 with respect to $\mathbb{P}_\mu$, if we prove that $\mu_\gamma : f \mapsto \mu(e^{-\gamma\xi} f)$ is in $(\mathbb{L}^b(\pi))^*$, that the function $\gamma \mapsto \mu_\gamma$ is continuous from J_0 to $(\mathbb{L}^b(\pi))^*$, and finally that Condition (14) holds. Observe that (14) holds from Proposition 3.8 since the initial distribution μ is assumed to be absolutely continuous with respect to π in Theorem 5.1. Next, since the density g_μ of μ with respect to π is supposed to be in $\mathbb{L}^p(\pi)$, we have

$$\forall f \in \mathbb{L}^b(\pi), \quad \mu_\gamma(f) = \int_{\mathbb{R}^d} f(y)\, e^{-\gamma\xi(y)}\, g_\mu(y)\, d\pi(y)$$

thus $\mu_\gamma \in (\mathbb{L}^b(\pi))^*$ since $e^{-\gamma\xi}\, g_\mu \in \mathbb{L}^p(\pi)$. Moreover the norm in $(\mathbb{L}^b(\pi))^*$ of $(\mu_\gamma - \mu_{\gamma'})$ equals to $\|(e^{-\gamma\xi} - e^{-\gamma'\xi})\, g_\mu\|_p$, which converges to 0 as $\gamma' \to \gamma$ from Lebesgue's theorem.

Now we apply the assertion (ii) of Theorems 2.4 to prove that ν defined in (2) is finite under Condition (29). To that effect we need to study the spectral radius $r(\gamma)$ of P_γ. First observe that the non-increasingness of $r(\cdot)$ follows from Proposition 3.9 since $\mathbb{L}^a(\pi)$ is a Banach lattice. Consequently the set $J_0 := \{\gamma > 0 : r(\gamma) > 2(1-\alpha)\}$ is an interval with $\min J_0 = 0$ since $r(0) = 2$. Now set $h_\gamma := e^{-\gamma\xi}$ for $\gamma \ge 0$ and $h_\infty := \mathbf{1}_{\{\xi=0\}}$. Recall that $P_\gamma f = 2[\alpha\pi(f\, h_\gamma) + (1-\alpha)U(f\, h_\gamma)]$, that $U_\gamma(\cdot) = U(\cdot \times h_\gamma)$, and denote by $r(U_\gamma)$ the spectral radius of $(U_\gamma)_{|\mathbb{L}^a(\pi)}$. Note that $r(U_\gamma) \le 1$ since U is Markov.

Lemma 5.3. *Let $\gamma \in [0, \infty]$, $a \in (1, +\infty)$ and $\lambda \in \mathbb{C}$ be such that $|\lambda| > 2(1-\alpha)r(U_\gamma)$. Then λ is an eigenvalue of $(P_\gamma)_{|\mathbb{L}^a(\pi)}$ if and only if*

$$\lambda = 2\alpha \sum_{n\ge 0} \frac{2^n(1-\alpha)^n}{\lambda^n} \pi(h_\gamma U_\gamma^n(\mathbf{1}_{\mathbb{X}})). \tag{30}$$

In particular $r(\gamma) > 2(1-\alpha)$ if and only if (30) admits a solution belonging to $(2(1-\alpha), +\infty)$.

Proof of Lemma 5.3. First, let $f \in \mathbb{L}^a(\pi)$, $f \ne 0$, be such $P_\gamma f = \lambda f$ in $\mathbb{L}^a(\pi)$, i.e. $\lambda f = 2[\alpha\pi(f\, h_\gamma) + (1-\alpha)U_\gamma(f)]$, thus

$$\left[I - \frac{2(1-\alpha)}{\lambda} U_\gamma\right](f) = \frac{2\alpha\pi(f\, h_\gamma)}{\lambda}\, \mathbf{1}_{\mathbb{X}}\,.$$

Observe that $\pi(f\, h_\gamma) \ne 0$, otherwise f would satisfy $\lambda f = 2(1-\alpha)U_\gamma(f)$, which contradicts the fact that $\lambda/(2-2\alpha)$ is not in the spectrum of U_γ. Hence

$$f = \frac{2\alpha\, \pi(f h_\gamma)}{\lambda} \left[I - \frac{2(1-\alpha)}{\lambda} U_\gamma\right]^{-1} (\mathbf{1}_{\mathbb{X}}) = \frac{2\alpha\, \pi(f h_\gamma)}{\lambda} \sum_{n\ge 0} \frac{2^n(1-\alpha)^n}{\lambda^n} U_\gamma^n \mathbf{1}_{\mathbb{X}},$$

and so

$$\lambda = \lambda\pi\left(\frac{f\, h_\gamma}{\pi(f\, h_\gamma)}\right) = 2\alpha \sum_{n\ge 0} \frac{2^n(1-\alpha)^n}{\lambda^n} \pi\left(h_\gamma\, U_\gamma^n\, \mathbf{1}_{\mathbb{X}}\right)\,,$$

which leads to (30).
Second, if (30) holds, then we consider $g \in \mathbb{L}^a(\pi)$ be given by $g := \frac{2\alpha}{\lambda}\sum_{n\geq 0}\frac{2^n(1-\alpha)^n}{\lambda^n}U_\gamma^n \mathbf{1}_{\mathbb{X}}$. This g satisfies $\lambda g = 2[\alpha \mathbf{1}_{\mathbb{X}} + (1-\alpha)U_\gamma(g)]$ and, due to (30), $\pi(h_\gamma\, g) = 1$, so $P_\gamma g = \lambda g$. This proves the first equivalence of Lemma 5.3. For the second one, assume that $r(\gamma) > 2(1-\alpha)$. Then $r(\gamma)$ is an eigenvalue of $(P_\gamma)_{|\mathbb{L}^a(\pi)}$ since we know that Hypothesis 2.1 holds on J_0 and $\mathcal{B}_0 = \mathbb{L}^a(\pi)$. The first part of Lemma 5.3 then implies that $\lambda = r(\gamma)$ satisfies (30) (note that $r(\gamma) > 2(1-\alpha)r(U_\gamma)$ from $r(U_\gamma) \leq 1$). Now suppose that there exists $\lambda \in (2(1-\alpha), +\infty)$ satisfying (30). Then $\lambda > 2(1-\alpha)r(U_\gamma)$, thus λ is an eigenvalue of $(P_\gamma)_{|\mathbb{L}^a(\pi)}$ from the first part of Lemma 5.3, so that $r(\gamma) \geq \lambda > 2(1-\alpha)$. □

Remark 5.4. *Lemma 5.3 implies that, if $\gamma > 0$ is such that $r(\gamma) > 2(1-\alpha)$, then $\lambda = r(\gamma)$ is the unique positive solution of* (30). *Indeed, under the condition $r(\gamma) > 2(1-\alpha)$, we know that $r(\gamma)$ satisfies* (30) *(see the above arguments). Moreover Equation* (30) *admits at most one solution $\lambda \in (0, +\infty)$ since the left (resp. right) hand side of* (30) *is an increasing (resp. decreasing) function of the variable λ.*

Assume $\alpha > 1/2$ and (29) (which implies that $\pi(\xi = 0) < 1$). For every $\gamma \in [0, \infty]$ and every $\lambda > 0$, we write $\zeta(\gamma, \lambda)$ for the expression contained in the right hand side of (30), which rewrites as

$$\zeta(\gamma,\lambda) = 2\alpha \sum_{n\geq 0}\frac{2^n(1-\alpha)^n}{\lambda^n}\mathbb{E}_\pi\left[e^{-\gamma\sum_{k=0}^n \xi(Z_k)}\right] \quad \text{if} \quad \gamma \in (0,\infty)\,,$$

and

$$\zeta(\infty,\lambda) = 2\alpha \sum_{n\geq 0}\frac{2^n(1-\alpha)^n}{\lambda^n}\mathbb{P}_\pi\left(\sum_{k=0}^n \xi(Z_k) = 0\right)\,,$$

by using the Markov property. Note that $\zeta(\cdot, 1)$ is decreasing, continuous on $(0, \infty)$, with finite values (since $2(1-\alpha) < 1$). Note that $\zeta(0, 1) = \frac{\alpha}{\alpha - \frac{1}{2}} > 1$ and that $\zeta(\infty, 1)$ is the left hand side of (29), thus $\zeta(\infty, 1) < 1$. Therefore there exists a unique $\nu \in (0, \infty)$ such that $\zeta(\nu, 1) = 1$. Since $1 > 2(1-\alpha)$ it follows from the first equivalence of Lemma 5.3 that $\lambda = 1$ is an eigenvalue of $(P_\nu)_{|\mathbb{L}^a(\pi)}$. Thus $r(\nu) \geq 1$. In particular we have $r(\nu) > 2(1-\alpha)$, so that $r(\nu) = 1$ from Remark 5.4. Moreover $[0, \nu] \subset J_0 = \{\gamma \in [0, \infty) : r(\gamma) > 2(1-\alpha)\}$. The claimed statements on J_0 and ν in Theorem 5.1 are proved.

Now we apply Theorem 2.5 to prove the existence of the constant C_ν in (4). Assume that $\alpha > 1/2$, that (29) holds, and that $\pi(\xi^\tau) < \infty$ for some $\tau > 1$. Let $p > \frac{\tau}{\tau-1}$ and set $a_3 := \frac{p}{p-1}$ (i.e. $1/p + 1/a_3 = 1$). Note that $a_3 < \tau$. Let a_2 be such that $a_3 < a_2 < \tau$. Since $\lim_{a \to +\infty}\frac{\tau a}{\tau + a} = \tau$, we can chose $a_1 > a_2$ such that $a_2 < \frac{\tau a_1}{\tau + a_1}$. Next let $a_0 > a_1$. From Lemma 5.2 we deduce that the assumptions of Theorem 2.5 hold with the spaces $\mathcal{B}_i = \mathbb{L}^{a_i}(\pi)$ for $i = 0, 1, 2, 3$, so that we can apply Theorem 2.5: we conclude that r is C^1 on J_0. The fact that $r' < 0$ can easily be proved using Proposition 3.12 (to check the condition in Assertion (i) of Proposition 3.12, use the fact that Hypotheses 3.2 and 3.3 hold with $\mathcal{B} = \mathbb{L}^a(\pi)$ and apply the last assertion of Proposition 3.6). In this particular case, we can also use the fact that r is given by an implicit formula $F(r(\gamma), \gamma) = 0$ (see (30)), where F is C^1 with non-null derivatives at $(r(\gamma), \gamma)$. Moreover C_ν in (4) is well-defined, provided that $\mu_\gamma : f \mapsto \mu(e^{-\gamma\xi} f)$ is in $(\mathbb{L}^{a_3}(\pi))^*$ and that the function $\gamma \mapsto \mu_\gamma$ is continuous from J_0 to $(\mathbb{L}^{a_3}(\pi))^*$. These conditions hold since μ is absolutely continuous with respect to π with density in $\mathbb{L}^p(\pi)$ (see the proof of the multiplicative ergodicity). Moreover (14) has been proved together with the multiplicative ergodicity. The proof of Theorem 5.1 is then achieved. □

6. Application to the linear autoregressive model

Linear autoregressive model. *$\mathbb{X} := \mathbb{R}$ and $X_n = \alpha X_{n-1} + \vartheta_n$ for $n \geq 1$, where X_0 is a real-valued random variable, $\alpha \in (-1, 1)$, and $(\vartheta_n)_{n\geq 1}$ is a sequence of i.i.d. real-valued random variables independent of X_0. Let $r_0 > 0$. We assume that ϑ_1 has a continuous Lebesgue probability density function $p > 0$ on $\mathbb{R}$ satisfying the following condition: for all $x_0 \in \mathbb{R}$, there exist a neighbourhood V_{x_0} of x_0 and a non-negative function $q_{x_0}(\cdot)$ such that $y \mapsto (1 + |y|)^{r_0}\, q_{x_0}(y)$ is Lebesgue-integrable and such that :*

$$\forall y \in \mathbb{R},\ \forall v \in V_{x_0},\ p(y + v) \leq q_{x_0}(y). \tag{31}$$

The domination condition (31) means that p satisfies (32) under a (local) uniform domination way. This implies that p has a moment of order r_0, that is

$$\int |x|^{r_0} p(x)dx < \infty\,. \tag{32}$$

In other words ϑ_1 admits a moment of order r_0. Observe that $(X_n)_{n\in\mathbb{N}}$ is a Markov chain with transition kernel

$$P(x, A) = \int_{\mathbb{R}} \mathbf{1}_A(\alpha x + y)p(y)\, dy = \int_{\mathbb{R}} \mathbf{1}_A(y)p(y - \alpha x)\, dy \tag{33}$$

Set $V(x) := (1 + |x|)^{r_0}$, $x \in \mathbb{R}$. Recall that, under Assumption (32), P satisfies the following drift condition (see [21])

$$\forall \delta > |\alpha|^{r_0},\ \exists L \equiv L(\delta) > 0, \quad PV \leq \delta\, V + L\, \mathbf{1}_{\mathbb{R}}. \tag{34}$$

Moreover it is well-known that $(X_n)_{n\in\mathbb{N}}$ is V-geometrically ergodic, see [21]. Let $(\mathcal{B}_V, \|\cdot\|_V)$ be the weighted-supremum Banach space

$$\mathcal{B}_V := \Big\{\ f : \mathbb{R} \to \mathbb{C} \text{ measurable } : \|f\|_V := \sup_{x\in\mathbb{R}} |f(x)|V(x)^{-1} < \infty\ \Big\}. \tag{35}$$

Let $(\mathcal{C}_V, \|\cdot\|_V)$ denote the following subspace of $\mathcal{B}_V$:

$$\mathcal{C}_V := \left\{\ f \in \mathcal{B}_V :\ f \text{ is continuous and } \ \ell_V(f) := \lim_{|x|\to\infty} \frac{f(x)}{V(x)} \text{ exists in } \ \mathbb{C}\right\},$$

where the symbol $\lim_{|x|\to\infty}$ means that the limits when $x \to \pm\infty$ exist and are equal. Note that $V \in \mathcal{C}_V$ and that $\mathcal{C}_V$ is a closed subspace of $(\mathcal{B}_V, \|\cdot\|_V)$. Let $\mathcal{C}_{0,V}$ be the subspace of $\mathcal{C}_V$ defined by

$$\mathcal{C}_{0,V} := \{f \in \mathcal{C}_V : \ell_V(f) = 0\}.$$

Finally we denote by $(\mathcal{C}_1, \|\cdot\|_\infty)$ the space of bounded continuous complex-valued functions on $\mathbb{R}$ endowed with the supremum norm $\|\cdot\|_\infty$. We will see below that, for every $\gamma \in (0, +\infty]$, P_γ continuously acts on $\mathcal{C}_V$ (see Lemma 6.2). For $\gamma \in (0, +\infty]$, we denote by $r(\gamma)$ the spectral radius of P_γ on $\mathcal{C}_V$, that is:

$$r(\gamma) = r(P_{\gamma|\mathcal{C}_V}) := \lim_{n\to+\infty} \|P_\gamma^n\|_V^{1/n} = \lim_{n\to+\infty} \|P_\gamma^n V\|_V^{1/n}$$

where $\|\cdot\|_V$ also denotes the operator norm on $\mathcal{C}_V$. The last equality holds because, for every $n \geq 1$, we have $\|P_\gamma^n\|_V = \|P_\gamma^n V\|_V$. Indeed, since $V \in \mathcal{C}_V$ with $\|V\|_V = 1$, $\|P_\gamma^n\|_V \geq \|P_\gamma^n V\|_V$. Moreover, for every $f \in \mathcal{C}_V$ with $\|f\|_V = 1$, we have

$$\|P_\gamma^n f\|_V = \sup_{x\in\mathbb{R}} \frac{|(P_\gamma^n f)(x)|}{V(x)} \leq \sup_{x\in\mathbb{R}} \frac{(P_\gamma^n |f|)(x)}{V} \leq \|f\|_V \sup_{x\in\mathbb{R}} \frac{(P_\gamma^n V)(x)}{V(x)} = \|P_\gamma^n V\|_V\,.$$

We will also prove that $\lim_{\gamma\to 0_+} r(\gamma) \geq 2$.

Recall that $\xi : \mathbb{R} \to [0, +\infty)$ is a measurable function and that $S_n = \sum_{k=0}^n \xi(X_k)$. Let $\kappa : \mathbb{R} \to \{2, ...\}$ be a measurable function. The function $\xi : \mathbb{R} \to [0, +\infty)$ is said to be coercive if $\lim_{|x|\to+\infty} \xi(x) = +\infty$, i.e. if, for every β, the set $[\xi \leq \beta]$ is bounded.

Mention that Theorem 2.7 of [13] directly follows from the next theorem.

Theorem 6.1. *Assume that the previous assumptions hold. Assume that the distribution μ of X_0 belongs to $\mathcal{C}_V^*$, namely satisfies $\mu(V) < \infty$. Assume moreover that ξ is coercive, that κ is bounded, that p is continuous, and that $\sup_{x\in\mathbb{R}}(\xi(x)/V(x)) < \infty$. Then, under $\mathbb{P}_\mu$,*

a) $(S_n, \kappa(X_n))_n$ is multiplicatively ergodic on $(0, +\infty)$ with $\rho = r$ on $[0, +\infty)$.
b) If moreover the Lebesgue measure of the set $[\xi = 0]$ is zero, then $\lim_{\gamma\to+\infty} r(\gamma) = 0$. Hence ν is finite.
c) Moreover, if there exists $\tau > 0$ such that $\sup_{x\in\mathbb{R}}(\xi(x)^{1+\tau}/V(x)) < \infty$, then $\gamma \mapsto r(\gamma)$ admits a negative derivative on $[0, +\infty)$. Hence (4) holds also with $C_\nu \in (0, +\infty)$.

Recall that $\int_{\mathbb{R}} |x|^{r_0} \, d\pi(x) < \infty$ under the assumptions of Theorem 6.1 (see [4, 5]). Hence $\sup_{x\in\mathbb{R}} \frac{\xi(x)^{1+\tau}}{(1+|x|)^{r_0}} < \infty$ implies that $\int_{\mathbb{R}} |\xi|^{1+\tau} \, d\pi < \infty$.

The next subsections are devoted to the proof of Theorem 6.1. Here we apply the assertions (i)-(ii) of Theorems 2.4 and 2.5 by considering the action of the Laplace-type kernels P_γ on $\mathcal{C}_{V^a}$ for suitable $a \in [0, 1]$, in particular on $\mathcal{C}_V$ (i.e. $a = 1$) and $\mathcal{C}_{V^0} = \mathcal{C}_1$ (i.e. $a = 0$).

6.1. **Study of Hypothesis 2.3*.** In this subsection we prove that $((P_\gamma)_{\gamma\in J}, \mathcal{B}_0, \mathcal{B}_1)$ satisfies Hypothesis 2.3* with $J = (0, +\infty]$, $\mathcal{B}_0 = \mathcal{C}_1$, and $\mathcal{B}_1 = \mathcal{C}_V$. First we specify the action of P_γ on $\mathcal{C}_V$ and on $\mathcal{C}_1$.

Lemma 6.2. *Assume that Assumption (31) holds (thus (32)), that p is continuous, that ξ is coercive and that κ is bounded. Then, for every $\gamma \in [0, +\infty]$, P_γ continuously acts on both $\mathcal{C}_1$ and $\mathcal{C}_V$. For every $\gamma \in [0, +\infty]$, P_γ is compact from $\mathcal{C}_1$ into $\mathcal{C}_V$. For every $\gamma \in (0, +\infty]$, we have $P_\gamma(\mathcal{B}_V) \subset \mathcal{C}_{0,V}$.*

Proof. Let $\gamma \in [0, +\infty]$. From (34) it easily follows that

$$P_\gamma V \leq PV \leq (\delta + L)V \, , \tag{36}$$

so that P_γ continuously acts on $\mathcal{B}_V$. Let $f \in \mathcal{B}_V$. Then

$$\forall x \in \mathbb{R}, \quad (P_\gamma f)(x) = \int_{\mathbb{R}} \psi_\gamma(x, y) \, dy \tag{37}$$

$$\text{with} \quad \begin{cases} \psi_\gamma(x, y) := \kappa(y) e^{-\gamma\xi(y)} \, f(y) \, p(y - \alpha x) & \text{if} \quad 0 \leq \gamma < \infty \\ \psi_\gamma(x, y) := \kappa(y) \mathbf{1}_{\{\xi=0\}}(y) \, f(y) \, p(y - \alpha x) & \text{if} \quad \gamma = +\infty. \end{cases}$$

Let $A > 0$. We deduce from Assumption (31) and from a usual compactness argument ($[-A, A]$ is compact) that there exists a non-negative function $q \equiv q_A$ such that $y \mapsto V(y) \, q(y)$ is Lebesgue-integrable and

$$\forall v \in [-A, A], \ \forall y \in \mathbb{R}, \quad p(y + v) \leq q(y).$$

Thus we have for every $x \in [-A, A]$ and for every $y \in \mathbb{R}$

$$|\psi_\gamma(x,y)| \;\leq\; \|\kappa\|_\infty \|f\|_V \,(1+|y|)^{r_0}\, p(y-\alpha x) \leq \|\kappa\|_\infty \|f\|_V V(y)\, q(y). \tag{38}$$

Since $x \mapsto \psi_\gamma(x,y)$ is continuous for every $y \in \mathbb{R}$ from the continuity of p, we deduce from Lebesgue's theorem that the function $P_\gamma f$ is continuous on $\mathbb{R}$. We have proved that, if $\gamma \in [0,+\infty]$ and if $f \in \mathcal{B}_V$, then $P_\gamma f$ is continuous on $\mathbb{R}$. Thus P_γ continuously acts on $\mathcal{C}_1$.

Now, if $\gamma \in (0,+\infty]$, then

$$\forall x \in \mathbb{R}, \quad \frac{(P_\gamma f)(x)}{V(x)} = \int_{\mathbb{R}} \chi_\gamma(x,y)\, dy \qquad \text{with } \; \chi_\gamma(x,y) := \theta_\gamma(\alpha x+y)\, \frac{f(\alpha x+y)}{V(x)}\, p(y)$$

$$\text{where} \quad \theta_\gamma(\alpha x+y) := \begin{cases} \kappa(y) e^{-\gamma\xi(\alpha x+y)} & \text{if} \quad 0<\gamma<\infty \\ \kappa(y)\mathbf{1}_{\{\xi=0\}}(\alpha x+y) & \text{if} \quad \gamma=+\infty. \end{cases}$$

For every $(x,y) \in \mathbb{R}^2$, we obtain that

$$|\chi_\gamma(x,y)| \;\leq\; \theta_\gamma(\alpha x+y)\, \|f\|_V \left(\frac{1+|x|+|y|}{1+|x|}\right)^{r_0} p(y) \;\leq\; \|\kappa\|_\infty \|f\|_V \big(1+|y|\big)^{r_0} p(y).$$

Moreover $\lim_{|x|\to+\infty} \theta_\gamma(\alpha x+y) = 0$ since ξ is coercive. It follows again from Lebesgue's theorem that

$$\lim_{|x|\to+\infty} \frac{(P_\gamma f)(x)}{V(x)} = 0,$$

thus $P_\gamma f \in \mathcal{C}_{0,V}$. We have proved that, if $\gamma \in (0,+\infty]$, then $P_\gamma(\mathcal{B}_V) \subset \mathcal{C}_{0,V}$, thus the last assertion of Lemma 6.2 holds and P_γ continuously acts on $\mathcal{C}_V$. Finally, to prove the compactness property stated in Lemma 6.2, let $\gamma \in [0,+\infty]$ and consider P_γ as written in (37). Since p is continuous, the image by P_γ of the unit ball $\{f \in \mathcal{C}_1 : \|f\|_\infty \leq 1\}$ is equicontinuous from Scheffé's lemma. Then P_γ is compact from $\mathcal{C}_1$ into $\mathcal{C}_V$ from Ascoli's theorem and from $\lim_{|x|\to\infty} V(x) = +\infty$. □

The second inequality in (36) combined with the Jensen inequality (since $P(x,dy)$ is a probability measure) implies the following useful inequality

$$\forall \alpha \in (0,1), \quad \sup_{x\in\mathbb{R}} \frac{(PV^\alpha)(x)}{V^\alpha(x)} \leq (\delta+L)^\alpha\,. \tag{39}$$

Then the continuity of $\gamma \mapsto P_\gamma$ from $(0,+\infty]$ into $\mathcal{L}(\mathcal{C}_1,\mathcal{C}_V)$ required in Hypothesis 2.3* follows from the following.

Lemma 6.3. *Let $0 \leq a < a+b \leq 1$. Assume that $\xi \leq cV$ for some positive constant c. Then the following operator-norm inequality holds for every $(\gamma,\gamma') \in [0,+\infty)^2$*

$$\|P_\gamma - P_{\gamma'}\|_{\mathcal{C}_{V^a},\mathcal{C}_{V^{a+b}}} \;:=\; \sup_{f\in\mathcal{C}_{V^a},\|f\|_{V^a}\leq 1} \|P_\gamma f - P_{\gamma'} f\|_{V^{a+b}} \quad \leq \|\kappa\|_\infty (c|\gamma-\gamma'|)^b (\delta+L)^{a+b}\,.$$

Proof. Let $(\gamma,\gamma') \in [0,+\infty)^2$. For all $(u,v) \in [0,+\infty)^2$, we have $|e^{-u}-e^{-v}| \leq |e^{-u}-e^{-v}|^b \leq |u-v|^b$ from Taylor's inequality. Thus we obtain for any $f \in \mathcal{C}_{V^a}$

$$\begin{aligned} \big|(P_\gamma f)(x) - (P_{\gamma'} f)(x)\big| &\leq \|\kappa\|_\infty \|f\|_{V^a} \int_{\mathbb{R}} \big|e^{-\gamma\xi(y)} - e^{-\gamma'\xi(y)}\big| (V(y))^a p(y-\alpha x)\, dy \\ &\leq \|\kappa\|_\infty \|f\|_{V^a} (c\,|\gamma-\gamma'|)^b \int_{\mathbb{R}} (V(y))^{a+b}\, p(y-\alpha x)\, dy \\ &\leq \|\kappa\|_\infty \|f\|_{V^a} (c\,|\gamma-\gamma'|)^b PV^{a+b}(x), \end{aligned}$$

from which we deduce the desired inequality, due to (39). □

Lemma 6.4. *Assume that Assumption (31) holds (thus (32)) and that ξ is coercive. Then*

$$\|P_\gamma - P_\infty\|_{\mathcal{C}_1,\mathcal{C}_V} := \sup_{f\in\mathcal{C}_1, \|f\|_\infty\le 1} \|P_\gamma f - P_\infty f\|_V \ \longrightarrow 0 \quad \textit{when } \gamma\to+\infty.$$

Proof. Let $\varepsilon > 0$. Let $f \in \mathcal{C}_1$ be such that $\|f\|_\infty \le 1$. From $|P_\gamma f| \le P\mathbf{1}_\mathbb{R} = \mathbf{1}_\mathbb{R}$ it follows that there exists $A \equiv A(\varepsilon)$ such that :

$$|x| > A \ \Rightarrow\ \forall\gamma\in(0,+\infty),\ \frac{|(P_\gamma f)(x)|}{V(x)} \le \varepsilon. \tag{40}$$

Moreover, for any $\beta > 0$ and $x\in\mathbb{R}$ such that $|x| \le A$, we obtain that

$$\begin{aligned}\big|(P_\gamma f - P_\infty f)(x)\big| &\le \|\kappa\|_\infty e^{-\gamma\beta}\int_{[\xi>\beta]} p(y-\alpha x)\,dy + \|\kappa\|_\infty \int_{[0<\xi\le\beta]} p(y-\alpha x)\,dy \\ &\le \|\kappa\|_\infty\left(e^{-\gamma\beta} + \int_{[0<\xi\le\beta]} q(y)\,dy\right)\end{aligned}$$

where $q \equiv q_A$ is the function given in (38). Since q is Lebesgue-integrable on $\mathbb{R}$, we have $\int_{[0<\xi\le\beta]} q(y)\,dy \to 0$ when $\beta\to 0$, so that there exists $\beta_0 \equiv \beta_0(\varepsilon) > 0$ such that

$$\|\kappa\|_\infty \int_{[0<\xi\le\beta_0]} q(y)\,dy \le \frac{\varepsilon}{2}.$$

Finally let $\gamma_0 \equiv \gamma_0(\varepsilon) > 0$ be such that : $\forall\gamma>\gamma_0,\ \|\kappa\|_\infty e^{-\gamma\beta_0} \le \varepsilon/2$. Then

$$|x| \le A \ \Rightarrow\ \forall\gamma\in(\gamma_0,\infty),\ \frac{|(P_\gamma f - P_\infty f)(x)|}{V(x)} \le |(P_\gamma f)(x) - (P_\infty f)(x)| \le \varepsilon. \tag{41}$$

Inequalities (40) and (41) provide the desired statement. □

To study Conditions (10a) and (10b) of Hypothesis 2.3* with $J = (0,+\infty]$, $\mathcal{B}_0 = \mathcal{C}_1$, and $\mathcal{B}_1 = \mathcal{C}_V$, we use the duality arguments of [12, prop. 5.4]. The topological dual spaces of $\mathcal{C}_V$ and $\mathcal{C}_1$ are denoted by $(\mathcal{C}_V^*, \|\cdot\|_V)$ and $(\mathcal{C}_1^*, \|\cdot\|_\infty)$ respectively (for the sake of simplicity we use the same notation for the dual norms). For any $\gamma > 0$, we denote by P_γ^* the adjoint operator of P_γ on $\mathcal{C}_V$. Note that each P_γ^* is a contraction with respect to the dual norm $\|\cdot\|_\infty$ because so is P_γ on $\mathcal{C}_1$.

In the sequel, $\delta > |\alpha|^{r_0}$ is fixed, as well as the associated constant $L \equiv L(\delta)$ in (34).

Lemma 6.5. *Assume that (32) holds, that κ is bounded, and that ξ is coercive. Then, for every $\gamma > 0$ and for every $\beta > 0$, there exists a positive constant L_β such that*

$$P_\gamma V \le \|\kappa\|_\infty (e^{-\gamma\beta}\,\delta\, V + L_\beta\,\mathbf{1}_\mathbb{R}). \tag{42}$$

Moreover

$$P_\infty V \le \|\kappa\|_\infty \left(\sup_{[\xi=0]} V\right)\mathbf{1}_\mathbb{R}. \tag{43}$$

Proof. We have for every $\gamma > 0$ and for every $\beta > 0$

$$\begin{aligned} P_\gamma V &= P(\kappa e^{-\gamma\xi} V) = P\big(\kappa e^{-\gamma\xi}\mathbf{1}_{[\xi>\beta]}V\big) + P\big(\kappa e^{-\gamma\xi}\mathbf{1}_{[\xi\le\beta]}V\big) \\ &\le \|\kappa\|_\infty \left(e^{-\gamma\beta}\big(\delta\, V + L\,\mathbf{1}_{\mathbb{R}}\big) + \int_{[\xi\le\beta]} V(y)P(\cdot,dy)\right) \qquad \text{(from (34))} \\ &\le \|\kappa\|_\infty \left(e^{-\gamma\beta}\,\delta\, V + \big(L + \sup_{[\xi\le\beta]} V\big)\mathbf{1}_{\mathbb{R}}\right) \end{aligned}$$

from which we deduce the first desired statement. For P_∞, we have

$$P_\infty V = P(\kappa \mathbf{1}_{\{\xi=0\}}V) \le \left(\sup_{[\xi=0]} V\right) P(\kappa) \le \left(\sup_{[\xi=0]} V\right) \|\kappa\|_\infty \mathbf{1}_{\mathbb{R}}.$$

□

Corollary 6.6. *Assume that Assumption (32) holds true, that κ is bounded, and that ξ is coercive. Then, for every $\gamma_1 > 0$ and for every $\varepsilon > 0$, there exists a constant $D > 0$ such that*

$$\forall \gamma \in [\gamma_1, +\infty],\ \forall f^* \in \mathcal{C}_V^*, \quad \|P_\gamma^* f^*\|_V \le \varepsilon\, \|f^*\|_V + D\, \|f^*\|_\infty. \tag{44}$$

Moreover, for every $\gamma \in (0, +\infty]$, the essential spectral radius $r_{ess}((P_{\gamma|\mathcal{C}_V})^*)$ is zero.

Proof. Choose $\beta = \beta(\gamma_1, \varepsilon) > 0$ such that $\|\kappa\|_\infty e^{-\gamma_1\beta}\,\delta < \varepsilon$. Then we deduce from Lemma 6.5 that $P_{\gamma_1} V \le \varepsilon\, V + D\,\mathbf{1}_{\mathbb{R}}$, where $D \equiv D(L, \gamma_1, \varepsilon)$ is a positive constant. Now let $\gamma \in [\gamma_1, +\infty]$. Since $P_\gamma V \le P_{\gamma_1} V$, we also have $P_\gamma V \le \varepsilon\, V + D\,\mathbf{1}_{\mathbb{R}}$. This inequality easily rewrites as (44) (see the proof in [7, p. 190]). Finally, since P_γ^* is compact from $\mathcal{C}_V^*$ into $\mathcal{C}_1^*$ (Lemma 6.2), we deduce from [10] that $r_{ess}(P_\gamma^*) \le \varepsilon$. We obtain $r_{ess}(P_\gamma^*) = 0$ because ε is arbitrary. □

Remark 6.7. *Let $\gamma_1 > 0$, $\varepsilon > 0$ and $0 \le a \le a+b \le 1$. Observe that Corollary 6.6 holds also if we replace V by V^{a+b} (since ϑ_1 admits a moment of order $r_0(a+b)$). Moreover notice that (44) with V^{a+b} instead of V directly gives that there exists a constant $D_{\varepsilon,a+b} > 0$ such that*

$$\forall \gamma \in [\gamma_1, +\infty],\ \forall f^* \in \mathcal{C}_{V^{a+b}}^*, \quad \|P_\gamma^* f^*\|_{V^{a+b}} \le \varepsilon\, \|f^*\|_{V^{a+b}} + D_{\varepsilon,a+b}\, \|f^*\|_{V^a} \tag{45}$$

since $\|f^*\|_\infty \le \|f^*\|_{V^a}$.

The previous statements ensure that Hypothesis 2.3* holds with $J = (0, +\infty]$, $\mathcal{B}_0 = \mathcal{C}_1$, and $\mathcal{B}_1 = \mathcal{C}_V$. The use of Theorems 2.4 and 2.5 also requires, first to study the function $\gamma \mapsto r(\gamma) = r(P_{\gamma|\mathcal{C}_V})$, in particular the limit of $r(\gamma)$ when γ tends to 0 and $+\infty$, second to check Hypothesis 2.1 on the space $\mathcal{C}_V$. This is the purpose of the next subsections.

6.2. **Preliminary useful statements on $r(\gamma)$.** For every $\gamma > 0$ we set $r(\gamma) = r(P_{\gamma|\mathcal{C}_V})$.

Proposition 6.8. *Assume that Assumption (32) holds true, that κ is bounded, that ξ is coercive and finally that the function ξ/V is bounded on $\mathbb{R}$. Then $\lim_{\gamma\to 0_+} r(\gamma) \ge 2$.*

Proof. We need the following lemma concerning the special case $\kappa \equiv 2$. To avoid confusion we write below $\widetilde{P}_\gamma$ and $\widetilde{r}(\gamma)$ in place of P_γ and $r(\gamma)$ when $\kappa \equiv 2$.

Lemma 6.9. *Assume that (32) holds, that $\kappa \equiv 2$, and that ξ is coercive. Then $\widetilde{P}_0$ continuously acts on $\mathcal{C}_V$. Moreover the function $\gamma \mapsto \widetilde{r}(\gamma)$ is continuous at $\gamma = 0$, with $\widetilde{r}(0) = 2$.*

Proposition 6.8 follows from Lemma 6.9. Indeed, from $P_\gamma^n \geq \widetilde{P}_\gamma^n$ (since $\kappa \geq 2$), we deduce that $r(\gamma) = r(P_\gamma) \geq r(\widetilde{P}_\gamma) = \widetilde{r}(\gamma)$. It then follows from Proposition 3.9 that $\lim_{\gamma \to 0_+} r(\gamma) \geq \lim_{\gamma \to 0_+} \widetilde{r}(\gamma) = \widetilde{r}(0) = 2$. □

Proof of Lemma 6.9. Note that $\widetilde{P}_0 = 2P$. The fact that P, thus $\widetilde{P}_0$, continuously acts on $\mathcal{C}_V$ follows from Lemma 6.2 applied with $\kappa \equiv 2$ and $\gamma = 0$. Moreover iterating Inequality (34) proves that P is power-bounded on $\mathcal{C}_V$ (i.e. $\sup_{n \geq 1} \|P^n V\|_V < \infty$), thus $r(P) = 1$ since P is Markov. Moreover (34) rewrites as the following (dual) Doeblin-Fortet inequality (see the proof in [7, p. 190]):

$$\forall f^* \in \mathcal{C}_V^*, \quad \|P^* f^*\|_V \leq \delta \, \|f^*\|_V + L \, \|f^*\|_\infty. \tag{46}$$

Since P is compact from $\mathcal{C}_1$ into $\mathcal{C}_V$ (apply the same argument as in Lemma 6.2), so is P^* from $\mathcal{C}_V^*$ into $\mathcal{C}_1^*$. Then we deduce from [10] and by duality that, under Assumption (32), P is a quasi-compact operator on $\mathcal{C}_V$ and that its essential spectral radius $r_{ess}(P)$ satisfies the following bound (see also [24, Sect. 8]): $r_{ess}(P) \leq \delta$. It follows that

$$\widetilde{r}(0) = r(\widetilde{P}_0) = 2 \quad \text{and} \quad r_{ess}(\widetilde{P}_0) \leq 2\delta. \tag{47}$$

Observe that (34) and Inequality $\widetilde{P}_\gamma V \leq 2PV$ give $\widetilde{P}_\gamma V \leq 2\delta \, V + 2L \, \mathbf{1}_{\mathbb{R}}$, which rewrites as the following Doeblin-Fortet inequality:

$$\forall \gamma \in [0, +\infty), \ \forall f^* \in \mathcal{C}_V^*, \quad \|\widetilde{P}_\gamma^* f^*\|_V \leq 2\delta \, \|f^*\|_V + 2L \, \|f^*\|_\infty. \tag{48}$$

Using (47), (48) and Lemma 6.3 (with $a = 0$ et $b = 1$), it follows from Theorem 4.1 (applied with $\delta_0 = 2\delta$) that $\gamma \mapsto \widetilde{r}(\gamma)$ is continuous at $\gamma = 0$ since $\widetilde{r}(0) = 2 > 2\delta$. □

Proposition 6.10. *Assume that Assumption (31) holds (thus (32)), that p is continuous, that ξ is coercive, that κ is bounded, and finally that* $\mathrm{Leb}(\xi = 0) = 0$. *Then* $\lim\limits_{\gamma \to +\infty} r(\gamma) = 0$.

Proof. We apply Theorem 4.1 to the family $(P_\gamma)_{\gamma \in (0, +\infty]}$ at the neighborhood of $\gamma = +\infty$. Observe that $\mathrm{Leb}(\xi = 0) = 0$ implies that $P_\infty = 0$, in particular the spectral radius $r(\infty)$ of P_∞ is zero. From Lemma 6.4 and Corollary 6.6, Theorem 4.1 applies to the above family, and the Conclusion (20) and $r(\infty) = 0$ then give $\limsup_{\gamma \to +\infty} r(\gamma) \leq \varepsilon$ for every $\varepsilon > 0$ (given by (44)). Since moreover $r(\cdot)$ is non-increasing, we obtain $\lim_{\gamma \to +\infty} r(\gamma) = 0$. □

6.3. **Study of Hypothesis 2.1.** In this subsection we prove that Hypothesis 2.1 holds with respect to $(J_1, \mathcal{B}_1)$ with $\mathcal{B}_1 = \mathcal{C}_V$ and $J_1 := (0, \theta_1)$, where

$$\theta_1 := \sup\{\gamma > 0 \, : \, r(\gamma) > 0\}. \tag{49}$$

We know from Proposition 6.8 that $\theta_1 \in (0, +\infty]$. Note that $\mathcal{C}_V$ is a Banach lattice. Let us use Proposition 3.5 to prove Hypothesis 2.1.

Observe that, for any $\gamma \in (0, \theta_1)$, P_γ is quasi-compact on $\mathcal{C}_V$ from Corollary 6.6 since $r(\gamma) > 0$. The other conditions of Proposition 3.5 follow from Remark 6.12 and Lemmas 6.13-6.14 below. First we state the following.

Lemma 6.11. *For any non-null $e^* \in \mathcal{C}_V^*$, $e^* \geq 0$, there exists a nonnegative Borel measure $\mathfrak{m} \equiv \mathfrak{m}_{e^*}$ on $\mathbb{R}$ such that*

$$\forall f \in \mathcal{C}_V, \quad e^*(f) = \mathfrak{m}\left(\frac{f}{V} - \ell_V(f) \, \mathbf{1}_{\mathbb{R}}\right) + e^*(V) \, \ell_V(f). \tag{50}$$

Remark 6.12. *Since $\mathcal{B} = \mathcal{C}_V$ is a Banach lattice, $\mathcal{C}_V$ satisfies Hypothesis 3.2. Moreover, due to Lemma 6.11, Hypothesis 3.3 is fulfilled with J_1 and $\mathcal{B} = \mathcal{C}_V$. Indeed, let $\gamma \in J_1$ and let $\phi \in \mathcal{C}_V$ be non-null and non-negative. Then, we have $P_\gamma\phi > 0$ everywhere from the definition of P and the strict positivity of the function $p(\cdot)$. Now prove that, if $\psi \in \mathcal{B}^* \cap \mathrm{Ker}(P_\gamma^* - r(\gamma)I)$ is non-null and non-negative, then $\psi(P_\gamma\phi) > 0$, so ψ is positive. Let $\gamma > 0$. First observe that $\psi \neq c\,\ell_V$ for every $c \in \mathbb{C}$ because $r(\gamma) > 0$ and $P_\gamma^*(\ell_V) = 0$ from Lemma 6.2. Second note that $\mathfrak{m} = 0$ in (50) implies that $e^* = e^*(V)\,\ell_V$. Thus the nonnegative measure $\mathfrak{m} \equiv \mathfrak{m}_\psi$ associated with ψ in (50) is non-null. Since $\ell_V(P_\gamma\phi) = 0$ from Lemma 6.2, we deduce from (50) (applied with $e^* = \psi$) and from $P_\gamma\phi > 0$ that $\psi(P_\gamma\phi) = \mathfrak{m}(P_\gamma\phi/V) > 0$.*

Proof of Lemma 6.11. Let $(\mathcal{C}, \|\cdot\|)$ denote the subspace of $\mathcal{C}_1$ defined as follows

$$\mathcal{C} := \left\{ g \in \mathcal{C}_1 \ : \ \lim_{|x|\to\infty} g(x) \text{ exists in } \mathbb{C} \right\},$$

with the notation $\lim_{|x|\to\infty}$ having the same meaning as in the definition of $\mathcal{C}_V$. For every $g \in \mathcal{C}$, we set: $\ell(g) := \lim_{|x|\to\infty} g(x)$. We denote by $\mathcal{C}^*$ the topological dual space of $\mathcal{C}$. Let $e^* \in \mathcal{C}_V^*$, $e^* \geq 0$, and let $\widetilde{e}^* \in \mathcal{C}^*$ be defined by:

$$\forall g \in \mathcal{C}, \quad \widetilde{e}^*(g) := e^*(gV).$$

Next let $\widetilde{e}_0^*$ be the restriction of $\widetilde{e}^*$ to $\mathcal{C}_0 := \{g \in \mathcal{C} : \ell(g) = 0\}$. From the Riesz representation theorem, there exists a unique positive Borel measure $\mathfrak{m}$ on $\mathbb{R}$ such that

$$\forall g \in \mathcal{C}_0, \quad \widetilde{e}_0^*(g) = \mathfrak{m}(g) := \int_{\mathbb{R}} g\, d\mathfrak{m}.$$

Then, writing $g = (g - \ell(g)\,\mathbf{1}_{\mathbb{R}}) + \ell(g)\,\mathbf{1}_{\mathbb{R}}$ for any $g \in \mathcal{C}$, we obtain that

$$\widetilde{e}^*(g) = \mathfrak{m}\big(g - \ell(g)\,\mathbf{1}_{\mathbb{R}}\big) + \widetilde{e}^*(\mathbf{1}_{\mathbb{R}})\,\ell(g).$$

We conclude by observing that, for any $f \in \mathcal{C}_V$, we have $e^*(f) = \widetilde{e}^*(f/V)$. □

Lemma 6.13. *If $\gamma \in J_1$ (i.e. $r(\gamma) > 0$) and if $f, g \in \mathcal{C}_V$ are such that $P_\gamma f = r(\gamma)f$ and $P_\gamma g = r(\gamma)g$ with $f > 0$, then $g \in \mathbb{C} \cdot f$.*

Proof. Let $f, g \in \mathcal{C}_V \cap \mathrm{Ker}(P_\gamma - r(\gamma)I)$ with $f > 0$. Let $\beta \in \mathbb{C}$ be such that $h := g - \beta f$ vanishes at 0. Since $h \in \mathrm{Ker}(P_\gamma - r(\gamma)I)$, we deduce from a classical result for positive operators acting on a Banach lattice that $P_\gamma|h| = r(\gamma)|h|$ (see Proposition A.1). Then $|h|(0) = 0$, the positivity of $p(\cdot)$ and finally the continuity of $|h|$ show that $h = 0$. □

Lemma 6.14. *Let $h \in \mathcal{C}_V$ with $|h| > 0$ and $\lambda \in \mathbb{C}$ be such that $|\lambda| = 1$ and $P\frac{h}{|h|} = \lambda\frac{h}{|h|}$ in $\mathbb{L}^1(\pi)$. Then $\lambda = 1$.*

Proof. Observe that $\frac{h}{|h|}$ is in $\mathcal{C}_1$ so in $\mathcal{B}_V$. But it is known from [21] that $(X_n)_n$ is V-geometrically ergodic, so λ=1. □

6.4. **Proof of Theorem 6.1.** To prove Assertion $a)$ of Theorem 6.1, we apply Theorem 2.4. Let γ_1 be such that $0 < \gamma_1 < \theta_1$, with θ_1 is given in (49). Let $\varepsilon \in (0,1)$. Then, from the results of the previous subsections, the assumptions of Theorem 2.4 hold with $J = (\gamma_1, \theta_1)$, $\mathcal{B}_0 = \mathcal{C}_1$, $\mathcal{B}_3 = \mathcal{C}_V$, $\delta_0 = \varepsilon$, thus with $J_0 := \{\gamma \in J : r(\gamma) > \varepsilon\}$. Note that the space $\mathcal{B}$ of Theorem 2.4 is here $\mathcal{B} = \mathcal{B}_3 = \mathcal{C}_V$. A first consequence is that $\theta_1 = +\infty$. Indeed, for every $\gamma > 0$, P_γ continuously acts on $\mathcal{C}_V$ from Lemma 6.2. Moreover, since κ is bounded, the map

$f \mapsto \pi(\kappa e^{-\gamma\xi} f)$ is in $\mathcal{B}^* = (\mathcal{C}_V)^*$ and Condition (19) holds from Remark 3.15. Consequently the hypotheses of Lemma 3.14 are fulfilled with γ_1 as stated above (i.e. $0 < \gamma_1 < \theta_1$), with $\gamma_2 = \theta_1$, and for every $\gamma_3 > \theta_1$ (if $\theta_1 < \infty$). But Lemma 3.14 then ensures that $r(\gamma) > 0$ for every $\gamma \in (\gamma_1, \gamma_3)$, thus for every $\gamma \in (0, +\infty)$ since γ_3 is arbitrary large. This proves that $\theta_1 = +\infty$.

Now we prove the assumptions of Assertion (i) of Theorem 2.4. Recall that Hypotheses 3.2 and 3.3 hold (see Remark 6.12), and note that the additional condition in Assertion (*ii*) of Proposition 3.8 clearly holds from the form of P (see (33)) and from the positivity of the density p. Thus (14) is satisfied due to Proposition 3.8. Next let us prove that the continuity assumptions of Assertion (i) of Theorem 2.4 hold, namely that $\gamma \mapsto Ph_{\kappa,\gamma}$ and $\gamma \mapsto \mu(\kappa e^{-\gamma\xi})$ are continuous from J_0 to $\mathcal{C}_1$ and to $\mathcal{C}_V^*$ respectively. For every $\gamma, \gamma' > \gamma_1$ and every $\beta > 0$, we have

$$\|Ph_{\kappa,\gamma} - Ph_{\kappa,\gamma'}\|_\infty \le \|(\kappa - 1)(e^{-\gamma\xi} - e^{-\gamma'\xi})\|_\infty \le \|\kappa - 1\|_\infty \big(2\, e^{-\gamma_1\beta} + \beta|\gamma - \gamma'|\big)\,,$$

by using the sets $[\xi \ge \beta]$, $[\xi < \beta]$ and the Taylor inequality. Next let $\varepsilon > 0$ and choose β such that $2\,\|\kappa - 1\|_\infty e^{-\gamma_1\beta} < \varepsilon/2$. Then $\|Ph_{\kappa,\gamma} - Ph_{\kappa,\gamma'}\|_\infty < \varepsilon$ provided that $|\gamma - \gamma'| < \varepsilon/(2\beta\|\kappa - 1\|_\infty)$. Moreover, for every $\gamma \in J_0$, the map $\mu_\gamma : f \mapsto \mu(\kappa e^{-\gamma\xi} f)$ is in $(\mathcal{C}_V)^*$ from $\mu(V) < \infty$, and we have for every $\gamma, \gamma' \in J_0$ and for every $f \in \mathcal{C}_V$

$$\big|\mu_\gamma(f) - \mu_{\gamma'}(f)\big| \le \|\kappa\|_\infty \|f\|_V \mu\left(\big|e^{-\gamma\xi} - e^{-\gamma'\xi}\big|V\right)$$

so that the norm of $(\mu_\gamma - \mu_{\gamma'})$ in $(\mathcal{C}_V)^*$ is less than $\|\kappa\|_\infty\, \mu(|e^{-\gamma\xi} - e^{-\gamma'\xi}|V)$ which converges to 0 as $\gamma' \to \gamma$ from Lebesgue's theorem. This proves the assumptions of Assertion (i) of Theorem 2.4. Since γ_1 and ε are arbitrarily small, we deduce from Theorem 2.4 that, under $\mathbb{P}_\mu$, $(S_n, \kappa(X_n))_n$ is multiplicatively ergodic on $(0, +\infty)$ with $\rho(\gamma) = r(\gamma) > 0$ on $(0, +\infty)$. We have proved Assertion *a*) of Theorem 6.1.

For Assertion *b*), observe that $\lim_{\gamma \to +\infty} r(\gamma) = 0$ from Proposition 6.10. Consequently, from the continuity of $r(\cdot)$, ν is finite and satisfies (15), and so (2), with respect to $\mathbb{P}_\mu$, provided that μ is a probability distribution belonging to $\mathcal{C}_V^*$.

Finally, to prove Part *c)* of Theorem 6.1, we assume now that $\xi \in \mathcal{B}_{V^{\frac{1}{1+\tau}}}$ for some $\tau > 0$ and that $[\xi = 0]$ has Lebesgue measure 0, and we apply Theorem 2.5. Consider any

$$0 < a_0 < a_1 < a_1 + \frac{1}{1+\tau} < a_2 < a_3 = 1.$$

Let us prove that the additional assumptions of Theorem 2.5 hold true with $\mathcal{B}_i := \mathcal{C}_{V^{a_i}}$ for $i \in \{0, 1, 2, 3\}$. Let $i \in \{0, 1, 2\}$. The fact that $(P_\gamma)_\gamma$ satisfies the conditions of Hypothesis 2.3* on $(J, \mathcal{B}_i, \mathcal{B}_{i+1})$ comes from Lemma 6.3 and Remark 6.7. The fact that Hypothesis 2.1 is satisfied on $\mathcal{B}_{i+1}$ follows from Proposition 3.5: apply the results of Subsection 6.3 with $V^{a_{i+1}}$ in place of V. Observe that

$$\|\xi f\|_{\mathcal{B}_2} = \sup_{x\in\mathbb{R}} \frac{|\xi(x) f(x)|}{(V(x))^{a_2}} \le \sup_{x\in\mathbb{R}} \frac{|\xi(x)|}{(V(x))^{\frac{1}{1+\tau}}} \sup_{x\in\mathbb{R}} \frac{|f(x)|}{(V(x))^{a_1}} \le \|f\|_{\mathcal{B}_1} \sup_{x\in\mathbb{R}} \frac{|\xi(x)|}{(V(x))^{\frac{1}{1+\tau}}}.$$

Hence we have proved that $f \mapsto \xi f$ is in $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$. The fact that $\gamma \mapsto P_\gamma$ is C^1 from $(0, +\infty)$ to $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ and that $P'_\gamma := P_\gamma(-\xi f)$ can be established by adapting the operator-norm estimations used in the proof of Assertion (vii) of Lemma 5.2 to the present context of weighted-supremum spaces (use the techniques of [14, Lemma 10.4]). Finally, due to Proposition 3.12, we have $r'(\nu) < 0$. We conclude by Theorem 2.5.

Appendix A. Proof of Proposition 3.5

Proposition 3.5 directly follows from the following statement.

Proposition A.1. *Let $\mathcal{B}$ be a non null complex Banach lattice of functions $f : \mathbb{X} \to \mathbb{C}$ (or of classes of such functions modulo π). Let Q be a (nonnull) nonnegative quasicompact operator on $\mathcal{B}$ such that $r(Q) \neq 0$ and such that for every nonnull nonnegative $f \in \mathcal{B}$ and for every nonnull nonnegative $\psi \in \mathcal{B}^* \cap \mathrm{Ker}(Q^* - r(Q)I)$, we have $Qf > 0$ (modulo π) and $\psi(Qf) > 0$. Then*

(a) $r(Q)$ is a first order pole of Q, and there exists a positive $\phi \in \mathcal{B}$ and a positive $\psi \in \mathcal{B}^*$ such that

$$\psi(\phi) = 1, \qquad Q\phi = r(Q)\phi \qquad \text{and} \qquad Q^*\psi = r(Q)\psi. \tag{51}$$

(b) Let $\lambda \in \mathbb{C}$ and $h \in \mathcal{B}$ such that $|\lambda| = r(Q)$ and $Qh = \lambda h$. Then $Q|h| = r(Q)|h|$ in $\mathcal{B}$.

(c) If moreover Q is of the form $Q = P(\kappa e^{-\gamma \xi} \cdot)$ where P is the operator associated with a Markov kernel, if $\mathbf{1}_{\mathbb{X}} \in \mathcal{B} \hookrightarrow \mathbb{L}^1(\pi)$, if $\mathrm{Ker}(Q - r(Q)I) = \mathbb{C} \cdot \phi$ and if 1 is the only complex number λ of modulus 1 such that $P(h/|h|) = \lambda h/|h|$ in $\mathbb{L}^1(\pi)$ for some $h \in \mathcal{B}$ with $|h| > 0$, then $r(Q)$ is the only eigenvalue of modulus $r(Q)$ of Q.

Proof. The fact that $r(Q)$ is a finite pole of Q is classical for a nonnegative quasi-compact operator Q on a Banach lattice. Let us just remember the main arguments. From quasi-compactness we know that there exists a finite pole $\lambda \in \sigma(Q)$ such that $|\lambda| = r(Q)$. Thus, setting $\lambda_n := \lambda(1 + 1/n)$ for any $n \geq 1$, we deduce from $\lambda \in \sigma(Q)$ that

$$\lim_{n \to +\infty} \|(\lambda_n I - Q)^{-1}\|_{\mathcal{B}} = +\infty.$$

Since $\mathcal{B}$ is a Banach lattice, we deduce from the Banach-Steinhaus theorem that there exists a nonnegative and nonnull element $f \in \mathcal{B}$ such that

$$\lim_{n \to +\infty} \|(\lambda_n I - Q)^{-1} f\|_{\mathcal{B}} = +\infty.$$

Next define $r_n := r(Q)(1 + 1/n)$ and observe that

$$\big|(\lambda_n I - Q)^{-1} f\big| = \big|\sum_{k \geq 0} \lambda_n^{-(k+1)}\, Q^k f\big| \leq \sum_{k \geq 0} r_n^{-(k+1)}\, Q^k f.$$

Since $\mathcal{B}$ is a Banach lattice, the last inequality is true in norm, that is

$$\big\|(\lambda_n I - Q)^{-1} f\big\| \leq \big\|\sum_{k \geq 0} r_n^{-(k+1)}\, Q^k f\big\|$$

from which we deduce that $\lim_{n \to +\infty} \|(r_n I - Q)^{-1}\|_{\mathcal{B}} = +\infty$, thus $r(Q) \in \sigma(Q)$. Finally $r(Q)$ is a finite pole of Q from quasi-compactness.

Let q denote the order of the pole $r(Q)$, namely $r(Q)$ is a pole of order q of the resolvent function $z \mapsto (zI - Q)^{-1}$. Then there exists $\rho > 0$ such that $(zI - Q)^{-1}$ admits the following Laurent series provided that $|z - r(Q)| < \rho$ and $z \neq r(Q)$:

$$(zI - Q)^{-1} = \sum_{k=-q}^{+\infty} (z - r(Q))^k A_k,$$

where A_k are bounded linear operators on $\mathcal{B}$. By quasi-compactness, A_{-1} is a projection onto the finite subspace $\mathrm{Ker}(Q - r(Q)I)^q$. Moreover we know that

$$A_{-q} = \big(Q - r(Q)I\big)^{q-1} \circ A_{-1} = A_{-1} \circ \big(Q - r(Q)I\big)^{q-1}, \tag{52}$$

and that, setting $r_n := r(Q)(1 + 1/n)$,

$$\begin{aligned} A_{-q} &= \lim_{n \to +\infty} \big(r_n - r(Q)\big)^q \big(r_n I - Q\big)^{-1} \\ &= \lim_{n \to +\infty} \big(r_n - r(Q)\big)^q \sum_{k \geq 0} r_n^{-(k+1)}\, Q^k. \end{aligned} \tag{53}$$

Since Q is a nonnull nonnegative operator on $\mathcal{B}$, so is A_{-q}. Since $A_{-q} \neq 0$, we take a nonnegative $h_0 \in \mathcal{B}$ such that $\phi := A_{-q} h_0 \neq 0$ in $\mathcal{B}$. We have $(Q - r(Q)I)A_{-q} = 0$, so $r(Q)\phi = Q\phi$. Similarly there exists a nonnegative $\psi_0 \in \mathcal{B}^*$ such that $\psi_1 := A_{-q}^* \psi_0$ is a nonzero and nonnegative element of $\mathrm{Ker}(Q^* - r(Q)I)$, where A_{-q}^* is the adjoint operator of A_{-q}. Note that $\psi_1(\phi) = \psi_1(Q\phi)/r(Q) > 0$ from our hypotheses, so that ϕ and $\psi := \psi_1/\psi_1(\phi)$ satisfy (51). To conclude the proof of Assertion (a), let us prove by reductio ad absurdum that $q = 1$. Assume that $q \geq 2$. Then $A_{-q}^2 = 0$ from (52) and $A_{-1}(\mathcal{B}) = \mathrm{Ker}(Q - r(Q)I)^q$, so that $\psi_1(\phi) = (A_{-q}^* \psi_0)(A_{-q} h_0) = \psi_0(A_{-q}^2 h_0) = 0$. This contradicts the above fact.

To prove (b), recall that, from our hypotheses, we have $\psi(g) = \psi(Qg)/r(Q) > 0$ for every nonnull nonnegative $g \in \mathcal{B}$. Let $\lambda \in \mathbb{C}$ and $h \in \mathcal{B}$ such that $|\lambda| = r(Q)$ and $Qh = \lambda h$. The positivity of Q gives $|\lambda h| = r(Q)|h| = |Qh| \leq Q|h|$, thus $g_0 := Q|h| - r(Q)|h| \geq 0$. From $\psi(g_0) = 0$, it follows that $g_0 = 0$, that is: $Q|h| = r(Q)|h|$ in $\mathcal{B}$.

Now let us prove Assertion (c) of Proposition A.1. Recall that the above nonnull nonnegative function $\phi \in \mathcal{B}$ is such that $Q\phi = r(Q)\phi$. From our hypotheses we deduce that $\phi > 0$ (modulo π). Let $\lambda \in \mathbb{C}$ and $h \in \mathcal{B}$ be such that $|\lambda| = r(Q)$, $h \neq 0$ and $Qh = \lambda h$. Due to the previous point and to our assumptions, we obtain that $Q|h| = r(Q)\,|h|$ and $|h| = \beta\phi$ for some $\beta > 0$. In particular $h \neq 0$ $\pi-$ a.s.. One may assume that $\beta = 1$ for the sake of simplicity. Let $A = \{x \in \mathbb{X} : |h(x)| = \phi(x) > 0\}$ and

$$B = \{x \in \mathbb{X} : (Q\phi)(x) = r(Q)\phi(x)\}, \quad C = \{x \in \mathbb{X} : (Qh)(x) = \lambda h(x)\}.$$

Let $A^c = \mathbb{X} \setminus A$. It follows from $\pi(A^c) = 0$ and from the invariance of π that

$$\pi\Big(P\big(\mathbf{1}_{A^c}(\phi + |h|)\,\kappa e^{-\gamma\xi}\big)\Big) = \pi\Big(\mathbf{1}_{A^c}\big(\phi + |h|\big)\,\kappa e^{-\gamma\xi}\Big) = 0.$$

Let $D := \{x \in \mathbb{X} : \big(P(\mathbf{1}_{A^c}\,(\phi + |h|)\,\kappa e^{-\gamma\xi})\big)(x) = 0\}$. Then we have $\pi(D) = 1$. Now define $E = A \cap B \cap C \cap D$. Then $\pi(E) = 1$, and we obtain that

$$\forall x \in E, \quad |h(x)| = \phi(x) > 0 \tag{54a}$$

$$\forall x \in E, \quad \lambda\, h(x) = \big(P(\mathbf{1}_A h\, \kappa e^{-\gamma\xi})\big)(x) = \int_A h(y)\,\kappa(y) e^{-\gamma\xi(y)}\, P(x, dy) \tag{54b}$$

$$\forall x \in E, \quad r(Q)\,\phi(x) = \big(P(\mathbf{1}_A \phi\, \kappa e^{-\gamma\xi})\big)(x) = \int_A \phi(y)\,\kappa(y) e^{-\gamma\xi(y)}\, P(x, dy). \tag{54c}$$

Let $x \in E$ and define the probability measure: $\eta_x(dy) := (r(Q)\,\phi(x))^{-1}\phi(y)\,\kappa(y) e^{-\gamma\xi(y)}\, P(x, dy)$. We have

$$\int_A \frac{r(Q)\,\phi(x)\,h(y)}{\lambda\,\phi(y)\,h(x)}\ \eta_x(dy) = 1.$$

Since $|h(x)| = \phi(x)$ and $|h| = \phi$ on A, the previous integrand is of modulus one. Then a standard convexity argument ensures that the following equality holds for $P(x,\cdot)-$almost every $y \in \mathbb{X}$:

$$r(Q)\,\phi(x)\,h(y) = \lambda\,\phi(y)\,h(x).$$

This implies that $r(Q)P\frac{h}{|h|} = \lambda\frac{h}{|h|}$ everywhere on E, thus $r(Q)P\frac{h}{|h|} = \lambda\frac{h}{|h|}$ in $\mathbb{L}^1(\pi)$. So $\lambda = r(Q)$ from the hypothesis of Assertion (c) of Proposition A.1. $\square$

Appendix B. A counter-example

Assume that $(\mathbb{X}, d)$ is a metric space equipped with its Borel σ-algebra. Let $\mathcal{L}^\infty$ denote the set of bounded functions $f : \mathbb{X} \to \mathbb{C}$, endowed with the supremum norm.

Proposition B.1. *Assume that P is a Markov kernel satisfying the following condition : there exists $S \in (0,+\infty)$ such that, for every $x \in \mathbb{X}$, the support of $P(x,dy)$ is contained in the ball $B(x,S)$ centered at x with radius S. Assume that $\kappa \equiv 2$ and that $\xi(y) \to 0$ when $d(y,x_0) \to +\infty$, where x_0 is some fixed point in $\mathbb{X}$. Then, for every $\gamma \in [0,+\infty)$, the kernel $P_\gamma := 2P(e^{-\gamma\xi}\cdot)$ continuously acts on $\mathcal{L}^\infty$ and its spectral radius $r(\gamma) = r((P_\gamma)_{|\mathcal{L}^\infty})$ satisfies the following*

$$\forall\gamma \in [0,+\infty), \quad r(\gamma) = 2.$$

Proof. We clearly have $r(\gamma) \le 2$ since $P_\gamma \le 2P$ and P is Markov. For any $\beta > 0$, we obtain with $f = \mathbf{1}_{[\xi\le\beta]}$

$$\forall x \in \mathbb{X}, \quad (P_\gamma f)(x) = 2\int_{[\xi\le\beta]} e^{-\gamma\xi(y)}\,P(x,dy) \ge 2e^{-\gamma\beta}\,P\big(x,[\xi\le\beta]\big).$$

The set $[\xi \le \beta]$ contains $\mathbb{X}\setminus B(x_0,R)$ for some $R>0$ since $\xi(y)\to 0$ when $d(y,x_0)\to+\infty$. Thus, for $d(x,x_0)$ sufficiently large ($d(x,x_0) > R+S$), we have $P\big(x,[\xi\le\beta]\big) = 1$, so that $\|P_\gamma\|_{\mathcal{L}^\infty} \ge \|P_\gamma f\|_{\mathcal{L}^\infty} \ge 2e^{-\gamma\beta}$. This gives $\|P_\gamma\|_{\mathcal{L}^\infty} = 2$ when $\beta\to 0$. Similarly we obtain with $f = \mathbf{1}_{[\xi\le\beta]}$, that, $\forall x \in \mathbb{X}\setminus B(x_0,R+2S)$,

$$\begin{aligned}(P_\gamma^2 f)(x) &= 4\int e^{-\gamma(\xi(y)+\xi(z))}\,\mathbf{1}_{[\xi\le\beta]}(z)\,P(y,dz)\,P(x,dy)\\ &\ge 4e^{-\gamma\beta}\int_{\mathbb{X}\setminus B(x_0,R+S)} e^{-\gamma\xi(y)}\,P\big(y,[\xi\le\beta]\big)P(x,dy) \ge 4e^{-2\gamma\beta}\end{aligned}$$

and so $\forall\beta>0, \quad \|P_\gamma^2\|_{\mathcal{L}^\infty} \ge \|P_\gamma^2 f\|_{\mathcal{L}^\infty} \ge 4e^{-2\gamma\beta}$. Again this provides $\|P_\gamma^2\|_{\mathcal{L}^\infty} = 4$ since β can be taken arbitrarily small. Similarly we can prove that $\|P_\gamma^n\|_{\mathcal{L}^\infty} = 2^n$ for every $n \ge 1$, thus $r(\gamma) = 2$. $\square$

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