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Abstract

We consider a Markov kernel on a measurable space, satisfying a minorization condition and a modulated drift condition. Then we show that there exists a solution to the so-called Poisson equation whose norm can be bounded from above using the modulated drift condition. This new bound is very simple and can be easily computed. This result is obtained using the submarkov residual kernel given by the minorization condition. Such a bound allows us to provide new control on the weighted total variation norms of the deviation between the invariant probability measure π_{θ_0} of a Markov kernel P_{θ_0} and the invariant probability measure π_θ of some perturbation P_θ of P_{θ_0}. From the standard connexion between Poisson's equation and the central limit theorem, a simple and computable bound on the asymptotic variance is also derived.

AMS subject classification :

Keywords : Poisson's equation; Drift conditions; Invariant probability measure; Perturbed Markov kernels; Asymptotic variance

Basic definitions used throughout the paper. Let $(\mathbb{X}, \mathcal{X})$ be a measurable space, and let $\mathcal{M}^+$ (resp. $\mathcal{M}^+_*$) denote the set of finite nonnegative (resp. positive) measures on $(\mathbb{X}, \mathcal{X})$.

- Any measurable function $V : \mathbb{X} \to [1, +\infty)$ is called a Lyapunov function. For every measurable function $g : \mathbb{X} \to \mathbb{R}$, we set $\|g\|_V := \sup_{x\in\mathbb{X}} |g(x)|/V(x) \in [0, +\infty]$, and we define the space $\mathcal{B}_V := \{g : \mathbb{X} \to \mathbb{R}, \text{measurable such that } \|g\|_V < \infty\}$.

- For any $\mu \in \mathcal{M}^+$ and any μ-integrable function $g : \mathbb{X} \to \mathbb{R}$, $\mu(g)$ denotes the integral $\int_{\mathbb{X}} g d\mu$. If $(\mu_1, \mu_2) \in (\mathcal{M}^+)^2$ is such that $\mu_i(V) < \infty, i = 1, 2$ for some Lyapunov function V, then the V-weighted total variation norm $\|\mu_1 - \mu_2\|_V$ is defined by
$$\|\mu_1 - \mu_2\|_V := \sup_{|g|\le V} \big|\mu_1(g) - \mu_2(g)\big|. \tag{1}$$

- A nonnegative kernel $K(x, dy) \in \mathcal{M}^+$, $x \in \mathbb{X}$ is said to be a Markov (respectively submarkov) kernel if $K(x, \mathbb{X}) = 1$ (respectively $K(x, \mathbb{X}) \le 1$) for any $x \in \mathbb{X}$. We denote by K its functional action defined by
$$\forall x \in \mathbb{X}, \quad (Kg)(x) := \int_{\mathbb{X}} g(y)\, K(x, dy),$$

*Univ Rennes, INSA Rennes, CNRS, IRMAR-UMR 6625, F-35000, France. Loic.Herve@insa-rennes.fr, James.Ledoux@insa-rennes.fr

where $g : \mathbb{X} \to \mathbb{R}$ is any $K(x,\cdot)-$integrable function. For every $n \geq 1$ the $n-$th iterate kernel of $K(x,dy)$ is denoted by $K^n(x,dy)$, $x \in \mathbb{X}$, and K^n stands for its functional action. As usual K^0 is the identity map I by convention.

1 Introduction

Let (V_0, V_1) be a couple of Lyapunov functions. For a Markov kernel P satisfying standard minorization and V_1-modulated drift conditions (see $\boldsymbol{D}(V_0,V_1)$ below), Glynn and Meyn proved in [GM96, Th. 2.3] that there exists a $P-$invariant probability measure π such that $\pi(V_1) < \infty$, and that there exists a positive constant c_0 such that, for any $g \in \mathcal{B}_{V_1}$ satisfying $\pi(g) = 0$, the Poisson equation

$$(I - P)\widehat{g} = g \tag{2}$$

admits a solution $\widehat{g} \in \mathcal{B}_{V_0}$ such that $\pi(\widehat{g}) = 0$ and

$$\|\widehat{g}\|_{V_0} \leq c_0 \, \|g\|_{V_1}. \tag{3}$$

Also see [MT09, Th. 17.7.1]. Note that the function g is not assumed to be $\pi-$centred in the original Glynn-Meyn's statement. Throughout our paper, the condition $\pi(g) = 0$ will be used to simplify the statements. Simply apply the results to function $g - \pi(g)1_{\mathbb{X}}$ to restore the general context. Under the aperiodicity condition, Glynn-Meyn's theorem is related to the pointwise convergence of the series $\sum_{k=0}^{+\infty} P^k g$, see [MT09, Th. 14.0.1]. We point out that the constant c_0 in (3) is unknown in general. In Section 2, the following theorem is proved (see Theorem 2.3).

Theorem 1 *Assume that P satisfies the following minorization condition*

$$\exists S \in \mathcal{X},\ \exists \nu \in \mathcal{M}_*^+, \quad \forall x \in \mathbb{X},\ \forall A \in \mathcal{X}, \quad P(x,A) \geq \nu(1_A)\, 1_S(x), \tag{$\boldsymbol{S}$}$$

and the following V_1-modulated drift condition with respect to the set S in ($\boldsymbol{S}$) and some couple (V_0, V_1) of Lyapunov functions

$$\exists b > 0, \quad PV_0 \leq V_0 - V_1 + b1_S. \tag{$\boldsymbol{D}(V_0,V_1)$}$$

Then P admits an invariant probability measure π such that $\pi(1_S) > 0$ and $\pi(V_1) < \infty$. Moreover let us introduce the submarkov residual kernel $R := P - \nu(\cdot)1_S$. Then, for every $g \in \mathcal{B}_{V_1}$ such that $\pi(g) = 0$, the function $\widetilde{g} := \sum_{k=0}^{+\infty} R^k g$ belongs to $\mathcal{B}_{V_0}$ and satisfies the Poisson equation (2), that is $(I-P)\widetilde{g} = g$, with

$$\|\widetilde{g}\|_{V_0} \leq a\|g\|_{V_1} \quad \textit{where} \quad a := 1 + \max\left(0\,,\, \frac{b - \nu(V_0)}{\nu(1_{\mathbb{X}})}\right). \tag{4}$$

Condition $\boldsymbol{D}(V_0,V_1)$ is the so-called V_1-modulated drift condition (e.g. see [MT09, Chap. 14 and Condition (V3)]. Such a condition has been widely used to analyse the geometric or sub-geometric rate of convergence in total variation norms of the Markov chain to its invariant probability measure π (e.g. see [DMPS18, Chap. 16, 17 and the references therein] for an overview and various examples, and [Del17] for an alternative operator-type approach). To the best of our knowledge, an estimate of the constant c_0 in (3) is only provided in [LL18, Prop. 1] for a discrete state-space $\mathbb{X}$ and in [Mas19] for a continuous-time Markov chain with a general state-space $\mathbb{X}$. In both [LL18] and [Mas19] the existence of an atom is assumed, and standard regeneration approach is then applied under the V_1-modulated drift condition to obtain the existence and a bound of a $\pi-$centred solution to Poisson's equation. Here, we use a quite different approach that does not require the existence of an atom.

Let us comment the conclusions of Theorem 1. The $P-$invariant probability measure π satisfying $\pi(1_S) > 0$ is derived from a Nummelin-type representation (see [Num84, Th. 5.2, Cor. 5.2]), that is: $\pi = \mu(1_{\mathbb{X}})^{-1}\mu$ with $\mu := \sum_{k=0}^{+\infty} \nu R^k \in \mathcal{M}_*^+$. This representation is classical under various hypotheses (see Appendix A for a direct proof under the sole minorization condition ($\boldsymbol{S}$)). The original trick in this work is that, under Assumption $\boldsymbol{D}(V_0, V_1)$, the submarkov residual kernel $R := P - \nu(\cdot)1_S$ satisfies the following drift condition

$$RV_{0,c} \leq V_{0,c} - V_1 \tag{5}$$

with $V_{0,c} := V_0 + d1_{\mathbb{X}}$ for some explicit positive constant d (see Lemma 2.2). Then the residual-type drift condition (5) enables us to define the function $\widetilde{g}$ and to prove the bound (4), while the Nummelin-type representation of π is proved to be crucial here to obtain that $\widetilde{g}$ is a solution to the Poisson equation. The innovative point in Theorem 1 is that the bound (4) is simple and explicit. Note that a similar bound can be obtained when Assumptions ($\boldsymbol{S}$) and $\boldsymbol{D}(V_0, V_1)$ are assumed to hold for P^ℓ for some $\ell \geq 2$ (see Remark 2.4).

In Section 3, Theorem 1 is proved to be relevant for perturbation issues. Indeed, when P and P' are two Markov kernels on $(\mathbb{X}, \mathcal{X})$ with respective invariant probability measures π and π', the following formula is of interest to control $\pi'(g) - \pi(g)$:

$$\pi'(g) - \pi(g) = \pi'\big((P' - P)\widehat{g}\big) \tag{6}$$

where $\widehat{g}$ is the solution to Poisson's equation used in Glynn-Meyn's theorem. This was first observed in [Sch68] for finite irreducible stochastic matrices (see also [Sen93]). Formula (6) may be subsequently used in any problem which can be thought of as a perturbation problem of Markov kernels (e.g. see [GM96, LL18, and references therein] and [MT09, Sec. 17.7]). Actually, from Theorem 1 Formula (6) still holds replacing $\widehat{g}$ with $\widetilde{g} := \sum_{k=0}^{+\infty} R^k(g - \pi(g)1_{\mathbb{X}})$, and the explicit bound (4) is then of great interest (recall that the bound in (3) is not explicit). This perturbation issue is addressed in Section 3 for a general family $\{P_\theta\}_{\theta\in\Theta}$ of Markov kernels, each of them satisfying a minorization condition and a V_1-modulated drift condition w.r.t. some Lyapunov functions V_0 and V_1 (independent of θ). Thus, denoting by π_θ the $P_\theta-$invariant probability measure provided by Theorem 1, and fixing some $\theta_0 \in \Theta$, it is proved in Theorem 3.2 that $\|\pi_\theta - \pi_{\theta_0}\|_{V_1} \to 0$ when $\theta \to \theta_0$, provided that for every $x \in \mathbb{X}$ we have $\Delta_{\theta,V_0}(x) := \|P_\theta(x,\cdot) - P_{\theta_0}(x,\cdot)\|_{V_0} \to 0$ when $\theta \to \theta_0$ (see (1) for the definition of the V-weighted total variation norm). Moreover a bound for $\|\pi_\theta - \pi_{\theta_0}\|_{V_1}$ is given in terms of the quantity $\pi_\theta(\Delta_{\theta,V_0})$. Here P_{θ_0} may be considered as the Markov kernel of interest, and the P_θ's for $\theta \neq \theta_0$ must be thought of some perturbed Markov kernels which are more tractable than P_{θ_0}. In particular the term $\pi_\theta(\Delta_{\theta,V_0})$ is expected to be known or at least computable for $\theta \neq \theta_0$.

The general approach in Theorem 1, which is based on the residual-type drift condition (5), is proved to be relevant under the V_1-modulated drift condition $\boldsymbol{D}(V_0, V_1)$. To the best of our knowledge, the bound (4) for solution to Poisson's equation is new. Note that the proof of Theorem 1 does not involve the splitting technique. Such a bound (4) allows us to specify the constant c_0 in the bound (3) of Glynn-Meyn theorem [GM96, Th. 2.3] (see Corollary 2.5). Finally Property (4) provides a simple bound for the asymptotic variance in the central limit theorem (see Corollary 2.6). These results apply whenever explicit modulated drift conditions are known: for such examples, e.g. see [FM00, FM03, DFM16] in the context of Metropolis algorithm, [TT94, DFM16] for autoregressive models, [LH07, LH12] for queueing systems, [JT02] for Markov chains associated with the mean of Dirichlet processes. The standard geometric drift condition involved in the so-called $V-$geometrically ergodic Markov chains, is a special case of modulated drift condition (see (15)). In this specific geometric case, the bounds obtained in this paper for solutions to Poisson's equation greatly improve those given in [HL23a] (see Corollary 2.7). Classical instances of $V-$geometrically ergodic Markov chains can be found in [MT09, RR04, DMPS18].

The perturbation theory for Markov chains has been widely developed in the last decades. Recall that the strong continuity assumption introduced in [Kar86] is suitable when $P_\theta = P_{\theta_0} + \theta D$, e.g. see [AANQ04, Mou21]. Note that neither the specific investigation of uniformly ergodic Markov chains (see [Mit05, MA10, AFEB16, JMMD]), nor that of reversible transition kernels (e.g. see [MALR16, NR21, and the references therein]), are addressed here. Recently Keller's approach [Kel82] involving a weak continuity assumption (for perturbed dynamical systems) has been adapted to V-geometrically ergodic Markov models, either using the Keller-Liverani perturbation theorem [KL99] (see [FHL13, HL14, HL23b]), or using the elegant idea of [HM11] involving a suitable Wasserstein distance (see [RS18, MARS20, and references therein], also see [SS00]). In the perturbation results of Section 3, Schweitzer's approach [Sch68] combined with the bound (4) is proved to be a relevant alternative method to investigate the quantitative control of the deviation between the invariant probability measures of Markov kernels. To the best of our knowledge, the results of Section 3 are new. They can be compared in the specific context of $V-$geometrically ergodic Markov kernels, see Example 3.7. In Example 3.8 an application to perturbed random walk on the half line is addressed under polynomial drift conditions.

2 The drift conditions and Poisson's equation

Recall that a Markov kernel P on $(\mathbb{X}, \mathcal{X})$ satisfying the following minorization condition $(\boldsymbol{S})$ (e.g. see [MT09])

$$\exists S \in \mathcal{X},\ \exists \nu \in \mathcal{M}_*^+, \quad \forall x \in \mathbb{X},\ \forall A \in \mathcal{X}, \quad P(x, A) \geq \nu(1_A)\, 1_S(x) \tag{$\boldsymbol{S}$}$$

is said to have a small set. The positive measure ν in $(\boldsymbol{S})$ is often written in the literature as $\nu = \varepsilon \mathsf{p}$ for some probability measure p on $(\mathbb{X}, \mathcal{X})$ and some $\varepsilon \in (0, 1]$. This formulation is not used here (note that the number $\nu(1_{\mathbb{X}})$ used in some bounds below equals to ε). Actually, the main property derived from Condition $(\boldsymbol{S})$ here is that the following so-called residual kernel R is a submarkov kernel

$$\forall x \in \mathbb{X}, \quad R(x, \cdot) := P(x, \cdot) - \nu(\cdot) 1_S(x). \tag{7}$$

Assume that P satisfies Condition $(\boldsymbol{S})$ and that V_0 is a Lyapunov function such that the function PV_0 is everywhere finite (i.e. $\forall x \in \mathbb{X},\ (PV_0)(x) < \infty$). Then we have $\nu(V_0) \leq (PV_0)(x) < \infty$ for any $x \in S$ from $(\boldsymbol{S})$, so that the nonnegative function RV_0 is well-defined. Now, given another Lyapunov function V_1, let us introduce the following residual-type drift condition:

$$RV_0 \leq V_0 - V_1. \tag{$\boldsymbol{R}(V_0, V_1)$}$$

Note that this condition is a special case of Condition $\boldsymbol{D}(V_0, V_1)$ (i.e. $b = \nu(V_0)$), and that it implies that $V_1 \le V_0$ since $RV_0 \ge 0$, and that $PV_0 \le (1 + \nu(V_0))V_0$, hence $\|PV_0\|_{V_0} < \infty$.

Assuming the residual-type drift condition ($\boldsymbol{R}(V_0, V_1)$) and using a Nummelin-type representation for the $P-$invariant probability measure π, the next Proposition 2.1 is our key preliminary result. It states that, for any $g \in \mathcal{B}_{V_1}$ the series $\sum_{k=0}^{+\infty} R^k g$ pointwise converges in $\mathbb{X}$ and defines a function $\widetilde{g}$ in $\mathcal{B}_{V_0}$ satisfying the nice bound $\|\widetilde{g}\|_{V_0} \le \|g\|_{V_1}$. Moreover $\widetilde{g}$ is a solution to Poisson's equation when $\pi(g) = 0$. Proposition 2.1 will be central for obtaining the bound (13) for $\|\widetilde{g}\|_{V_0}$ under the general V_1-modulated drift condition $\boldsymbol{D}(V_0, V_1)$ (see Theorem 2.3).

Proposition 2.1 *Assume that P satisfies the minorization Condition ($\boldsymbol{S}$) and that V_0 is a Lyapunov function such that PV_0 is everywhere finite. If the residual kernel R given in (7) satisfies the drift condition $\boldsymbol{R}(V_0, V_1)$ with a Lyapunov function V_1, then the following assertions hold.*

(i) For any $g \in \mathcal{B}_{V_1}$, the function $\widetilde{g} := \sum_{k=0}^{+\infty} R^k g$ is well-defined on $\mathbb{X}$ and $\widetilde{g} \in \mathcal{B}_{V_0}$ with

$$\|\widetilde{g}\|_{V_0} \le \|g\|_{V_1}. \tag{8}$$

(ii) The positive measure $\mu := \sum_{k=0}^{+\infty} \nu R^k$ is such that $\mu(1_{\mathbb{X}}) \le \mu(V_1) \le \nu(V_0) < \infty$, and $\pi \equiv \pi_{\nu,R} := \mu(1_{\mathbb{X}})^{-1}\mu$ defines a $P-$invariant probability measure satisfying $\pi(1_S) = \mu(1_{\mathbb{X}})^{-1} > 0$ and $\pi(V_1) < \infty$.

(iii) For any $g \in \mathcal{B}_{V_1}$ such that $\pi(g) = 0$, the function $\widetilde{g}$ satisfies Poisson's equation

$$(I - P)\widetilde{g} = g. \tag{9}$$

Proof. Let $x \in \mathbb{X}$. From $\boldsymbol{R}(V_0, V_1)$, we derive that $V_1 \le V_0 - RV_0$ and we obtain

$$\forall n \ge 1, \quad \sum_{k=0}^{n} (R^k V_1)(x) \le V_0(x), \tag{10}$$

so that $\sum_{k=0}^{+\infty} (R^k V_1)(x) \le V_0(x)$. Now let $g \in \mathcal{B}_{V_1}$. Using $|g| \le \|g\|_{V_1} V_1$, it follows that

$$\sum_{k=0}^{+\infty} \big|(R^k g)(x)\big| \le \|g\|_{V_1}\, V_0(x).$$

This proves Assertion (i). Next it follows from $1_{\mathbb{X}} \le V_1$ and $\sum_{k=0}^{+\infty} R^k V_1 \le V_0$ that

$$\sum_{k=0}^{+\infty} \nu(R^k 1_{\mathbb{X}}) \le \sum_{k=0}^{+\infty} \nu(R^k V_1) \le \nu(V_0) < \infty.$$

Hence the positive measure $\mu := \sum_{k=0}^{+\infty} \nu R^k$ is such that

$$0 < \nu(1_{\mathbb{X}}) \le \mu(1_{\mathbb{X}}) \le \mu(V_1) \le \nu(V_0) < \infty.$$

Then the Nummelin-type formula $\pi = \mu(1_{\mathbb{X}})^{-1}\mu$ defines a $P-$invariant probability measure satisfying $\pi(1_S) = \mu(1_{\mathbb{X}})^{-1} > 0$ (see Proposition A.1). Assertion (ii) is proved.

Let $g \in \mathcal{B}_{V_1}$. Since $\pi(V_1) = \mu(1_{\mathbb{X}})^{-1}\mu(V_1) < \infty$, we have $\pi(|g|) < \infty$. Now define

$$\forall n \geq 1, \quad \widetilde{g}_n := \sum_{k=0}^{n} R^k g.$$

Then, using $P = R + \nu(\cdot)1_S$ and setting $\mu_n(g) := \nu(\widetilde{g}_n) = \sum_{k=0}^{n} \nu(R^k g)$ we have

$$(I - P)\widetilde{g}_n = \widetilde{g}_n - R\widetilde{g}_n - \mu_n(g)1_S = g - R^{n+1}g - \mu_n(g)1_S. \tag{11}$$

We know that $\lim_n R^{n+1}g = 0$ (pointwise convergence) from the convergence of the series $\sum_{k=0}^{+\infty} R^k g$. Moreover, using $\mu(V_1) < \infty$, we obtain that $\lim_{n\to+\infty} \mu_n(g) = \mu(g)$. Finally, for every $x \in \mathbb{X}$, we have $\lim_n (P\widetilde{g}_n)(x) = (P\widetilde{g})(x)$ from Lebesgue's theorem applied to the sequence $(\widetilde{g}_n)_n$ w.r.t. the probability measure $P(x, dy)$ since $\lim_n \widetilde{g}_n = \widetilde{g}$, $|\widetilde{g}_n| \leq \|g\|_{V_1} V_0$ and $(PV_0)(x) < \infty$. Taking the limit when n goes to infinity in (11), we obtain

$$(I - P)\widetilde{g} = g - \mu(g)1_S. \tag{12}$$

Next, if we assume that $\pi(g) = 0$, then Equality (12) rewrites as Equality (9) since $\mu(g) = \pi(g)/\pi(1_S) = 0$ from the representation of π. The proof of Proposition 2.1 is complete. $\square$

Now let us assume that P satisfies $(\boldsymbol{S})$ and the standard V_1-modulated drift condition for some couple (V_0, V_1) of Lyapunov functions

$$\exists b > 0, \quad PV_0 \leq V_0 - V_1 + b1_S. \tag{$\boldsymbol{D}(V_0,V_1)$}$$

This implies that the functions PV_1 and PV_0 are everywhere finite. Then we have $\nu(V_0) < \infty$ from $(\boldsymbol{S})$. Thus the nonnegative function RV_0 is well-defined where R is the residual kernel defined in (7). If Condition $\boldsymbol{D}(V_0, V_1)$ holds with an atom S (i.e. $\forall x \in S,\ P(x, \cdot) = \nu(\cdot)$), then $b = \nu(V_0)$ may be chosen, so that Condition $\boldsymbol{R}(V_0, V_1)$ holds too. In the non-atomic case, the drift condition $\boldsymbol{D}(V_0, V_1)$ on P may not directly provide the residual-type condition $\boldsymbol{R}(V_0, V_1)$ since the constant b may be strictly larger than $\nu(V_0)$. However, starting from Assumption $\boldsymbol{D}(V_0, V_1)$, the next lemma shows that the slight change of the Lyapunov function V_0 into $V_{0,c} = V_0 + d1_{\mathbb{X}}$, with some suitable positive constant d, does provide the residual-type drift condition $\boldsymbol{R}(V_{0,c}, V_1)$.

Lemma 2.2 *Assume that P satisfies Conditions $(\boldsymbol{S})$ and $\boldsymbol{D}(V_0, V_1)$ w.r.t. some couple (V_0, V_1) of Lyapunov functions. Let $c \geq (b - \nu(V_0))/\nu(1_{\mathbb{X}})$. Then Condition $\boldsymbol{R}(V_{0,c}, V_1)$, with $V_{0,c} := V_0 + \max(0, c)1_{\mathbb{X}} \geq V_0$, holds for the residual kernel R defined in (7).*

Proof. We already know that the function RV_0 is well-defined and is finite from Assumptions $\boldsymbol{D}(V_0, V_1)$ and $(\boldsymbol{S})$. Set $d := \max(0, c)$ and $V_{0,c} = V_0 + d1_{\mathbb{X}}$. Note that $\nu(V_{0,c}) = \nu(V_0) + d\nu(1_{\mathbb{X}}) < \infty$ and that $PV_{0,c} = PV_0 + d1_{\mathbb{X}} < \infty$. We have

$$\begin{aligned} RV_{0,c} = PV_{0,c} - \nu(V_{0,c})1_S &= PV_0 + d1_{\mathbb{X}} - \big(\nu(V_0) + d\,\nu(1_{\mathbb{X}})\big)1_S \\ &\leq V_0 - V_1 + b1_S + d1_{\mathbb{X}} - \big(\nu(V_0) + d\,\nu(1_{\mathbb{X}})\big)1_S \\ &\leq V_{0,c} - V_1 + \big(b - \nu(V_0) - d\,\nu(1_{\mathbb{X}})\big)1_S. \end{aligned}$$

from the definitions of R and $V_{0,c}$, and from Assumption $\boldsymbol{D}(V_0, V_1)$. Hence the expected statement. $\square$

Under the standard V_1-modulated drift condition $\boldsymbol{D}(V_0, V_1)$ on P, the following statement is derived from Lemma 2.2 and Proposition 2.1. Theorem 2.3 below can be thought of as an extension of [GM96, Th 2.3] (see also [MT09, Th 17.7.1]) in that it provides an explicit and simple bound on the V_0-norm of a solution to Poisson's equation. To the best of our knowledge, the joint use of Lemma 2.2 and Proposition 2.1, as well as the bound (13) below, are new.

Theorem 2.3 *Assume that P satisfies the minorization Condition ($\boldsymbol{S}$) and the V_1-modulated drift condition $\boldsymbol{D}(V_0, V_1)$ w.r.t. some couple (V_0, V_1) of Lyapunov functions.*
Then the conclusions stated in Assertions (i) – (iii) of Proposition 2.1 hold true with the following bound in place of (8)

$$\forall g \in \mathcal{B}_{V_1}, \quad \|\widetilde{g}\|_{V_0} \leq a\|g\|_{V_1} \quad \textit{with} \quad a := 1 + \max\left(0\,,\, \frac{b - \nu(V_0)}{\nu(1_{\mathbb{X}})}\right) \tag{13}$$

where $\nu \in \mathcal{M}_*^+$ is given in ($\boldsymbol{S}$) and b is the positive constant given in $\boldsymbol{D}(V_0, V_1)$.

Let us mention that, when $g \in \mathcal{B}_{V_1}$ is such that $\pi(g) \neq 0$, the centred function $g_0 := g - \pi(g)1_{\mathbb{X}}$ is such that $\widetilde{g}_0 := \sum_{k=0}^{+\infty} R^k g_0$ is solution in $\mathcal{B}_{V_0}$ to Poisson's equation $(I - P)\widetilde{g}_0 = g_0 = g - \pi(g)1_{\mathbb{X}}$. The bound (13) is similar to those in [LL18, Prop. 1] and [Mas19], which have been obtained under V_1-modulated drift conditions, but assuming the existence of an atom. In this work, we do not assume the existence of an atom. Nor do we use the splitting method for passing from the atomic to the general case. Actually the atomic case appears here as a special case where the constant b in $\boldsymbol{D}(V_0, V_1)$ can be chosen equal to $\nu(V_0)$, so that the constant a in (13) is one. In other words, the atomic case is encompassed by the assumptions of Proposition 2.1.

Proof. Let $V_{0,c} := V_0 + d1_{\mathbb{X}}$ where $d := \max(0, \widehat{c})$ with $\widehat{c} := (b - \nu(V_0))/\nu(1_{\mathbb{X}})$ (see Lemma 2.2). Note that V_0 and $V_{0,c}$ are equivalent Lyapunov functions in the sense that $V_0 \leq V_{0,c} \leq (1 + d)V_0$. Proposition 2.1 applied with the drift condition of Lemma 2.2 shows that, for any $g \in \mathcal{B}_{V_1}$, the function $\widetilde{g} := \sum_{k=0}^{+\infty} R^k g$ belongs to $\mathcal{B}_{V_{0,c}}$ with

$$\|\widetilde{g}\|_{V_{0,c}} \leq \|g\|_{V_1},$$

and that $\widetilde{g}$ satisfies $(I - P)\widetilde{g} = g$ when $\pi(g) = 0$. Next (13) holds since $\|\cdot\|_{V_0} \leq (1 + d)\|\cdot\|_{V_{0,c}}$ from the inequality $V_{0,c} \leq (1 + d)V_0$. □

Remark 2.4 *Assume that Conditions ($\boldsymbol{S}$) and $\boldsymbol{D}(V_0, V_1)$ are only satisfied for the Markov kernel P^ℓ with some $\ell \geq 2$. Moreover suppose that π is the unique invariant probability measure for both P and P^ℓ. Recall that $M := \|PV_0\|_{V_0} < \infty$ from $\boldsymbol{D}(V_0, V_1)$. Set $R_\ell := P^\ell - \nu(\cdot)1_S$. Then, for every $g \in \mathcal{B}_{V_1}$ such that $\pi(g) = 0$, the function*

$$\tilde{g} := \sum_{k=0}^{\ell-1} P^k \widetilde{g}_\ell \quad \textit{with} \quad \widetilde{g}_\ell = \sum_{k=0}^{+\infty} {R_\ell}^k g$$

belongs to $\mathcal{B}_{V_0}$ and satisfies the Poisson equation $(I - P)\tilde{g} = g$. Moreover we have

$$\|\tilde{g}\|_{V_0} \leq \frac{a\big(M^\ell - 1\big)}{M - 1}\,\|g\|_{V_1} \quad \textit{with} \quad a := 1 + \max\left(0\,,\, \frac{b - \nu(V_0)}{\nu(1_{\mathbb{X}})}\right)$$

where $\nu \in \mathcal{M}_*^+$ and b are here given in Conditions ($\boldsymbol{S}$)-$\boldsymbol{D}(V_0, V_1)$ related to P^ℓ. Indeed Theorem 2.3 applied to the Markov kernel P^ℓ shows that, for every $g \in \mathcal{B}_{V_1}$ such that $\pi(g) = 0$, the function $\widetilde{g}_\ell \in \mathcal{B}_{V_0}$ and $\widetilde{g}_\ell$ satisfies $(I - P^\ell)\widetilde{g}_\ell = g$, with moreover

$$\|\widetilde{g}_\ell\|_{V_0} \leq a\, \|g\|_{V_1}.$$

The claimed statements then follow from $(I - P)\tilde{g} = (I - P^\ell)\widetilde{g}_\ell = g$ and from the inequality $\|\tilde{g}\|_{V_0} \leq \|\widetilde{g}_\ell\|_{V_0}(M^\ell - 1)/(M - 1)$.

Note that the invariant probability measure $\pi \equiv \pi_{\nu,R}$ involved in Proposition 2.1 and Theorem 2.3 only satisfies the moment condition $\pi(V_1) < \infty$, and there is no guarantee that $\pi(V_0) < \infty$. For Markov kernels satisfying a modulated drift condition, the existence and uniqueness of the $P-$invariant probability measure is investigated in many works under various hypothesis, e.g. see [MT09, DMPS18], and [FM03, Th. 1]. For instance, if P is $\psi-$irreducible for some positive measure ψ on $(\mathbb{X}, \mathcal{X})$ and satisfies Condition $\boldsymbol{D}(V_0, V_1)$, then P has a unique invariant probability measure, e.g. see [Mey22, Th 6.12].

In Corollary 2.5 below, we prove that, if the invariant probability measure $\pi \equiv \pi_{\nu,R}$ of Theorem 2.3 is such that $\pi(V_0) < \infty$, then it is the unique one integrating V_0. This statement is suitable to the perturbation results of the next Section 3, in which the moment condition $\pi(V_0) < \infty$ is involved.

Corollary 2.5 *Let P satisfying the assumptions of Theorem 2.3. Assume that the invariant probability measure $\pi \equiv \pi_{\nu,R}$ is such that $\pi(V_0) < \infty$. Then*

1. *π is the unique $P-$invariant probability measure which integrates V_0.*
2. *For any $g \in \mathcal{B}_{V_1}$ such that $\pi(g) = 0$, let $\widetilde{g} := \sum_{k=0}^{+\infty} R^k g$. Then the function $\widehat{g} = \widetilde{g} - \pi(\widetilde{g})1_{\mathbb{X}}$ is a $\pi-$centered solution on $\mathcal{B}_{V_0}$ to Poisson's equation $(I - P)\widehat{g} = g$. Moreover we have*

$$\|\widehat{g}\|_{V_0} \quad \leq \quad a\left(1 + \pi(V_0)\, \|1_{\mathbb{X}}\|_{V_0}\right) \|g\|_{V_1} \tag{14}$$

where the positive constant a is given in (13).

Note that, when Poisson's equation has a unique solution up to an additive constant, Inequality (14) gives a bound for the norm of the solution in Glynn-Meyn's theorem.

Proof. Let $g \in \mathcal{B}_{V_1}$. We know from Theorem 2.3 and Equality (12) that the associated function $\widetilde{g} := \sum_{k=0}^{+\infty} R^k g$ is in $\mathcal{B}_{V_0}$ and satisfies Equation $(I - P)\widetilde{g} = g - \mu(g)1_S$ with $\mu := \sum_{k=0}^{+\infty} \nu R^k \in \mathcal{M}_*^+$. Recall that $\pi = \mu(1_{\mathbb{X}})^{-1}\mu$. Consequently, if η is a $P-$invariant positive measure on $\mathbb{X}$ such that $\eta(V_0) < \infty$, then we have $\eta((I - P)\widetilde{g}) = 0 = \eta(g) - \mu(g)\eta(1_S)$, thus $\eta = \eta(1_S)\mu = \eta(1_S)\mu(1_{\mathbb{X}})\pi$. This proves the first assertion of Corollary 2.5.

To prove the second one, first note that $\widehat{g} \in \mathcal{B}_{V_0}$ and the property $\pi(\widehat{g}) = 0$ (under $\pi(V_0) < \infty$) are obvious. Moreover, if g is such that $\pi(g) = 0$, then we have $(I - P)\widehat{g} = (I - P)\widetilde{g} = g$ from Theorem 2.3 and $(I - P)1_{\mathbb{X}} = 0$. Finally we have

$$\|\widehat{g}\|_{V_0} \leq \left(1 + \pi(V_0)\, \|1_{\mathbb{X}}\|_{V_0}\right)\|\widetilde{g}\|_{V_0} \leq a\left(1 + \pi(V_0)\, \|1_{\mathbb{X}}\|_{V_0}\right)\|g\|_{V_1}$$

using the definition of $\widehat{g}$, the triangular inequality and $|\widetilde{g}| \leq \|\widetilde{g}\|_{V_0} V_0$ for the first inequality, the bound (13) applied to $\widetilde{g}$ for the second one. □

For a Markov model satisfying Assumption $\boldsymbol{D}(V_0, V_1)$, it is worth noticing that the condition $\pi(V_0) < \infty$ holds provided that P satisfies any preliminary V_0-modulated drift condition, that is: $PW \le W - V_0 + b1_S$ for some Lyapunov function W (apply Theorem 2.3 to the couple (W, V_0)). Recall that such nested modulated drift conditions $\boldsymbol{D}(W, V_0)$ and $\boldsymbol{D}(V_0, V_1)$ occur in most of the analysis of polynomial or subgeometric convergence rate of Markov models, e.g. see [JR02, FM03, AFV15], in particular see [JR02, Lem. 3.5]) in the polynomial case and [DFMS04, DMPS18] in the subgeometric case.

The next proposition provides a computable bound for the so-called asymptotic variance involved in the central limit theorem for Markov chains (e.g. see [MT09, Chap. 17], [DMPS18, Chap. 21] and [Jon04]). To the best of our knowledge, this bound is new, and this is achieved thanks to the nice bound (13) of Theorem 2.3, since the asymptotic variance is known to be closely related to Poisson's equation.

Corollary 2.6 *Assume that P satisfies Condition $(\boldsymbol{S})$ and $\boldsymbol{D}(V_0, V_1)$ with the additional condition $\pi({V_0}^2) < \infty$. For any $g \in \mathcal{B}_{V_1}$ such that $\pi(g) = 0$, set $\gamma_g^2 = \pi((\widetilde{g})^2 - (P\widetilde{g})^2)$ where $\widetilde{g}$ is the solution to Poisson's equation $(I - P)\widetilde{g} = g$ provided by Theorem 2.3. Then we have*

$$\gamma_g^2 \le 2\, a^2\, \pi({V_0}^2)\, \|g\|_{V_1}^2$$

where a is the positive constant given in (13).

Proof. From Theorem 2.3, we obtain that

$$\begin{aligned} \gamma_g^2 &\le \pi(\widetilde{g}^2) + \pi((P\widetilde{g})^2) \le a^2 \|g\|_{V_1}^2 \big(\pi({V_0}^2) + \pi((PV_0)^2)\big) \\ &\le 2\, a^2 \pi({V_0}^2)\, \|g\|_{V_1}^2 \end{aligned}$$

using successively $|\widetilde{g}| \le a \|g\|_{V_1} V_0$ from (13), the Cauchy-Schwarz inequality $(PV_0)^2 \le P{V_0}^2$ and finally the $P-$invariance of π. ☐

To conclude this section let us apply the previous statements in the case when P satisfies Condition $(\boldsymbol{S})$ and the following so-called $V-$geometric drift condition

$$\exists \delta \in (0,1),\ \exists K \in (0, +\infty), \quad PV \le \delta V + K 1_S \qquad (\boldsymbol{G}(\delta, V))$$

for some Lyapunov function V, where $S \in \mathcal{X}$ is the set in $(\boldsymbol{S})$. Then rewriting Condition $\boldsymbol{G}(\delta, V)$ as $PV \le V - (1-\delta)V + K1_S$, we obtain that P satisfies the following Condition $\boldsymbol{D}(V_0, V)$

$$PV_0 \le V_0 - V + b1_S \quad \text{with} \quad V_0 := \frac{V}{1-\delta} \quad \text{and} \quad b := \frac{K}{1-\delta}. \tag{15}$$

For the sake of simplicity, in addition to Conditions $(\boldsymbol{S})$ and $\boldsymbol{G}(\delta, V)$, we also suppose that $\nu(1_S) > 0$ for P to be $V-$geometrically ergodic (e.g. see [Bax05]). In particular we know that $\pi(V) < \infty$ and that two solutions to Poisson's equation in $\mathcal{B}_V$ differ from a constant. Observing that $\|\cdot\|_{V_0} = (1-\delta)\|\cdot\|_V$ and that $\pi(V_0)\, \|1_{\mathbb{X}}\|_{V_0} = \pi(V)\, \|1_{\mathbb{X}}\|_V$, the next statements are easily deduced from Theorem 2.3, Corollary 2.5 and Corollary 2.6.

Corollary 2.7 *Assume that P satisfies the minorization Condition $(\boldsymbol{S})$ with $\nu(1_S) > 0$ and the $V-$geometric drift condition $\boldsymbol{G}(\delta, V)$ w.r.t. some Lyapunov function V. Then*

1. *The conclusions stated in Assertions* $(i)-(iii)$ *of Proposition 2.1 hold true with the following bound in place of (8):*

$$\forall g \in \mathcal{B}_V, \quad \|\widetilde{g}\|_{V_0} \le a\|g\|_V \quad \text{with here} \quad a := 1 + \max\left(0\,,\, \frac{K-\nu(V)}{\nu(1_{\mathbb{X}})(1-\delta}\right)$$

   so that

$$\forall g \in \mathcal{B}_V, \quad \|\widetilde{g}\|_V \le \frac{a}{1-\delta}\,\|g\|_V. \tag{16}$$

   where $\nu \in \mathcal{M}_*^+$ *is given in* ($\boldsymbol{S}$) *and* δ, K *are the constants given in* $\boldsymbol{G}(\delta, V)$.

2. *For every* $g \in \mathcal{B}_V$ *such that* $\pi(g) = 0$, *the function* $\widehat{g} = \sum_{k=0}^{+\infty} P^k g$ *is the unique* $\pi-$*centered function in* $\mathcal{B}_V$ *solution to Poisson's equation* $(I-P)\widehat{g} = g$, *with*

$$\|\widehat{g}\|_V \quad \le \quad \frac{a\left(1+\pi(V)\,\|1_{\mathbb{X}}\|_V\right)}{1-\delta}\,\|g\|_V. \tag{17}$$

3. *If* $\pi(V^2) < \infty$ *then, for any* $g \in \mathcal{B}_V$ *such that* $\pi(g) = 0$, *the asymptotic variance* $\gamma_g^2 = \pi((\widetilde{g})^2 - (P\widetilde{g})^2)$ *with* $\widetilde{g} := \sum_{k=}^{+\infty} R^k g$ *solution of Poisson's equation* $(I-P)\widetilde{g} = g$, *satisfies*

$$\gamma_g^2 \le \frac{2\,a^2\pi(V^2)}{(1-\delta)^2}\,\|g\|_V^2.$$

In this geometric ergodicity context, bounds similar to (16) and (17) have been obtained in [HL23a, Eqs. (35) and (36a)] for the norm $\|\cdot\|_{V^{\alpha_0}}$ for some $\alpha_0 \in (0,1]$. Actually the method in [HL23a] consists in converting the $V-$geometric drift condition $\boldsymbol{G}(\delta, V)$ into the following residual-type geometric drift condition $RV^{\alpha_0} \le \delta^{\alpha_0}\,V^{\alpha_0}$. When the positive constant K in $\boldsymbol{G}(\delta, V)$ is such that $K \le \nu(V)$ (in particular in atomic case), the previous residual-type drift condition holds with $\alpha_0 = 1$, but otherwise we have $\alpha_0 < 1$. In practice α_0 may be close to zero (i.e. δ^{α_0} is close to one), so that the bounds in [HL23a, Eqs. (35) and (36a)] degrade since they depend on $(1-\delta^{\alpha_0})^{-1}$. By contrast, even in this geometric ergodicity context, the joint use of Lemma 2.2 and Proposition 2.1 simply involving here the $V-$modulated drift condition (15) is proved to provide the relevant bounds (16) and (17) for the norm $\|\cdot\|_V$ of solutions to Poisson's equation.

3 General perturbation results

In this section we deal with the quantitative control of the deviation between the invariant probability measures of Markov kernels. Let us first present a preliminary statement based on Theorem 2.3.

Proposition 3.1 *Assume that* P *satisfies Conditions* ($\boldsymbol{S}$)-$\boldsymbol{D}(V_0, V_1)$, *and let* $\pi \equiv \pi_{\nu,R}$ *be the* $P-$*invariant probability measure of Theorem 2.3. Let* P' *be another Markov kernel on* $(\mathbb{X}, \mathcal{X})$ *with invariant probability measure* π' *such that* $\|P'V_0\|_{V_0} < \infty$ *and* $\pi'(V_0) < \infty$. *Then*

$$\|\pi' - \pi\|_{V_1} \le a\big(1+\pi(V_1)\|1_{\mathbb{X}}\|_{V_1}\big)\,\pi'(\Delta_{V_0}) \tag{18}$$

where the positive constant a *is defined in (13) and where the function* Δ_{V_0} *is defined by*

$$\forall x \in \mathbb{X}, \quad \Delta_{V_0}(x) := \|P(x,\cdot) - P'(x,\cdot)\|_{V_0}.$$

Proof. Recall that $\|PV_0\|_{V_0} < \infty$ from $\boldsymbol{D}(V_0, V_1)$, so that Δ_{V_0} and $\pi'(\Delta_{V_0})$ are well-defined under the assumptions of Proposition 3.1.

Let $g \in \mathcal{B}_{V_1}$ such that $\|g\|_{V_1} \leq 1$. Since $\pi(V_1) < \infty$ from Theorem 2.3, $\pi(g)$ is well-defined. Define $g_0 = g - \pi(g)1_{\mathbb{X}}$ and $\widetilde{g_0} := \sum_{k=0}^{+\infty} R^k g_0$ with the residual kernel $R := P - \nu(\cdot)1_S$. Then we have

$$\pi'\big((P' - P)\widetilde{g_0}\big) = \pi'(\widetilde{g_0}) - \pi'(\widetilde{g_0} - g_0) = \pi'(g_0) = \pi'(g) - \pi(g) \tag{19}$$

using the $P'-$invariance of π', the Poisson equation $(I - P)\widetilde{g_0} = g_0$ from Theorem 2.3, and finally the definition of g_0. It follows from the definition of Δ_{V_0} that

$$|\pi'(g) - \pi(g)| \leq \int_{\mathbb{X}} \big|(P'\widetilde{g_0})(x) - (P\widetilde{g_0})(x)\big| \, d\pi'(x) \leq \|\widetilde{g_0}\|_{V_0} \int_{\mathbb{X}} \Delta_{V_0}(x) \, d\pi'(x).$$

Finally we know from Theorem 2.3 that $\|\widetilde{g}\|_{V_0} \leq a\|g_0\|_{V_1}$ with a defined in (13), so that

$$\|\widetilde{g_0}\|_{V_0} \leq a \, \|g - \pi(g)1_{\mathbb{X}}\|_{V_1} \leq a\big(1 + \pi(V_1)\|1_{\mathbb{X}}\|_{V_1}\big)$$

from which we deduce (18). □

Now let $\{P_\theta\}_{\theta\in\Theta}$ denote a family of transition kernels on $(\mathbb{X}, \mathcal{X})$, where Θ is an open subset of some metric space. Let us introduce the following minorization and modulated drift conditions w.r.t. this family $\{P_\theta\}_{\theta\in\Theta}$:

$$\forall\theta \in \Theta, \ \exists S_\theta \in \mathcal{X}, \ \exists \nu_\theta \in \mathcal{M}_*^+, \ \forall x \in \mathbb{X}, \ \forall A \in \mathcal{X}, \quad P_\theta(x, A) \geq \nu_\theta(1_A) \, 1_{S_\theta}(x), \tag{$\boldsymbol{S_\Theta}$}$$

and there exists a couple (V_0, V_1) of Lyapunov functions such that

$$\forall\theta \in \Theta, \ \exists b_\theta > 0, \quad P_\theta V_0 \leq V_0 - V_1 + b_\theta 1_{S_\theta}. \tag{$\boldsymbol{D_\Theta}(V_0, V_1)$}$$

Let us fix some $\theta_0 \in \Theta$. The family $\{P_\theta, \theta \in \Theta \setminus \{\theta_0\}\}$ must be thought of as a family of transition kernels which are perturbations of P_{θ_0} and converge (in a certain sense) to P_{θ_0} when $\theta \to \theta_0$. To that effect, under the Conditions $(\boldsymbol{S_\Theta})$-$\boldsymbol{D_\Theta}(V_0, V_1)$ we define

$$\forall\theta \in \Theta, \ \forall x \in \mathbb{X}, \quad \Delta_{\theta,V_0}(x) := \|P_\theta(x, \cdot) - P_{\theta_0}(x, \cdot)\|_{V_0}. \tag{20}$$

Finally, under the additional conditions $\sup_{\theta\in\Theta} b_\theta < \infty$ and $\inf_{\theta\in\Theta} \nu_\theta(1_{\mathbb{X}}) > 0$, we define the following positive constant

$$a := 1 + \max\big(0, c\big) \quad \text{with} \quad c := \sup_{\theta\in\Theta} \frac{b_\theta - \nu_\theta(V_0)}{\nu_\theta(1_{\mathbb{X}})}. \tag{21}$$

In Theorem 3.2 and Corollary 3.3 below, each Markov kernel P_θ is assumed to satisfy the assumptions of Theorem 2.3. Accordingly, the $P_\theta-$invariant probability measure denoted by π_θ in these two statements is $\pi_\theta \equiv \pi_{\theta,\nu,R}$ (see the definition of π in Assertion *(ii)* of Proposition 2.1 with $\nu = \nu_\theta$ and $R_\theta := P_\theta - \nu_\theta(\cdot)1_{S_\theta}$). Since π_θ is assumed below to satisfy $\pi_\theta(V_0) < \infty$, we know from the first assertion of Corollary 2.5 that there is no ambiguity about what π_θ is in Theorem 3.2 and Corollary 3.3.

Theorem 3.2 *Assume that the family $\{P_\theta\}_{\theta\in\Theta}$ satisfies Conditions $(\boldsymbol{S_\Theta})$-$\boldsymbol{D_\Theta}(V_0, V_1)$ with $b := \sup_{\theta\in\Theta} b_\theta < \infty$ and $\inf_{\theta\in\Theta} \nu_\theta(1_{\mathbb{X}}) > 0$. Moreover suppose that, for every $\theta \in \Theta$, the $P_\theta-$invariant probability measure $\pi_\theta \equiv \pi_{\theta,\nu,R}$ of Theorem 2.3 satisfies $\pi_\theta(V_0) < \infty$.*

Then we have

$$\|\pi_\theta - \pi_{\theta_0}\|_{V_1} \le a \min\left\{c_{\theta_0}\, \pi_\theta(\Delta_{\theta,V_0})\,,\, c_\theta\, \pi_{\theta_0}(\Delta_{\theta,V_0})\right\} \tag{22}$$

with a defined in (21) and with

$$\forall \theta \in \Theta, \quad c_\theta := 1 + \pi_\theta(V_1)\|1_{\mathbb{X}}\|_{V_1} \le 1 + b\,\|1_{\mathbb{X}}\|_{V_1}. \tag{23}$$

If the following additional assumption holds

$$\forall x \in \mathbb{X}, \quad \lim_{\theta \to \theta_0} \Delta_{\theta,V_0}(x) = 0, \tag{$\boldsymbol{\Delta_{V_0}}$}$$

then we have

$$\lim_{\theta \to \theta_0} \|\pi_\theta - \pi_{\theta_0}\|_{V_1} = 0.$$

Proof. Let $\theta \in \Theta$. We have $P_\theta V_0 \le (1+b)V_0$ from $\boldsymbol{D_\Theta}(V_0, V_1)$ and the definition of the positive constant b. Thus Proposition 3.1 can be applied to $(P, P') := (P_{\theta_0}, P_\theta)$ and to $(P, P') := (P_\theta, P_{\theta_0})$, which provide Inequality (22). Also observe that the bound in (23) follows from the inequality $\pi_\theta(V_1) \le b_\theta \le b$ which is easily deduced from $\boldsymbol{D_\Theta}(V_0, V_1)$ using $\pi_\theta(P_\theta V_0) = \pi_\theta(V_0)$ (recall that $\pi_\theta(V_0) < \infty$ by hypothesis). Next we have

$$\lim_{\theta \to \theta_0} \pi_{\theta_0}(\Delta_{\theta,V_0}) = \lim_{\theta \to \theta_0} \int_{\mathbb{X}} \Delta_{\theta,V_0}(x) d\pi_{\theta_0}(x) = 0 \tag{24}$$

from Lebesgue's theorem using $\Delta_{\theta,V_0} \le 2(1+b)V_0$, $\pi_{\theta_0}(V_0) < \infty$ and Assumption ($\boldsymbol{\Delta_{V_0}}$). Then we obtain that $\lim_{\theta \to \theta_0} \|\pi_\theta - \pi_{\theta_0}\|_{V_1} = 0$ from the second bound in (22) and from the inequality (23). □

When Condition $\boldsymbol{D_\Theta}(V_0, V_1)$ is satisfied, so is Condition $\boldsymbol{D}(V_0, 1_{\mathbb{X}})$ since $V_1 \ge 1_{\mathbb{X}}$. Thus, when Theorem 3.2 applies, then it also applies with $V_1 := 1_{\mathbb{X}}$ and then provides the control of the total variation error since $\|\pi_\theta - \pi_{\theta_0}\|_{TV} = \|\pi_\theta - \pi_{\theta_0}\|_{1_{\mathbb{X}}}$. Using $\pi_\theta(1_{\mathbb{X}}) = 1$, $\|1_{\mathbb{X}}\|_{1_{\mathbb{X}}} = 1$, so that we have here $c_\theta := 1 + \pi_\theta(1_{\mathbb{X}})\|1_{\mathbb{X}}\|_{1_{\mathbb{X}}} = 2$, we obtain the following estimate for $\|\pi_\theta - \pi_{\theta_0}\|_{TV}$.

Corollary 3.3 *Under the assumptions of Theorem 3.2 we have*

$$\|\pi_\theta - \pi_{\theta_0}\|_{TV} \le 2\,a\, \min\left\{\pi_\theta(\Delta_{\theta,V_0})\,,\, \pi_{\theta_0}(\Delta_{\theta,V_0})\right\} \tag{25}$$

with a defined in (21). If moreover $\{P_\theta\}_{\theta\in\Theta}$ satisfies Assumption ($\boldsymbol{\Delta_{V_0}}$), then we have $\lim_{\theta \to \theta_0} \|\pi_\theta - \pi_{\theta_0}\|_{TV} = 0$.

The convergence of $\pi_{\theta_0}(\Delta_{\theta,V_0})$ to 0 when $\theta \to \theta_0$ in (24) is of theoretical interest: it is used to prove that $\lim_{\theta \to \theta_0} \|\pi_\theta - \pi_{\theta_0}\|_{V_1} = 0$ in Theorem 3.2. However it is worth noticing that this term $\pi_{\theta_0}(\Delta_{\theta,V_0})$ in the bounds (22) and (25) is not computable in practice since the probability measure π_{θ_0} may be considered as unknown in our perturbation context. By contrast, the value of $\pi_\theta(\Delta_{\theta,V_0})$ in bounds (22) and (25) is expected to be known or at least computable for $\theta \ne \theta_0$, so that the bounds of interest in (22) and (25) are

$$\|\pi_\theta - \pi_{\theta_0}\|_{V_1} \le a\, c_{\theta_0}\, \pi_\theta(\Delta_{\theta,V_0}) \quad \text{and} \quad \|\pi_\theta - \pi_{\theta_0}\|_{TV} \le 2\,a\; \pi_\theta(\Delta_{\theta,V_0}) \tag{26}$$

with c_{θ_0} given in (23). However the bounds in (26) are relevant only if $\lim_{\theta \to \theta_0} \pi_\theta(\Delta_{\theta,V_0}) = 0$, which is not guaranteed under the conditions of Theorem 3.2. For this purpose note that, in the proof of Theorem 3.2, the conditions $(\boldsymbol{S_\Theta})$ and $\boldsymbol{D_\Theta}(V_0,V_1)$ for P_θ with $\theta \neq \theta_0$ are only used for obtaining the inequality $\|\pi_\theta - \pi_{\theta_0}\|_{V_1} \leq ac_\theta\, \pi_{\theta_0}(\Delta_{\theta,V_0})$ of (22). Consequently, if we are only interested in the two bounds in (26), then the assumptions of Theorem 3.2 can be relaxed as follows.

Proposition 3.4 *Assume that the (unperturbed) Markov kernel $P := P_{\theta_0}$ satisfies Conditions $(\boldsymbol{S})$ and $\boldsymbol{D}(V_0,V_1)$, and that, for every $\theta \in \Theta \setminus \{\theta_0\}$, we have $\|P_\theta V_0\|_{V_0} < \infty$. Moreover suppose that, for every $\theta \in \Theta \setminus \{\theta_0\}$, there exist a $P_\theta-$invariant probability measure π_θ on $(\mathbb{X},\mathcal{X})$ such that $\pi_\theta(V_0) < \infty$. Finally assume that the $P_{\theta_0}-$invariant probability measure π_{θ_0} of Theorem 2.3 satisfies $\pi_{\theta_0}(V_0) < \infty$. Then the two bounds in (26) hold.*

Indeed, under the assumptions of Proposition 3.4, the first bound in (26) directly follows from Proposition 3.1 applied to $(P,P') := (P_{\theta_0}, P_\theta)$ with $\theta \neq \theta_0$. The second bound in (26) is obtained by replacing V_1 with $1_{\mathbb{X}}$. In Proposition 3.4, the existence of a $P_\theta-$invariant probability measure π_θ must be assumed when $\theta \neq \theta_0$ since we do suppose that P_θ satisfies minorization and modulated drift condition for $\theta \neq \theta_0$. Actually, π_θ may be any $P_\theta-$invariant probability measure when $\theta \neq \theta_0$, while π_{θ_0} is the $P_{\theta_0}-$invariant probability measure of Theorem 2.3. In any case the assumption $\pi_\theta(V_0) < \infty$ is required for every $\theta \in \Theta$. Finally let's stress once again that the bounds in (26) are of interest only when the term $\pi_\theta(\Delta_{\theta,V_0})$ is computable and can be proved to converge to 0 when $\theta \to \theta_0$.

Remark 3.5 (Stability issue) *In some classical perturbation schemes, as the standard truncations of infinite stochastic matrices or the state space discretization procedure of non-discrete models, the whole family $\{P_\theta\}_{\theta\in\Theta}$ satisfies Condition $(\boldsymbol{S_\Theta})$-$\boldsymbol{D_\Theta}(V_0,V_1)$ provided that the unperturbed Markov kernel $P := P_{\theta_0}$ satisfies Conditions $(\boldsymbol{S})$-$\boldsymbol{D}(V_0,V_1)$. Moreover the set S and the constant b involved for $P := P_{\theta_0}$ in $(\boldsymbol{S})$-$\boldsymbol{D}(V_0,V_1)$ can often be used for the perturbed Markov kernels P_θ. In this case the conditions $b := \sup_{\theta\in\Theta} b_\theta < \infty$ and $\inf_{\theta\in\Theta} \nu_\theta(1_{\mathbb{X}}) > 0$ of Theorem 3.2 and Corollary 3.3 are straightforward. In the context of geometric drift conditions, the previous facts are proved to hold in many papers for truncatures of infinite stochastic matrices (e.g. see [LL18, HL14, and references therein]), and in [HL21] for the state space discretization procedure. The case of Markov models satisfying modulated drift conditions can be addressed similarly.*

Remark 3.6 *If P_θ is replaced with iterate P_θ^ℓ for some $\ell \geq 2$ in Conditions $(\boldsymbol{S_\Theta})$-$\boldsymbol{D_\Theta}(V_0,V_1)$ and if for every $\theta \in \Theta$ both P_θ and P_θ^ℓ admit a unique invariant probability measure π_θ, then all the previous perturbation results still hold replacing $\pi_\theta(\Delta_{\theta,V_0})$ with $\pi_\theta(\Delta_{\ell,\theta,V_0})$, where*

$$\forall x \in \mathbb{X}, \quad \Delta_{\ell,\theta,V_0}(x) := \|P_\theta^\ell(x,\cdot) - P_{\theta_0}^\ell(x,\cdot)\|_{V_0}.$$

Indeed, under the previous assumptions, Theorem 3.2 and Corollary 3.3 obviously apply to the family $\{P_\theta^\ell\}_{\theta\in\Theta}$. The same remark is valid in Proposition 3.4 when $P_{\theta_0}^\ell$ satisfies Conditions $(\boldsymbol{S})$ and $\boldsymbol{D}(V_0,V_1)$ for some $\ell \geq 2$.

Our perturbation results are discussed through the two following examples.

Example 3.7 (Geometric drift conditions) *In the perturbation context, under Condition $(\boldsymbol{S_\Theta})$, the standard geometric drift conditions for some Lyapunov function V are the* *following ones:*

$$\forall \theta \in \Theta, \quad K_\theta := \sup_{x \in S_\theta} (P_\theta V)(x) < \infty \quad \textit{and} \quad \delta_\theta := \sup_{x \in S_\theta^c} \frac{(P_\theta V)(x)}{V(x)} \in (0,1). \tag{27}$$

In addition to Assumptions $(\boldsymbol{S_\Theta})$ *and (27), we assume that, for every* $\theta \in \Theta$*, we have* $\nu_\theta(1_{S_\theta}) > 0$ *where* $(S_\theta, \nu_\theta) \in \mathcal{X} \times \mathcal{M}_*^+$ *is given in* $(\boldsymbol{S_\Theta})$*, so that each* P_θ *is* $V-$*geometrically ergodic, with unique* $P_\theta-$*invariant probability denoted by* π_θ *satisfying* $\pi_\theta(V) < \infty$ *(e.g. see [Bax05]). Moreover suppose that* $K := \sup_{\theta\in\Theta} K_\theta < \infty$ *and* $\delta := \sup_{\theta\in\Theta} \delta_\theta < 1$*. Then*

$$\forall \theta \in \Theta, \quad P_\theta V \leq \delta V + K 1_{S_\theta} \leq V - (1-\delta)V + K 1_{S_\theta}.$$

Note that the second inequality reads as the Condition $\boldsymbol{D}(V_0, V)$*,* $P_\theta V_0 \leq V_0 - V + b1_S$*, with* $V_0 = V/(1-\delta)$*,* $V_1 = V$ *and* $b = K/(1-\delta)$*, so that Theorem 3.2 could be applied here to control* $\|\pi_\theta - \pi_{\theta_0}\|_V$*. Mention that the bound of Theorem 3.2 then provides a generalization of the bounds [LL18, (10) in Th. 2] to the truncation of a transition kernel defined on a general state-space* $\mathbb{X}$ *without assuming the existence of an atom. Similarly the bound of Theorem 3.2 extends that in [LL18, (16) in Th. 3 with* $m = 1$*] to a general state-space* $\mathbb{X}$ *without assuming that the residual kernel is a contraction on* $\mathcal{B}_V$*, i.e.* $RV \leq \beta V$ *for some* $\beta < 1$*.*

The focus here is on the comparison of our results with [HL14, Prop. 2.1] and [RS18, Eq. (3.19)], Thus we only apply Corollary 3.3 in order to control the total variation norm $\|\pi_\theta - \pi_{\theta_0}\|_{TV}$*. Hence, we only use the following Condition* $\boldsymbol{D}(V_0, 1_{\mathbb{X}})$ *derived from* $\boldsymbol{D}(V_0, V)$ *using* $V \geq 1_{\mathbb{X}}$*:*

$$\forall \theta \in \Theta, \quad P_\theta V_0 \leq V_0 - 1_{\mathbb{X}} + b\, 1_{S_\theta} \quad \textit{with} \quad V_0 = \frac{V}{1-\delta} \quad \textit{and} \quad b := \frac{K}{1-\delta}.$$

Therefore, if $m := \inf_{\theta\in\Theta} \nu_\theta(1_{\mathbb{X}}) > 0$*, then* $\{P_\theta\}_{\theta\in\Theta}$ *satisfies the assumptions of Theorem 3.2 and we have from Corollary 3.3*

$$\|\pi_\theta - \pi_{\theta_0}\|_{TV} \leq \frac{2\,a}{1-\delta} \min\big\{\pi_\theta(\Delta_{\theta,V}), \pi_{\theta_0}(\Delta_{\theta,V})\big\} \quad \textit{with} \quad a = 1 + \max\big(0, \frac{b}{m}\big) \tag{28}$$

using the fact that $\Delta_{\theta,V_0}(x) = \Delta_{\theta,V}(x)/(1-\delta)$*. Moreover we have* $\lim_{\theta\to\theta_0} \|\pi_\theta - \pi_{\theta_0}\|_{TV} = 0$*, provided that Condition* $(\boldsymbol{\Delta_{V_0}})$ *is satisfied here with* $V_0 := V$ *(see Corollary 3.3). Recall that, if the term* $\pi_\theta(\Delta_{\theta,V})$ *can be computed and is proved to converge to 0 when* $\theta \to \theta_0$*, then the bound of interest in (28) is*

$$\|\pi_\theta - \pi_{\theta_0}\|_{TV} \leq \frac{2\,a}{1-\delta}\, \pi_\theta(\Delta_{\theta,V}) \tag{29}$$

and that (29) can be obtained under less restrictive assumptions focussing on P_{θ_0} *by using Proposition 3.4 (see also Remark 3.5).*

Now, let us compare Inequality (29) with the bound obtained in [HL14, Prop. 2.1] and [RS18, Eq. (3.19)] (see also [HL23b] for the iterated function systems), that is

$$\|\pi_\theta - \pi_{\theta_0}\|_{TV} \leq C\,\gamma_\theta\, \big|\ln \gamma_\theta\big| \quad \textit{with} \quad \gamma_\theta := \sup_{x\in\mathbb{X}} \frac{\Delta_{\theta,1_{\mathbb{X}}}(x)}{V(x)} \tag{30}$$

where the positive constant C *depends on the above constants* δ, K *and on the* $V-$*geometric rate of convergence of the iterates* P_θ^n *to the invariant distribution* π_θ*.*

- *The interest of the bound (30) is that it uses $\Delta_{\theta,1_{\mathbb{X}}}(x)$ rather than $\Delta_{\theta,V}(x)$ in (29). Note that the supremum bound over $x \in \mathbb{X}$ in the definition of γ_θ only requires to consider this supremum on a level set $\{x \in \mathbb{X} : V(x) \leq c\}$, observing that the supremum on the complementary set is arbitrarily small when c is large enough (use $\Delta_{\theta,1_{\mathbb{X}}}(x)/V(x) \leq 2/c$ when $V(x) > c$).*

- *The drawback of (30) is that it involves a logarithm term, but above all that the constant C in (30) depends on the $V-$geometric rate of convergence of P_θ^n to π_θ, which is unknown in general (or badly estimated).*

In conclusion, to prove that $\lim_{\theta\to\theta_0} \|\pi_\theta - \pi_{\theta_0}\|_{TV} = 0$, it is more relevant to use the results [HL14, RS18]. However, if the term $\pi_\theta(\Delta_{\theta,V})$ can be computed for $\theta \neq \theta_0$ and if $\pi_\theta(\Delta_{\theta,V})$ converges to 0 when $\theta \to \theta_0$, then the bound (29) is much more relevant than (30) since the multiplicative constant in (29) is simple and easily computable, in contrast to that in (30).

Example 3.8 (Random walk on the half line) *For θ belonging to some open metric space Θ, let us consider the random walk $\{X_n^{(\theta)}\}_{n\in\mathbb{N}}$ on the half line $\mathbb{X} := [0, +\infty)$ given by*

$$X_0^{(\theta)} \in \mathbb{X} \quad \textit{and} \quad \forall n \geq 1,\ X_n^{(\theta)} := \max\left(0, X_{n-1}^{(\theta)} + W_n^{(\theta)}\right) \tag{31}$$

where, for every $\theta \in \Theta$, $\{W_n^{(\theta)}\}_{n\geq 1}$ is a sequence of $\mathbb{R}$-valued i.i.d. random variables assumed to be independent of $X_0^{(\theta)}$, and to have a common parametric probability density function w.r.t. the Lebesgue measure on $\mathbb{R}$ which is denoted by $\mathfrak{p}_\theta$. Assume that

$$\beta := \sup_{\theta\in\Theta} \mathbb{E}[W_1^{(\theta)}] < 0 \quad \textit{and} \quad M := \sup_{\theta\in\Theta} \mathbb{E}\big[(\max(0, W_1^{(\theta)}))^m\big] < \infty \tag{32}$$

for some integer number $m \geq 2$. The transition kernel associated with $\{X_n\}_{n\in\mathbb{N}}$ is given by

$$\forall x \in \mathbb{X},\ \forall A \in \mathcal{X}, \quad P_\theta(x, A) = 1_A(0) \int_{-\infty}^{-x} \mathfrak{p}_\theta(y)\, dy + \int_{-x}^{+\infty} 1_A(x+y)\, \mathfrak{p}_\theta(y)\, dy. \tag{33}$$

Next define the following Lyapunov functions on $\mathbb{X}$:

$$\forall x \in \mathbb{X}, \quad V(x) = (1+x)^m, \quad V_0'(x) = (1+x)^{m-1} \quad \textit{and} \quad V_1(x) = (1+x)^{m-2}.$$

Applying to P_θ the modulated drift inequality [JR02, Eq. (59)] and using [JR02, Lem. 3.5], we obtain that there exist a finite interval $S = [0, s_0]$ for some $s_0 > 0$ and positive constants a, b', c, c' such that the following nested modulated drift conditions hold

$$\forall \theta \in \Theta, \quad P_\theta V \leq V - c\, V_0' + a 1_S \quad \textit{and} \quad P_\theta V_0' \leq V_0' - c'\, V_1 + b' 1_S. \tag{34}$$

Note that these constants a, b', c, c' may be chosen independently of θ from the computations in [JR02] and the uniform conditions in (32). Moreover c' in the second drift condition may be chosen such that $c' \in (0, 1]$, so that the following Condition $\boldsymbol{D}_{\boldsymbol{\Theta}}(V_0, V_1)$ holds

$$\forall \theta \in \Theta, \quad P_\theta V_0 \leq V_0 - V_1 + b 1_S \quad \textit{with} \quad V_0 := \frac{V_0'}{c'} \ \textit{and} \ \ b := \frac{b'}{c'}. \tag{35}$$

Next assume that the following function

$$\forall y \in \mathbb{R}, \quad h(y) := \inf_{\theta\in\Theta} \inf_{x\in S} \mathfrak{p}_\theta(y - x)$$

is positive on some open interval of $\mathbb{R}$. *Then the family* $\{P_\theta\}_{\theta\in\Theta}$ *satisfies Condition* $(\boldsymbol{S_\Theta})$ *with* $S_\theta := S$ *and* $\nu_\theta := \nu$, *where* ν *is the positive measure on* $\mathbb{R}$ *defined by*

$$\forall A \in \mathcal{X}, \quad \nu(1_A) := \int_{\mathbb{X}} 1_A(y)\, h(y)\, dy.$$

Thus, the family $\{P_\theta\}_{\theta\in\Theta}$ *satisfies Assumptions* $(\boldsymbol{S_\Theta})$-$\boldsymbol{D_\Theta}(V_0, V_1)$ *w.r.t. the Lyapunov functions* $V_0(x) = (1+x)^{m-1}/c'$ *and* $V_1(x) = (1+x)^{m-2}$, *with moreover* $\sup_{\theta\in\Theta} b_\theta = b < \infty$ *and* $\inf_{\theta\in\Theta} \nu_\theta(1_{\mathbb{X}}) = \nu(1_{\mathbb{X}}) > 0$. *The* $P_\theta-$*invariant probability measure* π_θ *provided by Theorem 2.3 applied to* $P := P_\theta$ *under* $\boldsymbol{D}(V, V_0')$ *in the first drift inequality in (34), satisfies* $\pi_\theta(V_0') < \infty$. *This gives* $\pi_\theta(V_0) < \infty$, *so that Theorem 3.2 and Corollary 3.3 apply. However, for these statements to be relevant, we have to investigate the function* $\Delta_{\theta,V_0}(x)$ *and the quantity* $\pi_\theta(\Delta_{\theta,V_0})$. *To that effect, fixing some* $\theta_0 \in \Theta$, *define*

$$\forall \theta \in \Theta,\ \forall y \in \mathbb{R}, \quad \delta_\theta(y) := |\mathfrak{p}_\theta(y) - \mathfrak{p}_{\theta_0}(y)|$$

$$\text{and} \ \ \forall \theta \in \Theta, \quad \varepsilon_\theta := \int_{\mathbb{R}} \delta_\theta(y)\, dy \quad \text{and} \quad \varepsilon_{\theta,m} := \int_{\mathbb{R}} |y|^{m-1}\, \delta_\theta(y)\, dy.$$

Note that $\varepsilon_\theta \le 2$ *and that* $\varepsilon_{\theta,m} < \infty$ *since* $P_\theta V_0'$ *is everywhere finite. Let* $g \in \mathcal{B}_{V_0}$ *such that* $|g| \le V_0$. *Then we have*

$$\begin{aligned} \forall x \in \mathbb{X}, \quad \big|(P_\theta g)(x) - (P_{\theta_0} g)(x)\big| &\le V_0(0) \int_{-\infty}^{-x} \delta_\theta(y)\, dy + \int_{-x}^{+\infty} V_0(x+y)\, \delta_\theta(y)\, dy \\ &\le \frac{\varepsilon_\theta}{c'} + \frac{c_m}{c'} \int_{\mathbb{R}} \big(1 + x^{m-1} + |y|^{m-1}\big)\, \delta_\theta(y)\, dy \\ &\le \frac{\varepsilon_\theta}{c'} + \frac{d_m\, \varepsilon_\theta}{c'} V_0(x) + \frac{c_m\, \varepsilon_{\theta,m}}{c'} \end{aligned}$$

with some positive constants c_m, d_m *only depending on the function* $t \mapsto (1+t)^{m-1}$ *for* $t \ge 0$. *Thus*

$$\forall x \in \mathbb{X}, \quad \Delta_{\theta,V_0}(x) \le \frac{\varepsilon_\theta + d_m\, \varepsilon_\theta V_0(x) + c_m\, \varepsilon_{\theta,m}}{c'}.$$

Therefore Assumption $(\boldsymbol{\Delta_{V_0}})$ *of Theorem 3.2 holds provided that*

$$\lim_{\theta \to \theta_0} \big(\varepsilon_\theta + \varepsilon_{\theta,m}\big) = 0.$$

Thus, such a condition ensures that $\lim_{\theta\to\theta_0} \|\pi_\theta - \pi_{\theta_0}\|_{V_1} = 0$.

Finally we have

$$\forall \theta \in \Theta, \quad \pi_\theta(\Delta_{\theta,V_0}) \le \frac{\varepsilon_\theta + d_m\, \pi_\theta(V_0)\, \varepsilon_\theta + c_m\, \varepsilon_{\theta,m}}{c'}.$$

Hence the following bounds (see (26))

$$\|\pi_\theta - \pi_{\theta_0}\|_{V_1} \le a\, c_{\theta_0}\, \pi_\theta(\Delta_{\theta,V_0}) \quad \text{and} \quad \|\pi_\theta - \pi_{\theta_0}\|_{TV} \le 2\, a\ \pi_\theta(\Delta_{\theta,V_0}) \tag{36}$$

$$\text{with} \quad a := 1 + \max\left(0, \frac{b - \nu(V_0)}{\nu(1_{\mathbb{X}})}\right) \quad \text{and} \quad c_{\theta_0} = 1 + b$$

are of interest, provided that the quantities ε_θ, $\varepsilon_{\theta,m}$ *and* $\pi_\theta(V_0)$ *are computable for* $\theta \ne \theta_0$ *and that both* ε_θ *and* $\varepsilon_{\theta,m}$ *converge to* 0 *when* $\theta \to \theta_0$.

Note that, for this specific model, it follows from [JT03, Prop. 3.5] that

$$\forall \gamma \in [2, +\infty), \quad \mathbb{E}\big[(\max(0, W_1^{(\theta)}))^\gamma\big] < \infty \iff \int_{\mathbb{R}} |x|^{\gamma-1}\, d\pi_\theta(x) < \infty.$$

Therefore, under Conditions (32), the Lyapunov function V_0 in the modulated drift condition (35) is the greatest possible one providing Condition $\boldsymbol{D}_{\boldsymbol{\Theta}}(V_0, V_1)$ with $\pi_\theta(V_0) < \infty$ for any $\theta \in \Theta$. Accordingly V_1 is the greatest possible Lyapunov function for which a bound can be obtained for $\|\pi_\theta - \pi_{\theta_0}\|_{V_1}$ via Theorem 3.2. Finally note that the property $\lim_{\theta \to \theta_0} \|\pi_\theta - \pi_{\theta_0}\|_{TV} = 0$ and the second bound in (36) hold when the moment condition in (32) is satisfied with $m = 2$.

A Existence of an $P-$invariant probability measure under the minorization Condition $(\boldsymbol{S})$

Let P be a Markov kernel on $(\mathbb{X}, \mathcal{X})$ satisfying Condition $(\boldsymbol{S})$. The next proposition provides, under Condition $(\boldsymbol{S})$, a simple characterization for P to have an invariant probability measure π such that $\pi(1_S) > 0$. Note that the Nummelin-type representation (37) of π is well-known under various recurrence assumptions on the underlying Markov chain $\{X_n\}_{n\in\mathbb{N}}$. The reader can consult [Num84, Th. 5.2, Cor. 5.2]), [MT09, Chap. 10]) where comments on the story of such kind of results are provided. An analytic proof of Proposition A.1 is provided below. Note that we do not need to introduce the concepts of irreducibility, recurrence, atom or splitted chain associated with $\{X_n\}_{n\in\mathbb{N}}$.

Proposition A.1 *If P satisfies the minorization condition $(\boldsymbol{S})$, then the following assertions are equivalent.*

(i) There exists an $P-$invariant probability measure π on $(\mathbb{X}, \mathcal{X})$ such that $\pi(1_S) > 0$.

(ii) $\displaystyle\sum_{k=1}^{+\infty} \nu(R^{k-1} 1_{\mathbb{X}}) < \infty$ with $R := P - \nu(\cdot)1_S$.

Under any of these two conditions

$$\pi := \mu(1_{\mathbb{X}})^{-1}\, \mu \quad \text{with} \quad \mu := \sum_{k=1}^{+\infty} \nu R^{k-1} \in \mathcal{M}_*^+ \tag{37}$$

is an $P-$invariant probability measure on $(\mathbb{X}, \mathcal{X})$ with $\mu(1_S) = 1$ and $\pi(1_S) = \mu(1_{\mathbb{X}})^{-1} > 0$.

Proof. Let P satisfying Condition $(\boldsymbol{S})$ and T be the following kernel

$$\forall x \in \mathbb{X},\ \forall A \in \mathcal{X}, \quad T(x, A) := \nu(1_A)\, 1_S(x)$$

so that $R = P - T$. Note that, for every $k \geq 1$, we have $\nu R^{k-1} \in \mathcal{M}^+$. Recall that for two nonnegative kernels K_1 and K_2, the inequality $K_1 \leq K_2$ means that for any measurable nonnegative function g, $K_1(g) \leq K_2(g)$. Set $T_0 := 0$ and $T_n := P^n - R^n$ for $n \geq 1$. Then

$$\forall n \geq 1,\ 0 \leq T_n \leq P^n,\ T_n - T_{n-1}P = (P^{n-1} - T_{n-1})T \text{ and } T_n = \sum_{k=1}^{n} \nu(R^{k-1}\cdot)P^{n-k}1_S. \tag{38}$$

The first property follows from $0 \le R \le P$. The second one is deduced from $P^n - T_n = (P^{n-1} - T_{n-1})(P - T)$. Finally, the last one is clear for $n = 1$ and it holds for $n \ge 2$ by an easy induction based on $T_n = P^{n-1}T + T_{n-1}R$.

Now, let us prove Proposition A.1. Assume that Assertion (i) holds. We deduce from (38) that $0 \le \pi\big((P^n - T_n)1_{\mathbb{X}}\big) = 1 - \pi(T_n 1_{\mathbb{X}}) = 1 - \pi(1_S)\sum_{k=1}^{n}\nu(R^{k-1}1_{\mathbb{X}})$ from which it follows that $\sum_{k=1}^{+\infty}\nu(R^{k-1}1_{\mathbb{X}}) \le \pi(1_S)^{-1} < \infty$ since $\pi(1_S) > 0$ by hypothesis. This gives Assertion (ii). Conversely, if Assertion (ii) holds, then $\mu := \sum_{k=1}^{+\infty}\nu R^{k-1} \in \mathcal{M}_*^+$ since $\mu(1_{\mathbb{X}}) \ge \nu(1_{\mathbb{X}}) > 0$. Moreover we have

$$\begin{aligned}
\forall A \in \mathcal{X}, \quad \mu(P1_A) &= \sum_{k=1}^{+\infty}\nu\big(P^k 1_A - T_{k-1}P1_A\big) \quad \text{from } R^{k-1} = P^{k-1} - T_{k-1} \\
&= \sum_{k=1}^{+\infty}\nu\big(P^k 1_A - T_k 1_A\big) + \sum_{k=1}^{+\infty}\nu\big(P^{k-1}T1_A - T_{k-1}T1_A\big) \quad \text{from (38)} \\
&= \mu(1_A) + \mu(T1_A) - \nu(1_A) \\
&= \mu(1_A) + \nu(1_A)\mu(1_S) - \nu(1_A) \qquad \text{from the definition of } T.
\end{aligned}$$

With $A = \mathbb{X}$ we obtain that $0 = \nu(1_{\mathbb{X}})\mu(1_S) - \nu(1_{\mathbb{X}})$, thus $\mu(1_S) = 1$ since $\nu(1_{\mathbb{X}}) > 0$. Consequently μ is $P-$invariant, so that $\pi := \mu(1_{\mathbb{X}})^{-1}\mu$ is an $P-$invariant distribution such that $\pi(1_S) = \mu(1_{\mathbb{X}})^{-1} > 0$. □

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