Markov kernels under minorization and modulated drift conditions

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Abstract

This paper is devoted to the study of Markov kernels on general measurable space under a first-order minorization condition and a modulated drift condition. The following issues can be addressed: Existence and uniqueness of invariant measures, recurrence/transience properties including Harris-recurrence property, convergence in total variation of iterates, Poisson's equation, perturbation schemes and rate of convergence of iterates including the so-called geometric ergodicity. All theses issues are discussed in the present document except the perturbation schemes and the non-geometric rate of convergence of iterates, both which will be included soon to form our final text. All the results reported here focus on Markov kernels using a residual kernel approach. This turns out to be a very simple and efficient way to deal with all mentioned issues on Markov kernels. In particular, the document is essentially self-contained.

AMS subject classification : 60J05, 47B34

Keywords : Small set/function; Minorization condition; Modulated drift condition; Invariant probability measure; Recurrence; Harris-recurrence; Poisson's equation; Rate of convergence; Perturbed Markov kernels

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1 Introduction

The purpose of this work is to study Markov kernels on general measurable space under the so-called Minorization and modulated Drift conditions, generically denoted here by M & D conditions. The following issues are addressed: Existence and uniqueness of invariant measures, recurrence/transience properties including Harris-recurrence property, convergence in total variation of iterates of the Markov kernel in the aperiodic and periodic cases, Poisson's equation, perturbation schemes, and finally rates of convergence in weighted total variation norms of iterates including the so-called geometric ergodicity. All these issues are discussed in the present document except the perturbation schemes and the non-geometric rates of convergence of iterates, both which will be included soon to form our final text on Markov kernels under conditions M & D. These two issues will be a revisited version of the material to be found in [HL24a, HL23a]. In this paper, the focus is on non-negative kernels, adopting in this sense the point of view in Seneta's book [Sen06] where discrete Markov chains are studied via non-negative matrices. It can also be thought of as a tribute to Nummelin's book [Num84] from which the idea of the treatment of Markov kernels via a residual kernel approach is borrowed. However, we decide here to keep a total focus on this kernel framework from the beginning to the end. This turns out to be a very simple and efficient way to deal with all mentioned issues on Markov kernels.

The M & D conditions are nowadays well known, widely illustrated and used in the literature on Markov chains via the splitting technique for extending the materials on atomic Markov chains to the non-atomic case, or via the coupling technique to derive convergence rates. Both techniques are based on a minorization condition. The reference books on this topic are [Num84, MT09] and more recently [DMPS18]. Here we use neither the splitting technique, nor the coupling construction. This also implies that no regeneration type-method is used here. Actually, with the exception of Section 6 which contains a few (fairly elementary) spectral theory arguments for studying the geometric ergodicity, the only prerequisite for this work is the handling of non-negative kernels. Indeed, the choice we have made to consider Markov kernels satisfying a minorization condition allows us to work immediately with the residual kernel, from which the issues on invariant measures, recurrence/transience including Harris-recurrence and convergence of iterates, can be treated simply. Then additional modulated drift conditions enable us to investigate series of residual kernel iterates, from which solutions to Poisson's equation and the perturbation issue as a by-product are easily deduced. Also mention that the recent book [BH22] proposes a relevant and interesting study under additional weak topological conditions, such as the weak Feller condition. This point of view is not addressed in our work.

The theory in [Num84, MT09, DMPS18] is developed under general minorization conditions involving, either the so-called definition of small-set (or small-function), or the even more general definition of petit sets. Both of these definitions are based on some $n-$th iterate of the transition kernel. In our work we have chosen to focus on the first order minorization condition with small-function, which corresponds to the definition [Num84, Def. 2.3] at first order ($n := 1$). This choice provides a relatively simple, straightforward, homogeneous and self-contained presentation, dealing first with the residual kernel, then with the Markov kernel. Note that the choice to deal with small-functions instead of small-sets requires here no additional effort. The choice of the order one is also motivated by the fact that most of classical examples of Markov chains verifying a minorization condition satisfy it at the first order. We therefore found it interesting to emphasise the order one, as long as the results are complete and the first-order minorization condition does not need to be strengthened by artificial assumptions.

All the results in this work apply to any discrete-time homogeneous Markov chain satisfying the M & D conditions. For such examples, readers can consult the reference books [Num84, MT09, DMPS18, BH22], as well as the following more specialized works: [FM00, FM03, DFM16] in the context of the Metropolis algorithm, [TT94, DFM16] for autoregressive models, [LH07, LH12] for queueing systems, [JT02] for Markov chains associated with the mean of Dirichlet processes, [Mey08] for Markov models in control. Classical instances of $V-$geometrically ergodic Markov chains can be found in e.g [MT09, RR04, DMPS18].

Although our method differs substantially from the splitting or coupling based methods, the conditions sometimes added to the M & D assumptions are related to the classic ones (e.g. accessibility, irreducibility, period). Here these additional assumptions can be directly introduced under their simplified form, i.e. expressed with the small-function. Finally, as previously quoted, the central point is that a non-negative kernel approach is used for deriving all the proposed material. All the needed prerequisites are recalled in Subsection 2.1. The few probabilistic material you need (see Subsection 2.2) is applying well-known formulas inducing the marginal laws of the Markov chain and the iterates of its transition kernel to deal with Harris-recurrence in Subsection 4.1. Of course, most of statements expressed in terms of Markov kernels in this work can be translated into a purely probabilistic form for discrete-time homogeneous Markov chains with general state space. To facilitate a comparative reading with the statements in reference probabilistic works as [Num84, MT09, DMPS18], the probabilistic interpretation of the main quantities used in this paper is reported in Appendix A. Further discussions are included in bibliographical comments at the end of each section.

2 Main notations and prerequisites

The main notations and definitions used throughout this document are gathered in this section. Most of them are concerned with non-negative kernel calculus. They are standard and the material of this section can be omitted in a first reading.

Let $(\mathbb{X}, \mathcal{X})$ be a measurable space and $\mathcal{X}^* := \mathcal{X} \setminus \{\emptyset\}$ be the subset of non-trivial elements of $\mathcal{X}$. For any $A \in \mathcal{X}^*$, we denote by 1_A the indicator function of A defined by $1_A(x) := 1$ if $x \in A$, and $1_A(x) := 0$ if $x \in A^c$, where $A^c := \mathbb{X} \setminus A$.

2.1 Measures and kernels

- We denote by $\mathcal{B}$ the sets of bounded measurable real-valued functions on $(\mathbb{X}, \mathcal{X})$. The subset of non-zero and non-negative functions in $\mathcal{B}$ is denoted by $\mathcal{B}^*_+$.

- **Non-negative measures on** $(\mathbb{X}, \mathcal{X})$**.** We denote by $\mathcal{M}_+$ (resp. $\mathcal{M}^*_{+,b}$) the set of non-negative (resp. finite positive) measures on $(\mathbb{X}, \mathcal{X})$. For any $\mu \in \mathcal{M}_+$ and any μ-integrable function $g : \mathbb{X} \to \mathbb{R}$, $\mu(g)$ denotes the integral $\int_{\mathbb{X}} g d\mu$. Let μ be a positive measure on $(\mathbb{X}, \mathcal{X})$. A set $A \in \mathcal{X}$ is said to be $\mu-$full if $\mu(1_{A^c}) = 0$.

- **Non-negative kernel on** $(\mathbb{X}, \mathcal{X})$**.** A non-negative kernel K on $(\mathbb{X}, \mathcal{X})$ is a map $K : \mathbb{X} \times \mathcal{X} \to [0, +\infty]$ satisfying the two following properties:

(i) For every $A \in \mathcal{X}$, the function $x \mapsto K(x, A)$ from $\mathbb{X}$ into $[0, +\infty]$ is a measurable function on $(\mathbb{X}, \mathcal{X})$,

(ii) For every $x \in \mathbb{X}$, the set function $A \mapsto K(x, A)$ from $\mathcal{X}$ into $[0, +\infty]$ is a non-negative measure on $(\mathbb{X}, \mathcal{X})$, denoted by $K(x, dy)$ or $K(x, \cdot)$.

The set of non-negative kernels on $(\mathbb{X}, \mathcal{X})$ is denoted by $\mathcal{K}_+$. An element $K \in \mathcal{K}_+$ is said to be **bounded** if the function $x \mapsto K(x, \mathbb{X})$ is bounded on $\mathbb{X}$.

- **Product of two non-negative kernels.** If K_1 and K_2 are in $\mathcal{K}_+$, then K_2K_1 is the element of $\mathcal{K}_+$ defined by

$$\forall x \in \mathbb{X},\ \forall A \in \mathcal{X}, \quad (K_2K_1)(x, A) := \int_{\mathbb{X}} K_1(y, A)\, K_2(x, dy). \tag{1}$$

The above term $(K_2K_1)(x, A)$ is well-defined in $[0, +\infty]$: indeed $y \mapsto K_1(y, A)$ is a measurable function from $\mathbb{X}$ into $[0, +\infty]$, and its integral is then computed w.r.t. the non-negative measure $K_2(x, dy)$. If K_1 and K_2 are both bounded, then so is K_2K_1.

- **Product of a non-negative measure by a non-negative measurable function.** For any $\mu \in \mathcal{M}_+$ and any measurable function $f : \mathbb{X} \to [0, +\infty]$, we define the following non-negative kernel, denoted by $f \otimes \mu$,

$$\forall x \in \mathbb{X},\ \forall A \in \mathcal{X}, \quad (f \otimes \mu)(x, A) := f(x)\, \mu(A). \tag{2}$$

- **Product of a non-negative kernel by a non-negative measure.** Any $\mu \in \mathcal{M}_+$ may be obviously considered as a non-negative kernel (i.e. $\forall x \in \mathbb{X},\ \mu(x, A) := \mu(1_A)$). If $\mu \in \mathcal{M}_+$ and $K \in \mathcal{K}_+$, then the product μK is given as a special case of Definition (1), that is

$$\forall x \in \mathbb{X},\ \forall A \in \mathcal{X}, \quad (\mu K)(x, A) := \int_{\mathbb{X}} K(y, A)\, \mu(dy). \tag{3}$$

Note that $\mu K \in \mathcal{M}_+$ since it does not depend on $x \in \mathbb{X}$. The measure μ is said to be $K-$invariant if $\mu K = \mu$.

- **Iterates of a non-negative kernel.** Let $K \in \mathcal{K}_+$. For every $n \geq 1$ the $n-$th iterate kernel of K, denoted by K^n, is the element of $\mathcal{K}_+$ defined by induction using the above formula (1). By convention K^0 is defined by: $\forall x \in \mathbb{X},\ \forall A \in \mathcal{X},\ K^0(x, A) = 1_A(x)$ (i.e. $K^0(x, \cdot)$ is the Dirac measure at x).

- **Functional action of a non-negative kernel.** Let $K \in \mathcal{K}_+$. We also denote by K its functional action defined by

$$\forall x \in \mathbb{X}, \quad (Kg)(x) := \int_{\mathbb{X}} g(y)\, K(x, dy), \tag{4}$$

where $g : \mathbb{X} \to \mathbb{R}$ is any measurable function assumed to be $K(x, \cdot)-$integrable for every $x \in \mathbb{X}$. For such a function g, we have

$$|Kg| \leq K|g|, \tag{5}$$

where $|g|$ denotes the absolute value of g (or its modulus if g is $\mathbb{C}-$valued) since

$$\forall x \in \mathbb{X}, \quad |(Kg)(x)| = \left| \int_{\mathbb{X}} g(y)\, K(x, dy) \right| \leq \int_{\mathbb{X}} |g(y)|\, K(x, dy) = (K|g|)(x).$$

Obviously K is a linear action.

If $K_1, K_2 \in \mathcal{K}_+$ and if $g : \mathbb{X} \to \mathbb{R}$ is a measurable function such that $g_1 := K_1 g$ is well-defined as well as $K_2 g_1$, then

$$(K_2 K_1)(g) = (K_2 \circ K_1)(g)$$

where the first term $(K_2 K_1)(g)$ denotes the functional action on g of the product kernel $K_2 K_1$ given in (1), while $K_2 \circ K_1$ denotes the usual composition of maps. In particular, for every $n \geq 1$, the functional action of the $n-$th iterate kernel of K^n of K is the $n-$th iterate for composition of the functional action of K. Finally note that the functional action of the kernel K^0 is the identity map I (i.e. $(K^0 g)(x) = g(x)$), which corresponds to the standard convention for linear operators.

Most questions involving a non-negative kernel can be addressed through its functional action, and this is the choice that will generally be made in this paper. In particular Property (5) *will be used repeatedly in this work.*

- **Functional action of a non-negative measure.** If $\mu \in \mathcal{M}_+$ (thus $\mu \in \mathcal{K}_+$), then its functional action (see (4)) is given by

  $$\forall x \in \mathbb{X},\ (\mu g)(x) := \int_{\mathbb{X}} g(y)\, \mu(dy), \quad \text{that is} \quad \mu g := \mu(g) 1_{\mathbb{X}},$$

  provided that g is $\mu-$integrable.

- **Order relation for non-negative kernels.** If K_1 and K_2 are in $\mathcal{K}_+$, the inequality $K_1 \leq K_2$ means that

  $$\forall x \in \mathbb{X},\ \forall A \in \mathcal{X}, \quad K_1(x, 1_A) \leq K_2(x, 1_A).$$

  In others words, $K_1 \leq K_2$ if $K := K_2 - K_1$ is a non-negative kernel, where K is defined by $K(x, 1_A) := K_2(x, 1_A) - K_1(x, 1_A)$ for any $x \in \mathbb{X}$ and $A \in \mathcal{X}$. Thus, for any $K_1, K_2 \in \mathcal{K}_+$, we have $K_1 \leq K_2$ if, and only if, the following property holds

  $$\forall g : \mathbb{X} \to [0, +\infty) \text{ measurable}, \quad 0 \leq K_1 g \leq K_2 g$$

  provided that $K_1 g$ and $K_2 g$ are well-defined (if not, this inequality still holds but in $[0, +\infty]$). In connection with this order relation, we shall often write $K \geq 0$ for recalling that $K \in \mathcal{K}_+$. Recall that

  $$K_1, K_2 \in \mathcal{K}_+ \implies K_1 K_2 \in \mathcal{K}_+ \text{ and } K_2 K_1 \in \mathcal{K}_+$$

  from the definition of the products of two elements of $\mathcal{K}_+$ (see (1)). From this, the following expected rules for sum and product can be easily deduced for any K, K_1, K_2, K_1', K_2' in $\mathcal{K}_+$ (i.e. each element in (6a)-(6c) is a non-negative kernel):

  $$K_1 \leq K_2,\ K_1' \leq K_2' \implies K_1 + K_1' \leq K_2 + K_2' \tag{6a}$$
  $$K_1 \leq K_2,\ K \in \mathcal{K}_+ \implies K K_1 \leq K K_2 \text{ and } K_1 K \leq K_2 K.. \tag{6b}$$
  $$K_1 \leq K_2, \implies \forall n \geq 0,\ K_1^n \leq K_2^n. \tag{6c}$$

  Properties (6a)–(6c) *will be used repeatedly hereafter, mainly through the functional action of the involved non-negative kernels.*

- **Series of kernels.** For any $(K_i)_{i\in I} \in \mathcal{K}_+^I$ where I is any countable set I, the element $K := \sum_{i\in I} K_i$ is defined in $\mathcal{K}_+$ by

$$\forall x \in \mathbb{X},\ \forall A \in \mathcal{X}, \quad K(x, A) := \sum_{i\in I} K_i(x, A).$$

The following formula holds for all sequences $(K_n)_{n\geq 0} \in \mathcal{K}_+^{\mathbb{N}}$ and $(K'_n)_{n\geq 0} \in \mathcal{K}_+^{\mathbb{N}}$:

$$\sum_{k,n=0}^{+\infty} K_n K'_k = K\,K' \quad \text{with} \quad K := \sum_{n=0}^{+\infty} K_n \quad \text{and} \quad K' := \sum_{k=0}^{+\infty} K'_k. \tag{7}$$

Since this formula is repeatedly used in this work, let us give a proof. Let $x \in \mathbb{X}$ and $A \in \mathcal{X}$. Then (7) is obtained from the following equalities in $[0, +\infty]$:

$$\begin{aligned}
\sum_{k,n=0}^{+\infty} (K_n K'_k)(x, A) &= \sum_{k,n=0}^{+\infty} \int_{\mathbb{X}} K'_k(y, A)\, K_n(x, dy) \\
&= \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{+\infty} \int_{\mathbb{X}} K'_k(y, A)\, K_n(x, dy) \right) \\
&= \sum_{n=0}^{+\infty} \int_{\mathbb{X}} \left(\sum_{k=0}^{+\infty} K'_k(y, A) \right) K_n(x, dy) \\
&= \sum_{n=0}^{+\infty} \int_{\mathbb{X}} K'(y, A) K_n(x, dy) = \int_{\mathbb{X}} K'(y, A) K(x, dy).
\end{aligned}$$

Indeed the first equality is just the definition of $K_n K'_k$, the second one is due to Fubini's theorem for double series of non-negative real numbers, the third one follows from the monotone convergence theorem w.r.t. each non-negative measure $K_n(x, dy)$, and finally the fourth and fifth ones are due to the definition of $K'(y, A)$ and $K(x, dy)$ respectively.

- **Markov and submarkov kernels.** A non-negative kernel K is said to be Markov (respectively submarkov) if $K(x, \mathbb{X}) = 1$ (respectively $K(x, \mathbb{X}) \leq 1$) for any $x \in \mathbb{X}$. In both cases, K is obviously a bounded kernel.

  If K is a Markov kernel, then an element $A \in \mathcal{X}$ is said to be $K-$absorbing if $K(x, A) = 1$ for any $x \in A$. An element $A \in \mathcal{X}$ is said to be an atom for K if the following condition holds: $\forall (x_1, x_2) \in A^2,\ K(x_1, dy) = K(x_2, dy)$ (such a set is sometimes called a proper atom too, e.g. see [Num84]).

  If K is a submarkov kernel, then $K(\mathcal{B}) \subset \mathcal{B}$. A function $g \in \mathcal{B}$ is said to be $K-$harmonic if $Kg = g$ on $\mathbb{X}$. When K is Markov, then the function $1_{\mathbb{X}}$ is always $K-$harmonic.

- **Restriction of functions, measures ans kernels to a subset.** For any $E \in \mathcal{X}$ we denote by $\mathcal{X}_E$ the $\sigma-$algebra induced by $\mathcal{X}$ on the set E, i.e. $\mathcal{X}_E := \{A \cap E, A \in \mathcal{X}\}$. For any $g \in \mathcal{B}$, the restriction g_E to E of g is the bounded $\mathcal{X}_E-$measurable function defined on E by: $\forall x \in E,\ g_E(x) = g(x)$. If $\eta \in \mathcal{M}_+$, then the restriction η_E to E of η is the non-negative measure on $(E, \mathcal{X}_E)$ defined by: $\forall A' \in \mathcal{X}_E,\ \eta_E(1_{A'}) = \eta(1_{A\cap E})$ where A is any element in $\mathcal{X}$ such that $A' = A \cap E$. If $K \in \mathcal{K}_+$, then the restriction K_E of K to E is the non-negative kernel on $(E, \mathcal{X}_E)$ defined by: $\forall x \in E,\ \forall A' \in \mathcal{X}_E,\ K_E(x, A') =$

$K(x, A\cap E)$ where A is any element in $\mathcal{X}$ such that $A' = A \cap E$. When the notation of the function/measure/kernel on $\mathbb{X}$ involves an index, the restriction to E is denoted by $\cdot_{|E}$ to avoid confusion (for instance, if $\eta_i \in \mathcal{M}_+$, the restriction of η_i to E is denoted by $\eta_{i|E}$). Finally observe that, if K is Markov on $(\mathbb{X}, \mathcal{X})$ and E is $K-$absorbing, then K_E is a Markov kernel on $(E, \mathcal{X}_E)$.

- **$V-$weighted space and $V-$weighted total variation norm.** Let $V : \mathbb{X} \to (0, +\infty)$ be any measurable function. For every measurable function $g : \mathbb{X} \to \mathbb{R}$, we set

$$\|g\|_V := \sup_{x\in\mathbb{X}} \frac{|g(x)|}{V(x)} \in [0, +\infty],$$

and we define the $V-$weighted space

$$\mathcal{B}_V := \big\{g : \mathbb{X} \to \mathbb{R}, \text{measurable such that } \|g\|_V < \ \infty\big\}.$$

Note that $\mathcal{B}_{1_{\mathbb{X}}} = \mathcal{B}$. The following obvious fact will be repeatedly used hereafter:

$$\forall g \in \mathcal{B}_V, \quad |g| \leq \|g\|_V\, V, \quad \text{ i.e. } \forall x \in \mathbb{X},\ |g(x)| \leq \|g\|_V V(x).$$

If $(\mu_1, \mu_2) \in (\mathcal{M}^*_{+,b})^2$ is such that $\mu_i(V) < \infty, i = 1, 2$, then the V-weighted total variation norm $\|\mu_1 - \mu_2\|'_V$ is defined by

$$\|\mu_1 - \mu_2\|'_V := \sup_{\|g\|_V \leq 1} \big|\mu_1(g) - \mu_2(g)\big|. \tag{8}$$

If $V = 1_{\mathbb{X}}$, then $\|\cdot\|'_{1_{\mathbb{X}}} = \|\cdot\|_{TV}$ is the standard total variation norm.

2.2 Markov chain

A Markov chain $(X_n)_{n\geq0}$ on the state space $\mathbb{X}$ with transition/Markov kernel P is a family of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\forall f \in \mathcal{B}, \quad \mathbb{E}[f(X_{n+1}) \mid \sigma(X_0, \ldots, X_n)] = (Pf)(X_n)$$

where $\sigma(X_0, \ldots, X_n)$ is the sub-$\sigma-$algebra of $\mathcal{F}$ generated by the r.v's $X_0, \ldots, X_n$. In particular, for any $A \in \mathcal{X}$,

$$\mathbb{E}[1_A(X_{n+1}) \mid \sigma(X_0, \ldots, X_n)] = (P1_A)(X_n) = \int_A P(x, dy) = P(x, A).$$

Assertions a)-b) below are relevant to link iterated kernels and the Markov chain. The classical statements c)-d) are prerequisites on occupation and hitting times of a set A, which are only used in Subsection 4.1 to study the Harris-recurrence property.

a) We have for any $k \geq 0$, $\mathbb{E}[f(X_{n+k}) \mid \sigma(X_0, \ldots, X_n)] = (P^k f)(X_n)$.

b) The probability $\mathbb{P}$ when $\mathbb{P}\{X_0 = x\} = 1$, is denoted by $\mathbb{P}_x$, and $\mathbb{E}_x$ is the expectation under $\mathbb{P}_x$.

c) Let $A \in \mathcal{X}$. Then the function defined by

$$\forall x \in \mathbb{X}, \quad g_A^\infty(x) := \mathbb{P}_x\Big\{\sum_{n=1}^{+\infty} 1_{\{X_n \in A\}} = +\infty\Big\} \tag{9}$$

is bounded on $\mathbb{X}$ and $P-$harmonic, e.g. see [DMPS18, Prop. 4.2.4], [Num84, Th. 3.4].

d) Let $A \in \mathcal{X}$ and let g_A be the function on $\mathbb{X}$ defined by

$$\forall x \in \mathbb{X}, \quad g_A(x) = \mathbb{P}_x\{T_A < \infty\} \tag{10}$$

where $T_A := \inf\{n \geq 0 : X_n \in A\}$ is the hitting time of the set A. Then g_A is superharmonic, i.e. $Pg_A \leq g_A$, and we have (e.g. see [Num84, Th. 3.4], [DMPS18, Th. 4.1.3]):

$$g_A^\infty = \lim_{n\to+\infty} \searrow P^n g_A. \tag{11}$$

3 Minorization condition, invariant measure and recurrence

In this section a standard first-order minorization condition on the Markov kernel P is introduced: $P \geq \psi \otimes \nu$ where $\nu \in \mathcal{M}_{+,b}^*$ and $\psi \in \mathcal{B}_+^*$. This allows us to decompose P as the sum of two submarkovian kernels $R := P - \psi \otimes \nu$, called the residual kernel, and $\psi \otimes \nu$. Two quantities of interest are defined from the residual kernel and its iterates: first the positive measure $\mu_R := \sum_{k=0}^{+\infty} \nu R^k$, second the $R-$harmonic function $h_R^\infty := \lim_n R^n 1_{\mathbb{X}}$. Then the existence of a $P-$invariant positive measure and the classical recurrence/transience dichotomy are studied according that $\mu_R(\psi) = 1$ or not (equivalently $\nu(h_R^\infty) = 0$ or not).

3.1 The minorization condition $(\boldsymbol{M}_{\nu,\psi})$ and the residual kernel

Recall that $\mathcal{B}_+^*$ is the set of non-negative and non-zero measurable bounded functions on $\mathbb{X}$ and that $\mathcal{M}_{+,b}^*$ is the set of finite positive measures on $(\mathbb{X}, \mathcal{X})$. Let P be a Markov kernel on $(\mathbb{X}, \mathcal{X})$. Let us introduce the *minorization condition* which is in force throughout this paper:

$$\exists(\nu, \psi) \in \mathcal{M}_{+,b}^* \times \mathcal{B}_+^* : \quad P \geq \psi \otimes \nu \quad (\text{i.e. } \forall x \in \mathbb{X},\ P(x, dy) \geq \psi(x)\,\nu(dy)). \tag{$\boldsymbol{M}_{\nu,\psi}$}$$

The function ψ is called a first-order *small-function* in the literature on the topic of Markov chains. That the non-negative function ψ in $(\boldsymbol{M}_{\nu,\psi})$ is bounded is required since $\psi(x)\,\nu(1_{\mathbb{X}}) \leq P(x, \mathbb{X}) = 1$ for any $x \in \mathbb{X}$ and $\nu(1_{\mathbb{X}}) > 0$. Moreover for any $(\psi, \phi) \in \mathcal{B}_+^* \times \mathcal{B}_+^*$ such that $\psi \geq \phi$, if $(\boldsymbol{M}_{\nu,\psi})$ is satisfied then so is $(\boldsymbol{M}_{\nu,\phi})$.

Under $(\boldsymbol{M}_{\nu,\psi})$, we can introduce the following submarkov kernel, called the *residual kernel*, which is central in our analysis of the Markov kernel P:

$$R \equiv R_{\nu,\psi} := P - \psi \otimes \nu \quad (\text{i.e. } \forall x \in \mathbb{X},\ R(x, dy) := P(x, dy) - \psi(x)\nu(dy)). \tag{12}$$

The most classical instance of minorization condition is when $\psi := 1_S$ for some $S \in \mathcal{X}^*$, that is

$$\exists(\nu, S) \in \mathcal{M}_{+,b}^* \times \mathcal{X}^* : \quad P \geq 1_S \otimes \nu \quad (\text{i.e. } \forall x \in \mathbb{X},\ P(x, dy) \geq 1_S(x)\,\nu(dy)), \tag{$\boldsymbol{M}_{\nu,1_S}$}$$

in which case the residual kernel is:

$$R \equiv R_{\nu,1_S} := P - 1_S \otimes \nu.$$

Such a set S is called a first-order *small-set.*

The following statement provides a general framework for Condition $(\boldsymbol{M}_{\nu,\psi})$ to hold. Moreover this proposition shows that, even if the minorizing measure ν is defined from $(\boldsymbol{M}_{\nu,1_S})$ with some set S, this condition $(\boldsymbol{M}_{\nu,1_S})$ is not the only one possible.

Proposition 3.1 *Assume that*

$$\forall x \in \mathbb{X}, \quad P(x, dy) \geq q(x, y)\, \lambda(dy) \tag{13}$$

where $q(\cdot, \cdot)$ is a non-negative measurable function on $\mathbb{X}^2$ and λ is a positive measure on $\mathbb{X}$. Let $S \in \mathcal{X}^*$ be such that the measurable non-negative function q_S defined by

$$\forall y \in \mathbb{X}, \quad q_S(y) := \inf_{x \in S} q(x, y)$$

is not $\lambda-$null, that is: $\lambda(1_A) > 0$ where $A := \{y \in \mathbb{X} : q_S(y) > 0\}$. Let $\nu \in \mathcal{M}^*_{+,b}$ and $\psi_S \geq 1_S$ be defined by

$$\nu(dy) := q_S(y)\lambda(dy) \quad \text{and} \quad \forall x \in \mathbb{X},\ \psi_S(x) := 1_S(x) \inf_{y \in A} \frac{q(x, y)}{q_S(y)}. \tag{14}$$

Then P satisfies Condition $(\boldsymbol{M}_{\nu,\psi_S})$ and so $(\boldsymbol{M}_{\nu,1_S})$.

Proof. For any fixed $x \in S$, we have $\nu(1_{\mathbb{X}}) \leq \int_{\mathbb{X}} q(x, y)\lambda(dy) \leq P(x, \mathbb{X}) = 1$ from the definition of ν, q_S and from (13). Thus ν is finite and $\nu(1_A) > 0$, so that $\nu \in \mathcal{M}^*_{+,b}$. Next, from the definition of ψ_S we obtain the following property: $\forall (x, y) \in S \times A,\ q(x, y) \geq q_S(y)\, \psi_S(x)$. In fact this inequality holds for every $(x, y) \in \mathbb{X}^2$ since $q(x, y) \geq 0$. Finally it follows from (13) that, for every $x \in \mathbb{X}$, we have $P(x, dy) \geq \psi_S(x)q_S(y)\lambda(dy)$, i.e. P satisfies $(\boldsymbol{M}_{\nu,\psi_S})$. Note that $\psi_S \geq 1_S$ from the definition of the function q_S, so that $(\boldsymbol{M}_{\nu,1_S})$ is satisfied. □

The next kernel identity (16) is the first key formula of this work. Recall that the residual kernel $R = P - \psi \otimes \nu$ is a submarkov kernel, so that the $n-$th iterate kernel R^n of R defined by induction using Formula (1) is a submarkov kernel too. Also recall that by convention $R^0(x, \cdot)$ is the Dirac measure at x. Finally note that, for every $k \geq 1$, we have $\nu\, R^k \in \mathcal{M}_{+,b}$ (see (3)).

Lemma 3.2 *Let P satisfy Condition $(\boldsymbol{M}_{\nu,\psi})$. Then we have*

$$\forall n \geq 1, \quad 0 \leq R^n \leq P^n, \tag{15}$$

$$P^n = R^n + \sum_{k=1}^{n} P^{n-k}\psi \otimes \nu R^{k-1}, \tag{16}$$

and

$$\sum_{n=0}^{+\infty} P^n = \sum_{n=0}^{+\infty} R^n + \Big(\sum_{n=0}^{+\infty} P^n\psi\Big) \otimes \Big(\sum_{k=0}^{+\infty} \nu R^k\Big). \tag{17}$$

Proof. We have $0 \leq R \leq P$, thus $0 \leq R^n \leq P^n$ using (6c). Set $T_0 := 0$ and $T_n := P^n - R^n$ for $n \geq 1$. Note that Property (16) is equivalent to

$$\forall n \geq 1, \quad T_n = \sum_{k=1}^{n} P^{n-k}\psi \otimes \nu R^{k-1}. \tag{18}$$

Equality (18) is clear for $n = 1$ since $T_1 = P - R = \psi \otimes \nu$. Next we have for any $n \geq 2$

$$P^n - T_n = R^n = R^{n-1}R = (P^{n-1} - T_{n-1})(P - T_1),$$

so that $T_n = P^{n-1}T_1 + T_{n-1}R$. Then (18) holds for $n \geq 2$ by an easy induction based on the previous equality for T_n: For instance use the functional action of kernels to check that, for every $g \in \mathcal{B}$, if $T_{n-1}g = \sum_{k=1}^{n-1} \nu(R^{k-1}g)P^{n-1-k}\psi$, then $T_n g = \sum_{k=1}^{n} \nu(R^{k-1}g)P^{n-k}\psi$.

From (16) and the convention for $P^0 = R^0$ we obtain that (see (7))

$$\begin{aligned}
\sum_{n=0}^{+\infty} P^n &= \sum_{n=0}^{+\infty} R^n + \sum_{n=1}^{+\infty}\sum_{k=1}^{n} P^{n-k}\psi \otimes \nu R^{k-1} = \sum_{n=0}^{+\infty} R^n + \sum_{k=1}^{+\infty}\sum_{n=k}^{+\infty} P^{n-k}\psi \otimes \nu R^{k-1} \\
&= \sum_{n=0}^{+\infty} R^n + \left(\sum_{n=0}^{+\infty} P^n\psi\right) \otimes \left(\sum_{k=0}^{+\infty} \nu R^k\right)
\end{aligned}$$

Thus (17) holds and the proof of Lemma 3.2 is complete. □

Under Condition ($\boldsymbol{M}_{\nu,\psi}$), we have $0 \leq R1_{\mathbb{X}} \leq 1_{\mathbb{X}}$. Since R is a non-negative kernel, we get $0 \leq R^{n+1}1_{\mathbb{X}} \leq R^n 1_{\mathbb{X}}$ for any $n \geq 0$. Thus the sequence $(R^n 1_{\mathbb{X}})_{n\geq 0}$ is non-increasing so that it converges point-wise. Consequently we can define the following measurable function $h_R^\infty : \mathbb{X} \to [0,1]$:

$$h_R^\infty := \lim_n \searrow R^n 1_{\mathbb{X}}. \tag{19}$$

Note that h_R^∞ is $R-$harmonic: indeed, for every $x \in \mathbb{X}$, we have $(R^{n+1}h_R^\infty)(x) = (RR^n h_R^\infty)(x)$, so that $h_R^\infty(x) = (Rh_R^\infty)(x)$ from Lebesgue's theorem applied to the finite non-negative measure $R(x, dy)$ observing that $R^n h_R^\infty \leq R^n 1_{\mathbb{X}} \leq 1_{\mathbb{X}}$.

Recall that, for every $k \geq 0$, we have $\nu R^k \in \mathcal{M}_{+,b}$ (see (3)). Under Condition ($\boldsymbol{M}_{\nu,\psi}$) let μ_R denote the positive measure on $(\mathbb{X}, \mathcal{X})$ (not necessarily finite) defined by

$$\mu_R := \sum_{k=0}^{+\infty} \nu R^k. \tag{20}$$

The measure μ_R is positive from $\mu_R(1_{\mathbb{X}}) \geq \nu(1_{\mathbb{X}}) > 0$. The measure μ_R as well as the function h_R^∞ are used throughout this section.

3.2 $P-$invariant measure

First prove the following simple lemma.

Lemma 3.3 *Assume that P satisfies Conditions ($\boldsymbol{M}_{\nu,\psi}$). Let g be a $P-$harmonic function. Then we have*

$$\forall n \geq 0, \quad \nu(g)\sum_{k=0}^{n} R^k\psi = g - R^{n+1}g. \tag{21}$$

In particular we have

$$\forall n \geq 0, \quad 0 \leq \nu(1_{\mathbb{X}}) \sum_{k=0}^{n} R^k \psi = 1_{\mathbb{X}} - R^{n+1} 1_{\mathbb{X}} \leq 1_{\mathbb{X}}. \tag{22}$$

Proof. Let $g \in \mathcal{B}$ be such that $Pg = g$. We have $\nu(g)\psi = (I - R)g$ from the definition (12) of R. Then Property (21) follows from

$$\forall n \geq 0, \quad \nu(g) \sum_{k=0}^{n} R^k \psi = \left(\sum_{k=0}^{n} R^k \right)(I - R)g = \sum_{k=0}^{n} R^k g - \sum_{k=1}^{n+1} R^k g = g - R^{n+1} g.$$

Since $P 1_{\mathbb{X}} = 1_{\mathbb{X}}$, Property (21) with $g := 1_{\mathbb{X}}$ is nothing else than (22). □

Recall that the positive measure ν in $(\boldsymbol{M}_{\nu,\psi})$ is finite (i.e. $\nu(1_{\mathbb{X}}) < \infty$).

Proposition 3.4 *Let P satisfy Condition $(\boldsymbol{M}_{\nu,\psi})$. Then the function series $\sum_{k=0}^{+\infty} R^k \psi$ point-wise converges and is bounded on $\mathbb{X}$. More precisely we obtain that*

$$0 \leq \nu(1_{\mathbb{X}}) \sum_{k=0}^{+\infty} R^k \psi = 1_{\mathbb{X}} - h_R^{\infty} \leq 1_{\mathbb{X}}. \tag{23}$$

Moreover we have $\mu_R(\psi) = \sum_{k=0}^{+\infty} \nu(R^k \psi) \in [0, 1]$, and the following equivalences hold

$$\mu_R(\psi) = 1 \iff \nu(h_R^{\infty}) = 0 \iff \mu_R(h_R^{\infty}) = 0. \tag{24}$$

The property $\mu_R(\psi) \leq 1$ proved above implies that there exists $A \in \mathcal{X}^*$ such that $\mu_R(1_A) < \infty$.

Proof. It follows from (22) that the series of non-negative functions $\sum_{k=0}^{+\infty} R^k \psi$ point-wise converges. When n growths to $+\infty$ in (22), we get the equality in (23) from the definition (19) of h_R^{∞}.

Next integrate w.r.t. the measure ν in (23) and apply the monotone convergence theorem to get $0 \leq \nu(1_{\mathbb{X}})\mu_R(\psi) = \nu(1_{\mathbb{X}}) - \nu(h_R^{\infty}) \leq \nu(1_{\mathbb{X}})$. Since $\nu(1_{\mathbb{X}}) > 0$, it follows that $\mu_R(\psi) \in [0, 1]$ and the first equivalence in (24) holds. Since $R h_R^{\infty} = h_R^{\infty}$, we have from (20) that $\nu(h_R^{\infty}) = 0$ implies that $\mu_R(h_R^{\infty}) = 0$. Finally, we have $\mu_R(h_R^{\infty}) \geq \nu(h_R^{\infty}) \geq 0$ from the definition (20) of μ_R so that $\mu_R(h_R^{\infty}) = 0$ implies that $\nu(h_R^{\infty}) = 0$. The proof of the second equivalence in (24) is complete. □

Theorem 3.5 ($P-$invariant positive measure) *Assume that P satisfies Condition $(\boldsymbol{M}_{\nu,\psi})$. Then the following assertions hold.*

1. *If $\mu_R(\psi) = 1$ (or equivalently $\nu(h_R^{\infty}) = 0$), then μ_R is a $P-$invariant positive measure.*
2. *If there exists $\zeta \in \mathcal{B}_+^*$ such that $\nu(\zeta) > 0$ and $\mu_R(P\zeta) = \mu_R(\zeta) < \infty$, then we have $\mu_R(\psi) = 1$.*

In particular, if $\nu(\psi) > 0$, then

$$\mu_R \textit{ is } P-\textit{invariant} \iff \mu_R(\psi) = 1 \iff \nu(h_R^{\infty}) = 0 \iff \mu_R(h_R^{\infty}) = 0.$$

Recall that the condition $\nu(\psi) > 0$ is the so-called *strong aperiodicity* property.

Proof. From the definitions (12) of R and (20) of μ_R, the following equalities hold in $[0, +\infty]$:

$$\forall A \in \mathcal{X}, \quad \mu_R(P1_A) = \mu_R(R1_A) + \nu(1_A)\mu_R(\psi) = \mu_R(1_A) + \nu(1_A)\big(\mu_R(\psi) - 1\big)$$

since we have $\mu_R(R1_A) = \mu_R(1_A) - \nu(1_A)$ in $[0, +\infty]$. Consequently, if $\mu_R(\psi) = 1$, then μ_R is a $P-$invariant positive measure and Assertion 1. is proved. Next, if $\zeta \in \mathcal{B}^*_+$ satisfies the assumptions in Assertion 2., then we deduce from $\mu_R(\zeta) = \mu_R(P\zeta) = \mu_R(\zeta) + \nu(\zeta)\big(\mu_R(\psi) - 1\big)$ that $\mu_R(\psi) = 1$. In the last assertion, that $\mu_R(\psi) = 1$ implies the $P-$invariance of μ_R is just Assertion 1. Next, if $\nu(\psi) > 0$ and μ_R is $P-$invariant, then Assertion 2. can be applied to $\zeta := \psi$ since we know that $\mu_R(\psi) < \infty$ from Proposition 3.4, so that we have $\mu_R(\psi) = 1$. The two last equivalences are (24). □

Theorem 3.6 ($P-$invariant probability measure) *If P satisfies Condition $(\boldsymbol{M}_{\nu,\psi})$, then the following assertions are equivalent.*

1. *There exists a $P-$invariant probability measure η on $(\mathbb{X}, \mathcal{X})$ such that $\eta(\psi) > 0$.*
2. $\mu_R(1_{\mathbb{X}}) = \sum_{k=0}^{+\infty} \nu(R^k 1_{\mathbb{X}}) < \infty$.

Under any of these two conditions, the following probability measure on $(\mathbb{X}, \mathcal{X})$

$$\pi_R := \mu_R(1_{\mathbb{X}})^{-1}\, \mu_R \quad \textit{with} \quad \mu_R := \sum_{k=0}^{+\infty} \nu R^k \in \mathcal{M}^+_{*,b} \tag{25}$$

is $P-$invariant with $\mu_R(\psi) = 1$ and $\pi_R(\psi) = \mu_R(1_{\mathbb{X}})^{-1} > 0$.

Proof. Assume that Assertion 1. holds. Then apply Formula (16) to $1_{\mathbb{X}}$ and compose on the left by η to get $1 = \eta(R^n 1_{\mathbb{X}}) + \eta(\psi)\sum_{k=1}^{n} \nu(R^{k-1} 1_{\mathbb{X}})$. It follows that

$$0 \le \eta\big(R^n 1_{\mathbb{X}}\big) = 1 - \eta(\psi) \sum_{k=1}^{n} \nu(R^{k-1} 1_{\mathbb{X}})$$

from which we deduce that $\mu_R(1_{\mathbb{X}}) = \sum_{k=1}^{+\infty} \nu(R^{k-1} 1_{\mathbb{X}}) \le \eta(\psi)^{-1} < \infty$ since $\eta(\psi) > 0$ by hypothesis. This proves that Assertion 1. implies Assertion 2.

Conversely, if Assertion 2. holds, then Assertion 2. of Theorem 3.5 can be applied with $\zeta := 1_{\mathbb{X}}$. Indeed, $\nu(1_{\mathbb{X}}) > 0$ and $\mu_R(P1_{\mathbb{X}}) = \mu_R(1_{\mathbb{X}}) < \infty$ since P is Markov. Hence we have $\mu_R(\psi) = 1$, so that μ_R is $P-$invariant from Assertion 1. of Theorem 3.5. Thus $\pi_R := \mu_R(1_{\mathbb{X}})^{-1}\, \mu_R$ is a $P-$invariant probability measure such that $\pi_R(\psi) = \mu_R(1_{\mathbb{X}})^{-1} > 0$. □

The following standard example of uniform ergodicity illustrates Theorem 3.6. Moreover, the well-known rate of convergence of $\|P^n(x, \cdot) - \pi_R(\cdot)\|_{TV}$ is obtained from Formula (16).

Example 3.7 (Uniform ergodicity) *Let P satisfy Condition $(\boldsymbol{M}_{\nu,1_{\mathbb{X}}})$, that is there exists $\nu \in \mathcal{M}^*_{+,b}$ such that $P \ge 1_{\mathbb{X}} \otimes \nu$. In other words the whole state space $\mathbb{X}$ is a first-order small-set for P. Then Condition 2. of Theorem 3.6 holds and we have*

$$\forall n \ge 1,\ \forall x \in \mathbb{X}, \quad \|P^n(x, \cdot) - \pi_R(\cdot)\|_{TV} \le 2(1 - \nu(1_{\mathbb{X}}))^n$$

where π_R is the $P-$invariant probability measure given by (25)*. Indeed the residual kernel $R \equiv R_{\nu,1_{\mathbb{X}}}$ is here $R = P - 1_{\mathbb{X}} \otimes \nu$ so that we have $R1_{\mathbb{X}} = (1 - \nu(1_{\mathbb{X}}))1_{\mathbb{X}}$. Consequently we obtain that*

$$\forall n \geq 1, \quad R^n 1_{\mathbb{X}} = (1 - \nu(1_{\mathbb{X}}))^n 1_{\mathbb{X}}.$$

Thus $\mu_R(1_{\mathbb{X}}) = \sum_{k=1}^{+\infty} \nu(R^{k-1}1_{\mathbb{X}}) = 1$, and it follows from Theorem 3.6 that the probability measure π_R given in (25) *is $P-$invariant ($\pi_R = \mu_R$ here). Moreover Formula* (16) *gives*

$$\forall n \geq 1, \quad P^n = R^n + 1_{\mathbb{X}} \otimes \mu_n \quad \textit{with} \quad \mu_n := \sum_{k=1}^{n} \nu R^{k-1}.$$

Consequently we have

$$\forall n \geq 1, \quad P^n - 1_{\mathbb{X}} \otimes \pi_R = R^n - 1_{\mathbb{X}} \otimes \sum_{k=n+1}^{+\infty} \nu R^{k-1},$$

from which we derive that

$$\begin{aligned}
\forall n \geq 1,\ \forall x \in \mathbb{X}, \quad \|P^n(x,\cdot) - \pi_R\|_{TV} \ &\leq\ \|R^n(x,\cdot)\|_{TV} + \Big\| \sum_{k=n+1}^{+\infty} \nu R^{k-1} \Big\|_{TV} \\
&= R^n(x, 1_{\mathbb{X}}) + \sum_{k=n+1}^{+\infty} \nu(R^{k-1}1_{\mathbb{X}}) \\
&= 2(1 - \nu(1_{\mathbb{X}}))^n.
\end{aligned}$$

3.3 Recurrence/Transience

If P satisfies Condition ($\boldsymbol{M}_{\nu,\psi}$), then P is said to be *recurrent* if the following condition holds:

$$\forall A \in \mathcal{X} :\ \mu_R(1_A) > 0 \Longrightarrow \sum_{k=0}^{+\infty} P^k 1_A = +\infty \text{ on } \mathbb{X} \text{ (i.e. } \forall x \in \mathbb{X},\ \sum_{k=0}^{+\infty} P^k(x, A) = +\infty), \quad (26)$$

where μ_R is the positive measure on $(\mathbb{X}, \mathcal{X})$ defined in (20). Note that if $A \in \mathcal{X}$ is such that $\nu(1_A) > 0$ then $\mu_R(1_A) > 0$. Observe that Equality (17) reads as

$$\sum_{n=0}^{+\infty} P^n = \sum_{n=0}^{+\infty} R^n + \left(\sum_{n=0}^{+\infty} P^n \psi \right) \otimes \mu_R \quad (27)$$

and is relevant in this section. To get a complete picture of recurrence/transience property for P satisfying Condition ($\boldsymbol{M}_{\nu,\psi}$) in the next statement, let us introduce the following definition. The Markov kernel P is said to be *irreducible* if

$$\sum_{n=1}^{+\infty} P^n \psi > 0 \text{ on } \mathbb{X}, \text{ i.e. } \forall x \in \mathbb{X},\ \exists q \equiv q(x) \geq 1, \quad (P^q \psi)(x) > 0. \quad (28)$$

Recall that under ($\boldsymbol{M}_{\nu,\psi}$), we have $\mu_R(\psi) \in [0, 1]$ from Proposition 3.4, and that μ_R is a $P-$invariant positive measure when $\mu_R(\psi) = 1$, or equivalently $\nu(h_R^\infty) = 0$ (see (24)), from Theorem 3.5. Finally, recall that $\| \cdot \|_{1_{\mathbb{X}}}$ denotes the supremum norm on $\mathcal{B}$ (i.e. $\|g\|_{1_{\mathbb{X}}} := \sup_{x \in \mathbb{X}} |g(x)|$).

Theorem 3.8 *Let P satisfy Condition ($\boldsymbol{M}_{\nu,\psi}$). Then the following assertions hold.*

1. *Case $\mu_R(\psi) = 1$ (or equivalently $\nu(h_R^\infty) = 0$). The Markov kernel P is recurrent if and only if P is irreducible (see (28)). When P is recurrent, μ_R is the unique $P-$invariant positive measure η (up to a multiplicative positive constant) such that $\eta(\psi) < \infty$.*

2. *Case $\mu_R(\psi) < 1$ (or equivalently $\nu(h_R^\infty) > 0$). The function series $\sum_{k=0}^{+\infty} P^k\psi$ is bounded on $\mathbb{X}$. If P is irreducible, then P is not recurrent, more precisely P is transient in the following sense: Defining for every $k \geq 1$ the set $A_k := \{x \in \mathbb{X} : \sum_{j=0}^{k}(R^j\psi)(x) \geq 1/k\}$ we have*

$$\mathbb{X} = \cup_{k=1}^{+\infty} A_k \quad \textit{and} \quad \forall k \geq 1,\ c_k := \|\sum_{n=0}^{+\infty} P^n 1_{A_k}\|_{1_{\mathbb{X}}} < \infty.$$

Actually we have: $\forall k \geq 1,\ c_k \leq k(k+1)(\nu(1_{\mathbb{X}})^{-1} + M)$ with $M := \|\sum_{k=0}^{+\infty} P^k\psi\|_{1_{\mathbb{X}}}$.

When P is irreducible, we have the following characterization of recurrence.

Corollary 3.9 *Assume that P satisfies Conditions ($\boldsymbol{M}_{\nu,\psi}$) and is irreducible. Then*

$$P \textit{ is recurrent} \iff \mu_R(\psi) = 1 \iff \nu(h_R^\infty) = 0 \iff \mu_R(h_R^\infty) = 0.$$

Proof. Assume that $\mu_R(\psi) \in [0, 1)$. Then P is not recurrent from the second assertion of Theorem 3.8. This proves the first direct implication. The converse one follows from the first assertion of Theorem 3.8. The two last equivalences are (24). □

The proof of Theorem 3.8 is based on the the two following lemmas.

Lemma 3.10 *Let P satisfy Condition ($\boldsymbol{M}_{\nu,\psi}$). If P is irreducible then the following statements hold:*

1. *$\sum_{n=0}^{+\infty} R^n\psi > 0$ on $\mathbb{X}$.*

2. *If $\mu_R(\psi) = 1$ (or equivalently $\nu(h_R^\infty) = 0$) then $\sum_{n=0}^{+\infty} P^n\psi = +\infty$ on $\mathbb{X}$.*

Proof. We prove Assertion 1. by contradiction. Assume that there exists $x \in \mathbb{X}$ such that $\sum_{n=0}^{+\infty}(R^n\psi)(x) = 0$. Then we have $h_R^\infty(x) = 1$ from (23). From the definition of $h_R^\infty(x)$ and $R^n 1_{\mathbb{X}} \leq 1$, it then follows that: $\forall n \geq 1,\ (R^n 1_{\mathbb{X}})(x) = 1$. Hence we deduce from Formula (16) and $(P^n 1_{\mathbb{X}})(x) = 1$ that

$$\forall n \geq 1, \quad \sum_{k=1}^{n}(P^{n-k}\psi)(x)\,\nu(R^{k-1}1_{\mathbb{X}}) = 0.$$

In particular the first term of this sum of non-negative real numbers is zero, that is we have: $\forall n \geq 1,\ (P^{n-1}\psi)(x)\,\nu(1_{\mathbb{X}}) = 0$. Since P is irreducible (see (28)), we know that there exists $q \equiv q(x) \geq 1$ such that $(P^q\psi)(x) > 0$. Then the previous equality with $n = q+1$ implies that $\nu(1_{\mathbb{X}}) = 0$: Contradiction. Assertion 1. is proved. Next, if $\mu_R(\psi) = 1$, then Equality (27) applied to ψ and Assertion 1. imply that $\sum_{n=0}^{+\infty} P^n\psi = +\infty$ on $\mathbb{X}$. □

Lemma 3.11 *Let P satisfy Condition ($\boldsymbol{M}_{\nu,\psi}$) with $\mu_R(\psi) > 0$. If P is recurrent, then $\sum_{k=0}^{+\infty} P^k\psi = +\infty$ on $\mathbb{X}$.*

Proof. Since $\mu_R(\psi) > 0$, there exists $\varepsilon > 0$ such that the set $F_\varepsilon := \{x \in \mathbb{X} : \psi(x) \geq \varepsilon\}$ satisfies $\mu_R(1_{F_\varepsilon}) > 0$ (otherwise we would have $\mu_R(\{x \in \mathbb{X} : \psi(x) > 0\}) = 0$, thus $\mu_R(\psi) = 0$). From recurrence and $1_{F_\varepsilon} \leq \psi/\varepsilon$, we obtain that $\sum_{n=0}^{+\infty} P^n\psi = +\infty$ on $\mathbb{X}$. □

Now, let us provide a proof of Theorem 3.8.

Proof of Theorem 3.8. Assume that $\mu_R(\psi) = 1$. If P is irreducible, then $\sum_{k=0}^{+\infty} P^k\psi = +\infty$ on $\mathbb{X}$ from Assertion 2. of Lemma 3.10. It follows from (27) applied to 1_A that $\sum_{k=0}^{+\infty} P^k 1_A = +\infty$ for every $A \in \mathcal{X}$ such that $\mu_R(1_A) > 0$, i.e. P is recurrent. Conversely, if P is recurrent, then it follows from $\mu_R(\psi) = 1$ and Lemma 3.11 that $\sum_{n=0}^{+\infty} P^n\psi = +\infty$ on $\mathbb{X}$. Thus P satisfies (28) (i.e. P is irreducible). Now let η be a $P-$invariant positive measure on $(\mathbb{X}, \mathcal{X})$ such that $\eta(\psi) < \infty$. From (16) we have

$$\forall n \geq 1, \quad \eta \geq \eta(\psi) \sum_{k=1}^{n} \nu R^{k-1}.$$

We deduce from the definition (20) of μ_R that $\eta \geq \eta(\psi)\mu_R$. Hence $\lambda := \eta - \eta(\psi)\mu_R$ is a non-negative measure, which is $P-$invariant since μ_R and η are. We have $\lambda(\psi) = 0$ since $\mu_R(\psi) = 1$. Thus we have $\lambda(P^k\psi) = 0$ for every $k \in \mathbb{N}$. From the monotone convergence theorem, it follows that

$$\lambda\Big(\sum_{k=0}^{+\infty} P^k\psi\Big) = \sum_{k=0}^{+\infty} \lambda(P^k\psi) = 0.$$

Moreover we know from the irreducibility definition (28) that $\sum_{k=0}^{+\infty} P^k\psi > 0$ on $\mathbb{X}$. It follows that $\lambda = 0$, i.e. $\eta = \eta(\psi)\mu_R$. Thus any $P-$invariant positive measure η such that $\eta(\psi) < \infty$ is such that $\eta = \eta(\psi)\mu_R$ (which implies that $\eta(\psi) > 0$). The second statement of Assertion 1. is proved.

Now assume that $\mu_R(\psi) < 1$. We have $\nu(h_R^\infty) > 0$ from (24) since $\mu_R(\psi) < 1$. Recall that $Rh_R^\infty = h_R^\infty$. Then, Formula (16) applied to h_R^∞ and $Rh_R^\infty = h_R^\infty$ give

$$\forall n \geq 1, \quad P^n h_R^\infty = h_R^\infty + \nu(h_R^\infty) \sum_{k=0}^{n-1} P^k\psi,$$

from which we deduce that: $\forall n \geq 1,\ \sum_{k=0}^{n-1} P^k\psi \leq \nu(h_R^\infty)^{-1} 1_{\mathbb{X}}$ since $h_R^\infty \geq 0$ and $P^n h_R^\infty \leq 1_{\mathbb{X}}$ from $h_R^\infty \leq 1_{\mathbb{X}}$. Consequently the function $\sum_{k=0}^{+\infty} P^k\psi$ is bounded on $\mathbb{X}$. Now assume that P is irreducible. Note that $\mu_R(\psi) = \nu(\sum_{n=0}^{+\infty} R^n\psi)$ from the monotone convergence theorem. Since ν is a positive measure, it follows from Lemma 3.10 that $\mu_R(\psi) > 0$. Thus, as in the proof of Lemma 3.11, there exists $\varepsilon > 0$ and a set F_ε such that $\mu_R(1_{F_\varepsilon}) > 0$ and $1_{F_\varepsilon} \leq \psi/\varepsilon$. We deduce that $\sum_{n=0}^{+\infty} P^n 1_{F_\varepsilon}$ is bounded on $\mathbb{X}$. Consequently P is not recurrent. Next let us prove that P is transient as defined in Theorem 3.8. We have $\mathbb{X} = \cup_{k=1}^{+\infty} A_k$. Indeed, otherwise there would exist $x \in \mathbb{X}$ such that: $\forall k \geq 1,\ \sum_{j=0}^{k} (R^j\psi)(x) < 1/k$, so that $\sum_{j=0}^{+\infty} (R^j\psi)(x) = 0$: This contradicts Lemma 3.10. Finally let $k \geq 1$. Observing that $1_{A_k} \leq k\sum_{j=0}^{k} R^j\psi$, we obtain that (see (7))

$$\sum_{n=0}^{+\infty} R^n 1_{A_k} \leq k \sum_{n=0}^{+\infty} R^n \bigg(\sum_{j=0}^{k} R^j\psi \bigg) = k \sum_{j=0}^{k} R^j \bigg(\sum_{n=0}^{+\infty} R^n\psi \bigg)$$
$$\leq k\,\nu(1_{\mathbb{X}})^{-1} \sum_{j=0}^{k} R^j 1_{\mathbb{X}} \leq k(k+1)\nu(1_{\mathbb{X}})^{-1} 1_{\mathbb{X}} \text{ (using (23) and } R1_{\mathbb{X}} \leq 1_{\mathbb{X}}).$$

Moreover, composing on the left the previous inequality by ν, it follows from the monotone convergence theorem that $\mu_R(1_{A_k}) \leq k(k+1)$. Then the last inequalities combined with Formula (27) applied to 1_{A_k} provide

$$\sum_{n=0}^{+\infty} P^n 1_{A_k} \leq k(k+1)\big[\nu(1_{\mathbb{X}})^{-1} + M\big]1_{\mathbb{X}} \quad \text{with} \quad M := \|\sum_{k=0}^{+\infty} P^k\psi\|_{1_{\mathbb{X}}}.$$

The proof of Theorem 3.8 is complete. □

When the positive measure μ_R is finite (i.e. $\mu_R(1_{\mathbb{X}}) < \infty$), then we have $\mu_R(\psi) = 1$ from Theorem 3.6. Moreover any $P-$invariant probability measure π is such that $\pi(\psi) < \infty$ since ψ is bounded. Therefore, the following statement is a direct consequence of Assertion 1. of Theorem 3.8.

Corollary 3.12 *Assume that P satisfies Condition ($\boldsymbol{M}_{\nu,\psi}$) with $\mu_R(1_{\mathbb{X}}) < \infty$ and is irreducible. Then P is recurrent, and the probability measure π_R given in* (25) *is the unique $P-$invariant probability measure.*

If Condition ($\boldsymbol{M}_{\nu,\psi}$) holds with $\mu_R(\psi) > 0$, then the following statement shows that the recurrence property actually implies that $\mu_R(\psi) = 1$, so that μ_R is $P-$invariant. Note that the condition $\mu_R(\psi) > 0$ is satisfied, either when P is irreducible from Lemma 3.10 since $\mu_R(\psi) = \nu(\sum_{k=0}^{+\infty} R^k\psi)$, or when the strong aperiodicity property $\nu(\psi) > 0$ holds since $\mu_R(\psi) \geq \nu(\psi)$.

Proposition 3.13 *Let P satisfy Condition ($\boldsymbol{M}_{\nu,\psi}$) with $\mu_R(\psi) > 0$. If P is recurrent, then μ_R is $P-$invariant.*

Proof. Since $\mu_R(\psi) > 0$ and P is assumed to be recurrent, we deduce from Lemma 3.11 that $\sum_{k=0}^{+\infty} P^k\psi = +\infty$ everywhere. Moreover, the sequence $(\nu(R^n 1_{\mathbb{X}}))_{n\geq 0}$ is non-increasing since $(R^n 1_{\mathbb{X}}))_{n\geq 0}$ is. Then, it follows from the kernel equality (16) applied with $1_{\mathbb{X}}$ that

$$\forall n \geq 1, \quad P^n 1_{\mathbb{X}} = 1_{\mathbb{X}} \geq \nu(R^{n-1}1_{\mathbb{X}}) \sum_{k=0}^{n-1} P^k\psi.$$

Since $\sum_{k=0}^{+\infty} P^k\psi = +\infty$ and $\nu(h_R^\infty) = \lim_n \nu(R^n 1_{\mathbb{X}})$ from the monotone convergence theorem, we deduce from the above inequality that $\nu(h_R^\infty) = 0$ which is equivalent to $\mu_R(\psi) = 1$ from (24). Then, the $P-$invariance of μ_R follows from Assertion 1. of Theorem 3.5. □

3.4 Further statements

The two first following propositions are used in the bibliographic discussions of Subsection 3.5. The second one may be relevant to check the condition $\mu_R(1_A) > 0$ in the definition (26) of recurrence. The third proposition is only used in the proof of Propositions 5.12 and 5.13 related to discussion on drift conditions in Section 5.

Proposition 3.14 *If P satisfies Condition ($\boldsymbol{M}_{\nu,\psi}$) with $\mu_R(\psi) > 0$, then P is irreducible (see* (28)*) if, and only if,*

$$\forall A \in \mathcal{X}: \quad \mu_R(1_A) > 0 \implies \sum_{n=1}^{+\infty} P^n 1_A > 0 \quad \textit{on } \mathbb{X}. \tag{29}$$

Proof. Equality (27) (i.e. (17)) reads also as $\sum_{n=1}^{+\infty} P^n = \sum_{n=1}^{+\infty} R^n + \big(\sum_{n=0}^{+\infty} P^n\psi\big) \otimes \mu_R$ since $P^0 = R^0$. Thus, we have

$$\forall A \in \mathcal{X},\ \forall x \in \mathbb{X}, \quad \sum_{n=1}^{+\infty} P^n(x, A) \geq \mu_R(1_A) \sum_{n=0}^{+\infty} (P^n\psi)(x),$$

from which we deduce that the irreducibility condition (28) implies Condition (29). Conversely assume that Condition (29) holds. Since there exists $\varepsilon > 0$ such that $\mu_R(\{\psi \geq \varepsilon\}) > 0$ from $\mu_R(\psi) > 0$, it follows from (29) that $\sum_{n=1}^{+\infty} P^n\psi \geq \varepsilon \sum_{n=1}^{+\infty} P^n 1_{\{\psi\geq\varepsilon\}} > 0$ on $\mathbb{X}$, i.e. (28) holds. □

Let us introduce the following Markov resolvent kernel

$$Q := \sum_{n=0}^{+\infty} 2^{-(n+1)} P^n. \tag{30}$$

Proposition 3.15 *If P satisfies Condition ($\boldsymbol{M}_{\nu,\psi}$), then the following equivalence holds:*

$$\forall A \in \mathcal{X} : \quad \mu_R(1_A) > 0 \iff \nu(Q1_A) > 0.$$

Proof. Let $A \in \mathcal{X}$. From (16) we obtain that

$$\begin{aligned} Q1_A &= \sum_{n=0}^{+\infty} 2^{-(n+1)} R^n 1_A + \sum_{n=1}^{+\infty} 2^{-(n+1)} \sum_{k=1}^{n} \nu(R^{k-1}1_A) P^{n-k}\psi \\ &= \sum_{n=0}^{+\infty} 2^{-(n+1)} R^n 1_A + \Big(\sum_{k=1}^{+\infty} 2^{-k} \nu(R^{k-1}1_A)\Big)\Big(\sum_{n=0}^{+\infty} 2^{-(n+1)} P^n\psi\Big). \end{aligned} \tag{31}$$

Then composing on the left by ν, it follows from the monotone convergence theorem that

$$\nu(Q1_A) = \sum_{n=0}^{+\infty} 2^{-(n+1)} \nu(R^n 1_A) + \Big(\sum_{k=1}^{+\infty} 2^{-k} \nu(R^{k-1}1_A)\Big)\Big(\sum_{n=0}^{+\infty} 2^{-(n+1)} \nu(P^n\psi)\Big).$$

Next from the definition (20) of μ_R we have: $\mu_R(1_A) = 0 \Leftrightarrow \forall k \geq 0,\ \nu(R^k 1_A) = 0$. It follows from the above equality that $\mu_R(1_A) = 0$ is equivalent $\nu(Q1_A) = 0$ since all the terms involved in this equality are non-negative. □

Proposition 3.16 *If P satisfies Condition ($\boldsymbol{M}_{\nu,\psi}$) and is irreducible, then every non-empty $P-$absorbing set is μ_R-full.*

Proof. Let $B \in \mathcal{X}^*$ be a $P-$absorbing set, that is satisfying: $\forall n \geq 1,\ \forall x \in B,\ P^n(x, B^c) = 0$. Let Q be defined in (30). Formula (31) applied to $A := B^c$ provides

$$\forall x \in B, \quad 0 = \sum_{n=1}^{+\infty} 2^{-(n+1)} R^n(x, B^c) + \Big(\sum_{k=1}^{+\infty} 2^{-k} \nu(R^{k-1}1_{B^c})\Big)(Q\psi)(x).$$

Since P is irreducible (see (28)), we know that $(Q\psi)(x) > 0$, so that we have: $\forall k \geq 1,\ \nu(R^{k-1}1_{B^c}) = 0$. Thus $\mu_R(1_{B^c}) = 0$ from the definition (20) of μ_R. □

3.5 Bibliographic comments

Here we discuss point by point the definitions and results concerning the classical concepts of this section, i.e. irreducibility, recurrence/transience properties and invariant measures, in link with the books [Num84, MT09, DMPS18]. A detailed historical background on these properties can be found in [Num84, pp. 141-144], [MT09, Sec. 4.5, 8.6,10.6] and [DMPS18, Sec. 9.6,10.4,11.6]. In discrete state space, we refer for example to [Nor97, Bré99, Gra14] (see also [Mey08, App. A] for an overview on Markov chains in modern terms).

A) *Small-set and small-functions.* Let $\ell \geq 1$. Recall that a set $S_\ell \in \mathcal{X}^*$ is said to be a $\ell-$order small-set for P in the standard literature on the topic of Markov chains (e.g. see [Num84, MT09, DMPS18]), if the following condition holds

$$\exists \nu_\ell \in \mathcal{M}_{+,b}^* : \quad P^\ell \geq 1_{S_\ell} \otimes \nu_\ell \quad (\text{i.e. } \forall x \in \mathbb{X},\ P^\ell(x, dy) \geq 1_{S_\ell}(x)\, \nu_\ell(dy)). \tag{32}$$

The extension to $\ell-$order small-functions writes as (see [Num84, Def. 2.3, p. 15])

$$\exists (\nu_\ell, \psi_\ell) \in \mathcal{M}_{+,b}^* \times \mathcal{B}_*^+ : \quad P^\ell \geq \psi_\ell \otimes \nu_\ell \quad (\text{i.e. } \forall x \in \mathbb{X},\ P^\ell(x, dy) \geq \psi_\ell(x)\, \nu_\ell(dy)). \tag{33}$$

Our minorization condition $(\boldsymbol{M}_{\nu,\psi})$ is nothing other than [Num84, Def. 2.3] with order one. Finally recall that $S \in \mathcal{X}^*$ is said to be petite (e.g. see [MT92]) if it is a small-set of order one for the Markov resolvent kernel $\sum_{n=0}^{+\infty} a_n P^n$ for some $(a_n)_n \in [0, +\infty)^{\mathbb{N}}$ such that $\sum_{n=0}^{+\infty} a_n = 1$. The notion of petite sets is not used in this work. The specific resolvent kernel $\sum_{n=0}^{+\infty} 2^{-(n+1)} P^n$ in (30) is only used in part D) below to support the current bibliographic discussion and to provide a sufficient condition for having $h_R^\infty = 0$ in Corollary 4.18.

B) *Residual kernels and invariant measure.* The representation (20) of $P-$invariant measure via the residual kernel was introduced in [Num84, Th. 5.2, Cor. 5.2] under the minorization condition $(\boldsymbol{M}_{\nu,\psi})$ and the recurrence assumption, so that the positive measure μ_R necessarily satisfies $\mu_R(\psi) = 1$ there. The $P-$invariance of μ_R under the sole Condition $(\boldsymbol{M}_{\nu,\psi})$ was proved in [HL23b] in the specific case when $\mu_R(1_{\mathbb{X}}) < \infty$: This corresponds to Theorem 3.6. This result is extended to the general case in Theorem 3.5, that is: under the single minorization Condition $(\boldsymbol{M}_{\nu,\psi})$, the $P-$invariance of μ_R is actually guaranteed when $\mu_R(\psi) = 1$, and is even equivalent to this condition under the additional strong aperiodicity assumption $\nu(\psi) > 0$. Consequently, contrary to the statement [Num84, Th. 5.2, Cor. 5.2, p. 73-74], the $P-$invariance of μ_R is here related directly to the condition $\mu_R(\psi) = 1$, which makes it possible to carry out this study independently of the recurrence property, and even independently of any irreducibility condition on P. Recall that the key point in the proof of Theorem 3.5 is the kernel identity (16).

C) *Accessibility and irreducibility conditions.* Recall that if P satisfies Condition $(\boldsymbol{M}_{\nu,1_S})$ then the set S is said to be a first-order small set. Let us comment Condition (28) in case $\psi := 1_S$. This condition then means that the set S is accessible according to [DMPS18, Def. 3.5.1, Lem. 3.5.2]. On the other hand recall that a Markov kernel P is said to be irreducible according to [DMPS18, Def. 9.2.1] if it admits an accessible small set. Thus our definition (28) of irreducibility for a Markov kernel P satisfying Condition $(\boldsymbol{M}_{\nu,1_S})$ coincides with that of [DMPS18] in case of a first-order small set. Now, thanks to Proposition 3.14, let us recall the link with the irreducibility notion used in [Num84, MT09]. First, in connection with the condition $\mu_R(1_S) = 0$ which is not addressed in Proposition 3.14, observe that this condition implies the transience of P from Theorem 3.8. Moreover this condition cannot hold under Condition (28) from Assertion 1. of Lemma 3.10 since $\mu_R(1_S) = \nu(\sum_{n=0}^{+\infty} R^n 1_S)$. Finally, nor can this condition be satisfied under the strong aperiodicity condition $\nu(1_S) > 0$ since $\mu_R \geq \nu$. Thus the discussion may be conducted assuming that P satisfies Condition ($\boldsymbol{M}_{\nu,1_S}$) with $\mu_R(1_S) > 0$ (i.e. $\exists k \geq 0,\ \nu(R^k 1_S) \neq 0$). Then it follows from Proposition 3.14 that our definition of P irreducible (see (28)) is equivalent to the μ_R-irreducibility of P as defined in [Num84, p. 11] and [MT09, p. 82], that is (29).

D) *Maximal irreducibility measures.* Although the notion of maximal irreducibility measures is not explicitly addressed in this work, it has to be discussed since it plays an important role in [Num84, MT09, DMPS18]. First note that, if P satisfies Conditions ($\boldsymbol{M}_{\nu,1_S}$) and (28), then μ_R is an irreducibility measure using the classical terminology in [MT09, DMPS18] (see Item C)). Actually μ_R is a maximal irreducibility measure according to the definition [DMPS18, Def. 9.2.2]: Every accessible set $A \in \mathcal{X}$ is such that $\mu_R(1_A) > 0$. Indeed A is accessible reads as $Q1_A > 0$ on $\mathbb{X}$ where Q is defined in (30). Next, if $Q1_A > 0$ on $\mathbb{X}$ then $\nu(Q1_A) > 0$, so that $\mu_R(1_A) > 0$ from Proposition 3.15. Of course Conditions ($\boldsymbol{M}_{\nu,1_S}$) and (28) also ensure that the minorizing measure ν is an irreducibility measure since $\nu(1_A) > 0$ implies that $\mu_R(1_A) > 0$. However ν is not maximal a priori. As is well known, any irreducibility measure η is absolutely continuous w.r.t. the maximal irreducibility measure μ_R since the condition $\eta(1_A) > 0$ implies that $Q1_A > 0$ on $\mathbb{X}$ from the definition of $\eta-$irreducibility, so that $\mu_R(1_A) > 0$ due to the above.

E) *Recurrence/transience and uniqueness of invariant measure in recurrence case.* Our definition (26) of recurrence corresponds to that in [Num84, pp. 27-28] and [MT09, p. 180] with μ_R as maximal irreducibility measure. From the discussion in Item C), this also corresponds to the recurrence definition [DMPS18, Def. 10.1.1]. The transience property used in Theorem 3.8 is that provided in [MT09, p. 171 and 180] and [DMPS18, Def. 10.1.3]. The Recurrence/Transience dichotomy stated in Theorem 3.8 is a well-known result for irreducible Markov chains, e.g. see [Num84, Th. 3.2, p. 28], [MT09, Th. 8.0.1] and [DMPS18, Th. 10.1.5]. The novelty in Theorem 3.8 is that this dichotomy can be simply declined according to whether $\mu_R(\psi) = 1$ or $\mu_R(\psi) \in [0, 1)$.

As indicated in Item B), the existence of $P-$invariant positive measures, which is obtained in our work independently of any irreducibility condition on P (Theorem 3.5), is classically proved under the recurrence assumption. In fact this is usually done together with the uniqueness issue. Under the recurrence assumption the existence and uniqueness (up to a positive multiplicative constant) of a $P-$invariant positive measure is obtained in [Num84, Th. 5.2, Cor. 5.2, p. 73-74] using the representation (20). This result is proved in [MT09, Th. 10.4.9] and [DMPS18, Th. 11.2.5] via splitting techniques, providing the classical regeneration-type representation of $P-$invariant positive measures.

F) *Strong aperiodicity condition $\nu(\psi) > 0$.* This condition is a particular case of the aperiodicity condition introduced in Subsection 4.2.

G) *The splitting construction.* To conclude this bibliographic discussion, it is worth remembering that the concept of small-set has a natural and crucial probabilistic interest in splitting or coupling techniques: This is the thread and backbone of the books

[Num84, MT09, DMPS18]. Here this probabilistic aspect is not addressed. In this work, the minorization Condition ($\boldsymbol{M}_{\nu,\psi}$) allows us to write the Markov kernel P as the sum of two non-negative kernels: the residual kernel $R := P - \psi \otimes \nu$ and the rank-one kernel $\psi \otimes \nu$. That R is non-negative is the crucial point to define all the quantities related to R in this section, especially the positive measure μ_R (see (20)) and the function h_R^∞ (see (19)). Actually one of the key points of the present section and of the next ones is the kernel identity (16). This formula is already present in Nummelin's book [Num84, Eq. (4.12)]. It seems that the sole way to obtain a probabilistic sense of this formula is to use the split Markov chain introduced in [Num78]. The idea is to introduce an appropriate enlargement of the state space of the original Markov chain in order to obtain a new Markov chain - the split chain - which has an atom. Then most of statements on the original chain are derived from applying results (obtained for example by the regeneration method) on atomic chains to this split chain. Thus, using the splitting construction requires switching from the original chain to the split chain for assumptions, and vice versa for results. The enlargement of the state space consists roughly in tagging the transitions of the original chain according to the occurrence of a $\psi-$dependent tossing coin in order to reflect the decomposition $R + \psi \otimes \nu$ of P in two submarkovian kernels. We refer to [Num84, Sec. 4.4], [CMR05, Sec. 14.2], [MT09, Chap. 5] for details. See also [Num02] for a readable survey on this topic in the case of Markov chain Monte Carlo (MCMC) kernels. Here, the kernel-based point of view allows us to study the general Markov chains in a single step. There is no need to resort to an intermediate class of Markov chains, e.g. atomic chains, before dealing with the general case via what may appear to be a technical device, e.g. the split chain. To turn back to our key formula (16), [Num84, Eq. (4.24)] provides a probabilistic interpretation from the splitting construction. What is new here is that we are exploiting Formula (16) solely as a kernel identity. The price to pay for this presentation is that we only consider Markov kernels satisfying a first-order minorization condition.

Appendix A gives the probabilistic interpretation of the main quantities used in this document. This should facilitate the comparative reading with the statements in reference probabilistic works as [Num84, MT09, DMPS18]. And, as for formula (16), all these probabilistic formulas are obtained from the split chain.

4 Harris recurrence and convergence of the iterates

Assume that P satisfies the minorization Condition ($\boldsymbol{M}_{\nu,\psi}$) and recall that $h_R^\infty := \lim_n R^n 1_{\mathbb{X}}$ (point-wise convergence, see (19)), where $R \equiv R_{\nu,\psi}$ is the residual kernel given in (12). Condition $h_R^\infty = 0$ is stronger than $\nu(h_R^\infty) = 0$. Under this condition $h_R^\infty = 0$, the results of the previous section are revisited in the following theorem with an additional result on the $P-$harmonic functions. Next, still under Condition $h_R^\infty = 0$, the Markov kernel P is shown to be Harris-recurrent, and the convergence in total variation norm of the iterates of P to its unique invariant probability measure is obtained when $\mu_R(1_{\mathbb{X}}) < \infty$ and P satisfies an aperiodicity condition. The periodic case is addressed in Subsection 4.3. Finally, introducing a drift inequality on P, a sufficient condition for the condition $h_R^\infty = 0$ to hold is presented in Subsection 4.4.

Theorem 4.1 *Let P satisfy Condition ($\boldsymbol{M}_{\nu,\psi}$). If $h_R^\infty = 0$, then the following assertions hold.*

1. The $P-$harmonic functions are constant on $\mathbb{X}$.

2. P is irreducible and is recurrent.

3. The positive measure $\mu_R := \sum_{k=0}^{+\infty} \nu R^k$ (see (20)) satisfies $\mu_R(\psi) = 1$, and is the unique $P-$invariant positive measure η (up to a multiplicative constant) such that $\eta(\psi) < \infty$. If $\mu_R(1_{\mathbb{X}}) < \infty$, then $\pi_R := \mu_R(1_{\mathbb{X}})^{-1}\mu_R$ (see (25)) is the unique $P-$invariant probability measure on $(\mathbb{X}, \mathcal{X})$.

Proof. It follows from (23) and $h_R^\infty = 0$ that

$$\sum_{k=0}^{+\infty} R^k\psi = \nu(1_{\mathbb{X}})^{-1}1_{\mathbb{X}}. \tag{34}$$

Let $g \in \mathcal{B}$ be such that $Pg = g$. Recall that, for every $n \geq 0$, we have $\nu(g)\sum_{k=0}^{n} R^k\psi = g - R^{n+1}g$ from (21). Moreover we have $\lim_n R^n g = 0$ since $|R^n g| \leq R^n|g| \leq \|g\|_{1_{\mathbb{X}}} R^n 1_{\mathbb{X}}$ and $h_R^\infty = 0$. Thus $g = \nu(g)\sum_{k=0}^{+\infty} R^k\psi$. We have proved that g is proportional to $1_{\mathbb{X}}$. This proves Assertion 1.

For Assertion 2., apply the kernel identity (27) to ψ to get

$$\sum_{n=0}^{+\infty} P^n\psi = \sum_{n=0}^{+\infty} R^n\psi + \mu_R(\psi)\sum_{n=0}^{+\infty} P^n\psi.$$

We have $\mu_R(\psi) = 1$ since $h_R^\infty = 0$ (see (24)). Then, we deduce from (34) and the previous equality that $\sum_{k=0}^{+\infty} P^k\psi = +\infty$. Thus the irreducibility property holds, as well as the recurrence property from Theorem 3.8.

The first part of Assertion 3. is a direct consequence of Assertion 1. of Theorem 3.8 using that $\nu(h_R^\infty) = 0$ (i.e. $\mu_R(\psi) = 1$) and that P is recurrent. The second part of Assertion 3. is Corollary 3.12. The proof of Theorem 4.1 is complete. □

The notations concerning restriction to a set $E \in \mathcal{X}$ of functions, measures ans kernels are provided in Section 2.

Lemma 4.2 *Assume that P satisfies Condition $(\boldsymbol{M}_{\nu,\psi})$ with $\mu_R(\psi) > 0$, where R is the residual kernel given in (12). Let $E \in \mathcal{X}$ be any μ_R-full $P-$absorbing set. Then the Markov kernel P_E on $(E, \mathcal{X}_E)$ satisfies Condition $(\boldsymbol{M}_{\nu_E,\psi_E})$. Moreover the associated residual kernel $P_E - \psi_E \otimes \nu_E$ is the restriction R_E to E of R, and the following equalities hold*

$$\forall x \in E,\ h_{R_E}^\infty(x) := \lim_n R_E^n(x, E) = h_R^\infty(x) \quad \textit{and} \quad \forall n \geq 0,\ \nu_E(R_E^n\psi_E) = \nu(R^n\psi).$$

Proof. Since $\mu_R(\psi) > 0$ and E is μ_R-full, we have $\mu_R(1_E\psi) = \mu_R(\psi) > 0$, thus ψ_E is non-zero. Moreover we have $\nu(1_E) = \nu(1_{\mathbb{X}}) > 0$ since $\mu_R(1_{E^c}) = 0$ implies that $\nu(1_{E^c}) = 0$ from the definition of μ_R. Then Condition $(\boldsymbol{M}_{\nu_E,\psi_E})$ for the Markov kernel P_E on $(E, \mathcal{X}_E)$ is deduced from the minorization condition $(\boldsymbol{M}_{\nu,\psi})$ for P since for every $A' \in \mathcal{X}_E$ and any $A \in \mathcal{X}$ such that $A' = A \cap E$ we have

$$\forall x \in E, \quad P_E(x, A') = P(x, A \cap E) \geq \nu(A \cap E)\psi(x) = \nu_E(A')\psi_E(x).$$

That $P_E - \psi_E \otimes \nu_E$ is the restriction of R to the set E is obvious. It follows that

$$\forall x \in E,\ \forall n \geq 1, \quad R_E^n(x, E) = R^n(x, E) = R^n(x, \mathbb{X})$$

since $R^n(x, E^c) = 0$ from $0 \leq R^n(x, E^c) \leq P^n(x, E^c) = 0$. Consequently we have for every $x \in E$: $\lim_n R_E^n(x, E) = h_R^\infty(x)$. Finally we have: $\forall n \geq 0, \forall x \in E,\ (R_E^n\psi_E)(x) = (R^n\psi)(x)$. Thus $\nu_E(R_E^n\psi_E) = \nu(R^n\psi)$ since $\nu(1_{E^c}) = 0$. □

4.1 Harris-recurrence

Let us present a first application of Theorem 4.1 to the so-called Harris-recurrence property. Let $(X_n)_{n\geq 0}$ be a Markov chain with transition kernel P. If P satisfies Condition $(\boldsymbol{M}_{\nu,\psi})$ and if $h_R^\infty = 0$, we know that P is recurrent from Theorem 4.1, that is (see (26))

$$\forall A \in \mathcal{X} :\ \mu_R(1_A) > 0 \Longrightarrow \forall x \in \mathbb{X},\ \mathbb{E}_x\Big[\sum_{k=0}^{+\infty} 1_{\{X_k \in A\}}\Big] = +\infty.$$

This recurrence property is proved below to be reinforced in

$$\forall A \in \mathcal{X} :\ \mu_R(1_A) > 0 \Longrightarrow\ \forall x \in \mathbb{X},\quad \mathbb{P}_x\Big\{\sum_{n=1}^{+\infty} 1_{\{X_n \in A\}} = +\infty\Big\} = 1. \tag{35}$$

Such a transition kernel P is said to be *Harris-recurrent.*

Theorem 4.3 *Let P satisfy Conditions $(\boldsymbol{M}_{\nu,\psi})$ and $h_R^\infty = 0$. Then the Markov chain $(X_n)_{n\geq 0}$ with transition kernel P is Harris-recurrent.*

First prove the following lemma.

Lemma 4.4 *Let P satisfy Conditions $(\boldsymbol{M}_{\nu,\psi})$ and $\mu_R(\psi) = 1$. If $g \in \mathcal{B}$ is such that $Pg \leq g$, then the non-negative function $g - Pg$ is μ_R-integrable and we have $\mu_R(g - Pg) = 0$.*

Lemma 4.4, which is used below in the proof of Theorem 4.3, has its own interest. Indeed, from the $P-$invariance of μ_R the conclusion of Lemma 4.4 is straightforward under the assumption $\mu_R(1_{\mathbb{X}}) < \infty$ since, for every $g \in \mathcal{B}$, the functions g and Pg are μ_R-integrable and $\mu_R(Pg) = \mu_R(g)$. However, if μ_R is not finite, the conclusion of Lemma 4.4 is no longer obvious.

Proof of Lemma 4.4. For every $n \geq 1$, it follows from $Pg = Rg + \nu(g)\psi$ that

$$\begin{aligned}
\sum_{k=0}^{n} \nu\big(R^k(g - Pg)\big) &= \sum_{k=0}^{n} \nu(R^k g) - \sum_{k=0}^{n} \nu(R^{k+1}g) - \nu(g)\sum_{k=0}^{n} \nu(R^k\psi) \\
&= \nu(g)\Big(1 - \sum_{k=0}^{n} \nu(R^k\psi)\Big) - \nu(R^{n+1}g) \\
&\leq 2\|g\|_{1_{\mathbb{X}}}\nu(1_{\mathbb{X}}) < \infty
\end{aligned} \tag{36}$$

using $0 \leq \sum_{k=0}^{n} \nu(R^k\psi) \leq \mu_R(\psi) = 1$ and $|g| \leq \|g\|_{1_{\mathbb{X}}} 1_{\mathbb{X}}$. Thus the series $\sum_{k=0}^{+\infty} \nu(R^k(g - Pg))$ of non-negative terms converges, that is $g - Pg$ is μ_R-integrable. Since $\mu_R(\psi) = 1$ we know that $\nu(h_R^\infty) = 0$ from (24). Thus we have $\lim_n \sum_{k=0}^{n} \nu(R^k\psi) = 1$ from the definition of μ_R. Moreover we have $|\nu(R^{n+1}g)| \leq \|g\|_{1_{\mathbb{X}}}\nu(R^{n+1}1_{\mathbb{X}})$ with $\lim_n \nu(R^{n+1}1_{\mathbb{X}}) = \nu(h_R^\infty) = 0$ from the definition of h_R^∞ and Lebesgue's theorem. Thus the property $\mu_R(g - Pg) = 0$ follows from (36). Lemma is proved. □

Proof of Theorem 4.3. Let $A \in \mathcal{X}$ be such that $\mu_R(1_A) > 0$. Recall that the function defined by $g_A^\infty(x) := \mathbb{P}_x\{\sum_{n=1}^{+\infty} 1_{\{X_n \in A\}} = +\infty\}$ for any $x \in \mathbb{X}$ is a $P-$harmonic function, see (9). Thus, under Condition $h_R^\infty = 0$, we know that g_A^∞ is constant on $\mathbb{X}$ from Theorem 4.1. We have to prove that $g_A^\infty = 1_{\mathbb{X}}$, namely that $g_A^\infty(x) = 1$ for at least one $x \in \mathbb{X}$.

Let g_A be defined by: $\forall x \in \mathbb{X},\ g_A(x) := \mathbb{P}_x\{T_A < \infty\}$ where $T_A := \inf\{n \geq 0 : X_n \in A\}$ is the hitting time of the set A. Recall that g_A is superharmonic, i.e. $Pg_A \leq g_A$, and that $g_A^\infty = \lim_n \searrow P^n g_A$, see (10)-(11). Let $n \geq 0$. It follows from $P(P^n g_A) \leq P^n g_A$ and Lemma 4.4 applies to $P^n g_A$ that the non-negative function $P^n g_A - P^{n+1} g_A$ is such that $\mu_R(P^n g_A - P^{n+1} g_A) = 0$. Thus there exists $E_n \in \mathcal{X}$ such that $\mu_R(1_{E_n^c}) = 0$ and $P^n g_A = P^{n+1} g_A$ on E_n. Now let $E := \cap_{n \geq 0} E_n$. Then we have $\mu_R(1_{E^c}) = 0$ and

$$\forall x \in E,\ \forall n \geq 0, \quad g_A(x) = (P^{n+1} g_A)(x).$$

Passing to the limit when $n \to +\infty$ we obtain that every $x \in E$ satisfies $g_A^\infty(x) = g_A(x)$. Finally we get from $\mu_R(1_{E^c}) = 0$ that $\mu_R(1_{A\cap E}) = \mu_R(1_A) > 0$, and we know that $g_A = 1$ on A from the definition of g_A. Therefore there exists a $x \in \mathbb{X}$ such that $g_A^\infty(x) = 1$. Thus $g_A^\infty = 1_{\mathbb{X}}$ since g_A^∞ is constant on $\mathbb{X}$. The proof of Theorem 4.3 is complete. □

Corollary 4.5 *If P satisfies Condition $(\boldsymbol{M}_{\nu,\psi})$, is irreducible and recurrent, then the restriction P_H of P to the μ_R-full $P-$absorbing set $H := \{h_R^\infty = 0\}$ is Harris-recurrent.*

The proof of Corollary 4.5 is based on Lemma 4.2 and on the following lemma.

Lemma 4.6 *Assume that P satisfies Condition $(\boldsymbol{M}_{\nu,\psi})$ and is irreducible. If $\nu(h_R^\infty) = 0$, then the set $H := \{h_R^\infty = 0\}$ is $P-$absorbing and μ_R-full.*

Proof. Since $\nu(h_R^\infty) = 0$ the set H is non-empty. Moreover it follows from $\nu(h_R^\infty) = 0$ and $Rh_R^\infty = h_R^\infty$ that $Ph_R^\infty = h_R^\infty$. Then we have

$$\forall x \in H, \quad 0 = h_R^\infty(x) = (Ph_R^\infty)(x) = \int_{\mathbb{X}} h_R^\infty(y) P(x, dy)$$

hence $P(x, H^c) = 0$, i.e. $P(x, H) = 1$, for any $x \in H$. Thus H is $P-$absorbing. That H is μ_R-full follows from Proposition 3.16. □

Proof of Corollary 4.5. We have $\nu(h_R^\infty) = 0$ and $\mu_R(\psi) = 1$ from Corollary 3.9. It follows from Lemma 4.6 that $H := \{h_R^\infty = 0\}$ is $P-$ is absorbing and μ_R-full. From Lemma 4.2 applied to the set H, we know that P_H satisfies Condition $(\boldsymbol{M}_{\nu_H,\psi_H})$ and that $h_{R_H}^\infty = 0$ on H from the definition of H. Consequently the last assertion of the corollary follows from Theorem 4.3 applied to the Markov kernel P_H on $(H, \mathcal{X}_H)$. □

4.2 Convergence of iterates: the aperiodic case

Set $\overline{D} := \{z \in \mathbb{C} : |z| \leq 1\}$. If P satisfies Condition $(\boldsymbol{M}_{\nu,\psi})$, then the following power series

$$\rho(z) := \sum_{n=1}^{+\infty} \nu(R^{n-1}\psi)\, z^n \tag{37}$$

absolutely converges for every $z \in \overline{D}$ since $\mu_R(\psi) = \sum_{k=0}^{+\infty} \nu(R^k\psi) < \infty$ from Proposition 3.4. If moreover P is irreducible, then this power series ρ is non-zero since $\sum_{n=0}^{+\infty} \nu(R^n\psi) = \nu(\sum_{n=0}^{+\infty} R^n\psi) > 0$ from monotone convergence theorem and Assertion 1. of Lemma 3.10.

If P satisfies Condition $(\boldsymbol{M}_{\nu,\psi})$ and is irreducible, then P is said to be *aperiodic* if $\rho(z)$ defined in (37) is not a power series in z^q for any integer $q \geq 2$. Using the notation g.c.d. for greatest common divisor, this aperiodicity condition is then equivalent to

$$\text{g.c.d.}\left\{n \geq 1 : \nu(R^{n-1}\psi) > 0\right\} = 1. \tag{38}$$

Note that this condition obviously holds when P is strongly aperiodic, i.e. $\nu(\psi) > 0$. In Subsection 4.3, under Conditions $(\boldsymbol{M}_{\nu,\psi})$ and $h_R^\infty = 0$, various equivalent conditions for aperiodicity are provided by Theorem 4.14. Actually, Assertion (b) of Theorem 4.14 shows that the aperiodicity condition does not depend on the choice of the couple (ν, ψ) in Condition $(\boldsymbol{M}_{\nu,\psi})$. Assertion (c) of Theorem 4.14 shows that aperiodicity condition is equivalent to the non-existence of d-cycle sets for P with $d \geq 2$.

When P satisfies Condition $(\boldsymbol{M}_{\nu,\psi})$ with $\mu_R(1_{\mathbb{X}}) < \infty$, is irreducible and aperiodic, the convergence of probability distributions $(\delta_x P^n)_{n\geq 0}$ to π_R in total variation norm is shown to be equivalent to the property $h_R^\infty = 0$ in the following theorem. As a corollary, the convergence of the probability distributions $(\delta_x P^n)_{n\geq 0}$ to π_R holds for π_R-almost $x \in \mathbb{X}$. Recall that under these assumptions, π_R is the unique $P-$invariant probability measure from Assertion 3. of Theorem 4.1.

Theorem 4.7 *Let P satisfy Condition $(\boldsymbol{M}_{\nu,\psi})$ with $\mu_R(1_{\mathbb{X}}) < \infty$. If P is irreducible and aperiodic, then the following equivalence holds:*

$$h_R^\infty = 0 \iff \forall x \in \mathbb{X}, \quad \lim_{n\to+\infty} \|\delta_x P^n - \pi_R\|_{TV} = 0.$$

Corollary 4.8 *Let P satisfy Condition $(\boldsymbol{M}_{\nu,\psi})$ with $\mu_R(1_{\mathbb{X}}) < \infty$. If P is irreducible and aperiodic, then*

$$\lim_{n\to+\infty} \|\delta_x P^n - \pi_R\|_{TV} = 0 \quad \textit{for } \pi_R-\textit{almost every } x \in \mathbb{X}.$$

Proof of Corollary 4.8. From Theorem 3.6 we have $\mu_R(\psi) = 1$, so that $\nu(h_R^\infty) = 0$ from (24). Then we know from Lemma 4.6 that the set $H := \{h_R^\infty = 0\}$ is $P-$absorbing and μ_R-full. From Lemma 4.2 applied to $E = H$, it follows that P_H satisfies Condition $(\boldsymbol{M}_{\nu_H,\psi_H})$ with $h_{R_H}^\infty = 0$ from the definition of H, and that g.c.d. $\{n \geq 1 : \nu_H(R_H^{n-1}\psi_H) > 0\} = 1$ since $\nu_H(R_H^{n-1}\psi_H) = \nu(R^{n-1}\psi)$. Thus P_H is irreducible from Theorem 4.1 applied to P_H, and P_H is aperiodic too. Finally note that the positive measure $\sum_{k=0}^{+\infty} \nu_H R_H^k$ is the restriction $\mu_{R|H}$ of μ_R to the set H, so that $\mu_{R|H}(\psi_H) = 1$ since $\mu_R(\psi) = 1$ and H is μ_R-full. Moreover the restriction $\pi_{R|H}$ of π_R to H is a P_H-invariant probability measure on $(H, \mathcal{X}_H)$. Hence Theorem 4.7 applied to P_H shows that, for every $x \in H$, we have $\lim_n \|\delta_x P_H^n - \pi_{R|H}\|_{TV} = 0$. Finally, since we have for every $x \in H$ and $A \in \mathcal{X}$

$$P^n(x, A) - \pi_R(1_A) = P^n(x, A \cap H) - \pi_R(1_{A\cap H}) = P_H^n(x, A \cap H) - \pi_{R|H}(1_{A\cap H})$$

we obtain that: $\forall x \in H$, $\lim_n \|\delta_x P^n - \pi_R\|_{TV} = 0$. This provides the expected conclusion since $\pi_R(1_H) = 1$. □

Proof of Theorem 4.7. The proof follows from the two next lemmas. Indeed assume that $h_R^\infty = 0$. Then $\lim_n P^n\psi = \pi_R(\psi)1_{\mathbb{X}}$ (point-wise convergence) from Lemma 4.9, thus the desired convergence in total variation norm holds from Lemma 4.11. Conversely assume that, for every $x \in \mathbb{X}$, we have $\lim_{n\to+\infty} \|\delta_x P^n - \pi_R\|_{TV} = 0$. Then it follows from the definition of $\|\cdot\|_{TV}$ that $\lim_{n\to+\infty}(P^n\psi)(x) = \pi_R(\psi)$ since ψ is bounded. Thus $h_R^\infty = 0$ from Lemma 4.9. □

Lemma 4.9 *Let P satisfy Condition $(\boldsymbol{M}_{\nu,\psi})$ with $\mu_R(1_{\mathbb{X}}) < \infty$. If P is irreducible and aperiodic, then*

$$h_R^\infty = 0 \iff \lim_{n\to+\infty}(P^n\psi) = \pi_R(\psi)1_{\mathbb{X}} \quad \textit{(point-wise convergence).}$$

Proof. The following power series

$$\mathcal{P}(z) := \sum_{n=0}^{+\infty} z^n P^n\psi \quad \text{and} \quad \mathcal{R}(z) := \sum_{n=0}^{+\infty} z^n R^n\psi$$

are well-defined on $\overline{D}$ since ψ is bounded. Note that $\mathcal{P}(z)$ and $\mathcal{R}(z)$ are function series. From the kernel identity (16) applied to ψ it follows that

$$\begin{aligned}\forall z\in\overline{D},\quad \mathcal{P}(z) = \sum_{n=0}^{+\infty} z^n P^n\psi &= \sum_{n=0}^{+\infty} z^n R^n\psi + \sum_{n=1}^{+\infty} z^n \sum_{k=1}^{n}\nu(R^{k-1}\psi)P^{n-k}\psi \\ &= \mathcal{R}(z) + \rho(z)\mathcal{P}(z).\end{aligned}$$

where $\rho(z)$ is the power series defined in (37). Using $\mu_R(\psi) = \sum_{k=1}^{+\infty}\nu(R^{k-1}\psi) = 1$ from Theorem 3.6, we have: $\forall z\in D,\ |\rho(z)|<1$ where $D = \{z\in\mathbb{C} : |z|<1\}$. Thus

$$\forall z\in D,\quad \mathcal{P}(z) = \mathcal{R}(z)\,U(z) \quad \text{with} \quad U(z) := \frac{1}{1-\rho(z)}. \tag{39}$$

Next, for any $k\geq 1$, we have $\nu(R^k1_{\mathbb{X}}) = \nu\big(R^{k-1}(R1_{\mathbb{X}})\big) = \nu(R^{k-1}1_{\mathbb{X}}) - \nu(1_{\mathbb{X}})\nu(R^{k-1}\psi)$ from $R1_{\mathbb{X}} = 1_{\mathbb{X}} - \nu(1_{\mathbb{X}})\psi$. Thus,

$$\forall k\geq 1,\quad \nu(1_{\mathbb{X}})\nu(R^{k-1}\psi) = \nu(R^{k-1}1_{\mathbb{X}}) - \nu(R^k1_{\mathbb{X}})$$

and

$$\begin{aligned}\forall n\geq 1,\quad \nu(1_{\mathbb{X}})\sum_{k=1}^{n} k\,\nu(R^{k-1}\psi) &= \sum_{k=1}^{n} k\big[\nu(R^{k-1}1_{\mathbb{X}}) - \nu(R^k1_{\mathbb{X}})\big] \\ &= \sum_{k=1}^{n} k\,\nu(R^{k-1}1_{\mathbb{X}}) - \sum_{k=2}^{n+1}(k-1)\,\nu(R^{k-1}1_{\mathbb{X}}) \\ &= \sum_{k=1}^{n}\nu(R^{k-1}1_{\mathbb{X}}) - n\,\nu(R^n1_{\mathbb{X}}).\end{aligned}$$

Hence $m := \sum_{k=1}^{+\infty} k\,\nu(R^{k-1}\psi) \leq \mu_R(1_{\mathbb{X}})\nu(1_{\mathbb{X}})^{-1} < \infty$. Now recall that $\sum_{k=1}^{+\infty}\nu(R^{k-1}\psi) = 1$ and that $\rho(z)$ is not a power series in z^q for any integer $q\geq 2$ since P is assumed to be aperiodic. Consequently the Erdös-Feller-Pollard renewal theorem [EFP49] provides the following property for the power series $U(z) = \sum_{k=0}^{+\infty} u_k z^k$ in (39):

$$\lim_{k\to+\infty} u_k = \frac{1}{m}.$$

Let $x\in\mathbb{X}$. Identifying the coefficients of the power series in Equation (39) (Cauchy product), we obtain that for every $n\geq 0$

$$(P^n\psi)(x) = \sum_{k=0}^{n} u_{n-k}(R^k\psi)(x) = \sum_{k=0}^{+\infty} v_n(k)(R^k\psi)(x) \quad \textit{with} \quad \forall k\geq 0,\ v_n(k) = u_{n-k}1_{[0,n]}(k).$$

For every $k \geq 1$, we have $\lim_n v_n(k) = 1/m$, and $|v_n(k)| \leq \sup_j |u_j| < \infty$. Moreover recall that $\sum_{k=0}^{+\infty}(R^k\psi)(x) < \infty$ from Proposition 3.4. Then it follows from Lebesgue theorem w.r.t. discrete measure that

$$\forall x \in \mathbb{X}, \quad \lim_n (P^n\psi)(x) = \frac{1}{m}\sum_{k=0}^{+\infty}(R^k\psi)(x). \tag{40}$$

Now we can prove Lemma 4.9. If $h_R^\infty = 0$, then we have $\sum_{k=0}^{+\infty}(R^k\psi)(x) = \nu(1_{\mathbb{X}})^{-1}$ from (34). Hence (40) provides: $\forall x \in \mathbb{X}$, $\lim_n (P^n\psi)(x) = (m\nu(1_{\mathbb{X}}))^{-1}$. Actually the constant $(m\nu(1_{\mathbb{X}}))^{-1}$ equals to $\pi_R(\psi)$ from Lebesgue theorem w.r.t. the $P-$invariant probability measure π_R. The direct implication in Lemma 4.9 is proved. Conversely, assume that $\lim_n P^n\psi = \pi_R(\psi)1_{\mathbb{X}}$ (point-wise convergence). Then we deduce from (40) that $\sum_{k=0}^{+\infty} R^k\psi = c\,1_{\mathbb{X}}$ with $c := m\pi_R(\psi)$. Thus $h_R^\infty = d\,1_{\mathbb{X}}$ with $d = 1 - c\nu(1_{\mathbb{X}})$ from (23). Finally recall that $\mu_R(\psi) = 1$, thus $\nu(h_R^\infty) = 0$ from (24). Hence $d\,\nu(1_{\mathbb{X}}) = 0$, from which we deduce that $h_R^\infty = 0$. □

Remark 4.10 *From the proof of Lemma 4.9 we deduce the following facts. If P satisfies Condition $(\boldsymbol{M}_{\nu,\psi})$ with $\mu_R(1_{\mathbb{X}}) < \infty$, then $m := \sum_{k=1}^{+\infty} k\,\nu(R^{k-1}\psi) < \infty$. If moreover P is irreducible and aperiodic and if $h_R^\infty = 0$, then $m = (\pi_R(\psi)\nu(1_{\mathbb{X}}))^{-1}$. Finally mention that, for the direct implication in the equivalence of Lemma 4.9, the renewal theorem in [Fel67, Th 1, p330] can be directly applied too.*

Lemma 4.11 *Assume that P satisfies Condition $(\boldsymbol{M}_{\nu,\psi})$ and $\mu_R(1_{\mathbb{X}}) < \infty$. If $h_R^\infty = 0$ and $\lim_n P^n\psi = \pi_R(\psi)1_{\mathbb{X}}$ (point-wise convergence), then $\lim_n \|\delta_x P^n - \pi_R\|_{TV} = 0$ for every $x \in \mathbb{X}$.*

Proof. Using (16) and $\pi_R = \pi_R(\psi)\sum_{k=1}^{+\infty} \nu R^{k-1}$ (see (25)), we have for every $n \geq 1$ and $g \in \mathcal{B}$

$$P^n g - \pi_R(g)1_{\mathbb{X}} = R^n g + \sum_{k=1}^{n} \nu(R^{k-1}g)\big(P^{n-k}\psi - \pi_R(\psi)1_{\mathbb{X}}\big) - \pi_R(\psi)\bigg(\sum_{k=n+1}^{+\infty} \nu(R^{k-1}g)\bigg)1_{\mathbb{X}}.$$

Thus

$$\|\delta_x P^n - \pi_R\|_{TV} \leq (R^n 1_{\mathbb{X}})(x) + \sum_{k=1}^{n} \nu(R^{k-1}1_{\mathbb{X}})\big|(P^{n-k}\psi)(x) - \pi_R(\psi)\big| + \pi_R(\psi)\sum_{k=n+1}^{+\infty} \nu(R^{k-1}1_{\mathbb{X}}).$$

We have $\lim_n (R^n 1_{\mathbb{X}})(x) = 0$ from $h_R^\infty = 0$. The last term in the right hand side of the previous inequality also converges to zero when $n \to +\infty$ since $\sum_{k=0}^{+\infty} \nu(R^k 1_{\mathbb{X}}) = \mu_R(1_{\mathbb{X}}) < \infty$. Next note that

$$\sum_{k=1}^{n} \nu(R^{k-1}1_{\mathbb{X}})\big|(P^{n-k}\psi)(x) - \pi_R(\psi)\big| = \sum_{k=1}^{+\infty} \nu(R^{k-1}1_{\mathbb{X}})f_n(k)$$

with $f_n(k) := |(P^{n-k}\psi)(x) - \pi_R(\psi)|1_{[1,n]}(k)$. Then, using $\sum_{k=1}^{+\infty} \nu(R^{k-1}1_{\mathbb{X}}) < \infty$, the above sum converges to zero when $n \to +\infty$ from Lebesgue's theorem w.r.t. discrete measure since, for every $k \geq 1$, we have $f_n(k) \leq 2\|\psi\|_{1_{\mathbb{X}}}$ and $\lim_n f_n(k) = 0$ by hypothesis. Lemma 4.11 is proved. □

4.3 Convergence of iterates: the periodic case

Assume that P satisfies Condition $(\boldsymbol{M}_{\nu,\psi})$ and is irreducible. Recall that the power series $\rho(z)$ given in (37), namely

$$\rho(z) := \sum_{n=1}^{+\infty} \nu(R^{n-1}\psi)\, z^n$$

is defined on $\overline{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ and is non-zero. Define

$$d := \text{g.c.d.}\,\{n \geq 1 : \nu(R^{n-1}\psi) > 0\} \tag{41}$$

where g.c.d. stands for greatest common divisor computed on a non-empty set. If $d = 1$, then P is aperiodic according to the definition of Subsection 4.2. If $d \geq 2$, then P is said to be *periodic*: In this case $\rho(z)$ is a power series in z^d. Under Conditions $(\boldsymbol{M}_{\nu,\psi})$ and $h_R^\infty = 0$, Integer d in (41) can be called the *period* of P without any ambiguity. Indeed under these two conditions, various equivalent characterizations of Integer d in (41) are presented in Theorem 4.14 below. Actually, from Assertion (b) of Theorem 4.14, the value of d does not depend on the choice of the couple (ν, ψ) in the minorization condition $(\boldsymbol{M}_{\nu,\psi})$.

Recall that Conditions $(\boldsymbol{M}_{\nu,\psi})$, $h_R^\infty = 0$ implies that P is irreducible and if moreover $\mu_R(1_{\mathbb{X}}) < \infty$ then π_R is the unique $P-$invariant probability measure from Theorem 4.1. Under these three conditions, the convergence in total variation norm of the probability measures $\sum_{r=0}^{d-1} \delta_x P^{nd+r}$ to π_R is obtained in the next theorem. In fact the two next statements are the natural extensions to the periodic case of Theorem 4.7 and Corollary 4.8.

Theorem 4.12 *Let P satisfy Condition $(\boldsymbol{M}_{\nu,\psi})$ with $\mu_R(1_{\mathbb{X}}) < \infty$ and $h_R^\infty = 0$. If P is periodic with period $d \geq 2$ (see (41)), then the following convergence holds:*

$$\forall x \in \mathbb{X}, \quad \lim_{n \to +\infty} \big\|\pi_R - \frac{1}{d}\sum_{r=0}^{d-1} \delta_x P^{nd+r}\big\|_{TV} = 0.$$

The proof of Theorem 4.12 is similar to that of the direct implication of Theorem 4.7 (where $d = 1$). When $d \geq 2$, the proof is just a little more technical, since we have to work with the sums $\frac{1}{d}\sum_{r=0}^{d-1} \delta_x P^{nd+r}$. This proof is postponed in Appendix B.

Corollary 4.13 *Let P satisfy Condition $(\boldsymbol{M}_{\nu,\psi})$ with $\mu_R(1_{\mathbb{X}}) < \infty$. If P is irreducible and periodic with $d \geq 2$ in (41), then the following convergence holds :*

$$\lim_{n \to +\infty} \big\|\pi_R - \frac{1}{d}\sum_{r=0}^{d-1} \delta_x P^{nd+r}\big\|_{TV} = 0 \quad \textit{for } \pi_R\textit{−almost every } x \in \mathbb{X}.$$

Proof. Using the restriction P_H of P to the μ_R-full $P-$absorbing set $H := \{h_R^\infty = 0\}$ from Lemma 4.6, Corollary 4.13 is deduced from Theorem 4.12 proceeding as for Corollary 4.8: Use $\text{g.c.d.}\,\{n \geq 1 : \nu_H(R_H^{n-1}\psi_H) > 0\} = d$ from $\nu_H(R_H^{n-1}\psi_H) = \nu(R^{n-1}\psi)$, and apply Theorem 4.12 to the sums $\frac{1}{d}\sum_{r=0}^{d-1} \delta_x P_H^{nd+r}$ to conclude. □

In the next statement the space $\mathcal{B} = \mathcal{B}_{1_{\mathbb{X}}}$ is extended to complex-valued functions, i.e.:

$$\mathcal{B}(\mathbb{C}) := \big\{g : \mathbb{X} \to \mathbb{C}, \text{measurable such that } \|g\|_{1_X} := \sup_{x\in\mathbb{X}} |g(x)| < \infty\big\}$$

where $|\cdot|$ stands here for the modulus in $\mathbb{C}$. Recall that $z \in \mathbb{C}$ is said to be an eigenvalue of P on $\mathcal{B}(\mathbb{C})$ if there exists a non-zero function $g \in \mathcal{B}(\mathbb{C})$ such that $Pg = zg$. Finally recall that P is irreducible under Conditions $(\boldsymbol{M}_{\nu,\psi})$ and $h_R^\infty = 0$ from Theorem 4.1, so that the positive integer $d = \text{g.c.d.}\,\{n \geq 1 : \nu(R^{n-1}\psi) > 0\}$ in (41) is well-defined in the next statement.

Theorem 4.14 *Assume that P satisfies Condition $(\boldsymbol{M}_{\nu,\psi})$ and $h_R^\infty = 0$. Let $\rho(z)$ be the power series given in* (37)*, and let $d := g.c.d.\,\{n \geq 1 : \nu(R^{n-1}\psi) > 0\}$. Then the following assertions holds and are equivalent:*

(a) The complex numbers z of modulus one satisfying $\rho(z) = 1$ are the $d-$th roots of unity.

(b) The eigenvalues of modulus one of P on $\mathcal{B}(\mathbb{C})$ are the $d-$th roots of unity.

(c) There exist a μ_R-full $P-$absorbing set $E \in \mathcal{X}$ and sets $C_0, \ldots, C_{d-1}$ in $\mathcal{X}$ such that

$$E = \bigsqcup_{\ell=0}^{d-1} C_\ell \quad \text{with} \quad \forall \ell = 0, \ldots, d-1,\ \forall x \in C_\ell, \quad P(x, C_{\ell+1}) = 1$$

using the convention $C_d = C_0$.

Under Condition $(\boldsymbol{M}_{\nu,\psi})$ and $h_R^\infty = 0$, that any of the three equivalent conditions (a)-(c) characterizes the period of P, is obvious. Indeed, assume that P satisfies Assertion (a) for some $d \geq 1$, and set $d' := \text{g.c.d.}\,\{n \geq 1 : \nu(R^{n-1}\psi) > 0\}$. Then the complex numbers z of modulus one satisfying $\rho(z) = 1$ are the $d'-$th roots of unity from Theorem 4.14, thus $d' = d$.

The proof of Theorem 4.14 is based on the following two lemmas.

Lemma 4.15 *Let P satisfy Condition $(\boldsymbol{M}_{\nu,\psi})$ and $h_R^\infty = 0$. Let $z \in \mathbb{C}$ be such that $|z| = 1$. Then z is an eigenvalue of P on $\mathcal{B}(\mathbb{C})$ if, and only if, we have $\rho(z) = 1$. Moreover, if any of these two conditions holds, then*

$$E_z := \{g \in \mathcal{B}(\mathbb{C}) : Pg = zg\} = \mathbb{C} \cdot \widetilde{\psi}_z \quad \text{with} \quad \widetilde{\psi}_z := \sum_{k=0}^{+\infty} z^{-(k+1)} R^k \psi.$$

Proof. First note that, for any $z \in \mathbb{C}$ such that $|z| = 1$, the above function $\widetilde{\psi}_z$ is well-defined and belongs to $\mathcal{B}(\mathbb{C})$ from Proposition 3.4. Moreover observe that

$$\nu(\widetilde{\psi}_z) = \sum_{k=0}^{+\infty} z^{-(k+1)} \nu(R^k\psi) = \rho(z^{-1}), \tag{42}$$

the exchange between series and $\nu-$integral being valid since $\sum_{k=0}^{+\infty} \nu(R^k\psi) < \infty$ from Proposition 3.4. Now, let $z \in \mathbb{C}$, $|z| = 1$, and let $g \in \mathcal{B}(\mathbb{C})$, $g \neq 0$, be such that $Pg = zg$. Thus we have $\nu(g)\psi = (zI - R)g$ from $P = R + \psi \otimes \nu$. Then we have for every $n \geq 0$

$$\begin{aligned} \nu(g) \sum_{k=0}^{n} z^{-(k+1)} R^k \psi = \Big(\sum_{k=0}^{n} z^{-(k+1)} R^k\Big)(zI - R)g &= \sum_{k=0}^{n} z^{-k} R^k g - \sum_{k=0}^{n} z^{-(k+1)} R^{k+1} g \\ &= g - z^{-(n+1)} R^{n+1} g. \end{aligned} \tag{43}$$

Moreover we have $|R^n g| \leq \|g\|_{1_{\mathbb{X}}} R^n 1_{\mathbb{X}}$, so $\lim_n R^n g = 0$ (point-wise convergence) from Condition $h_R^\infty = 0$. Hence $g = \nu(g)\widetilde{\psi}_z$, with $\nu(g) \neq 0$ since $g \neq 0$ by hypothesis. From (42) it follows that $\nu(g) = \nu(g)\rho(z^{-1})$, thus $\rho(z^{-1}) = 1$, or equivalently $\rho(z) = 1$ from $z^{-1} = \overline{z}$ (the conjugate of z) since $|z| = 1$ and the coefficients of the power series $\rho(\cdot)$ are real (even non-negative).

Conversely let $z \in \mathbb{C}$, $|z| = 1$, be such that $\rho(z) = 1$, thus $\rho(z^{-1}) = 1$. From (42) we have $\nu(\widetilde{\psi}_z) = 1$. Using $P = R + \psi \otimes \nu$ and Lebesgue's theorem w.r.t. $R(x, dy)$ for each $x \in \mathbb{X}$ we obtain that

$$P\widetilde{\psi}_z = z\sum_{k=0}^{+\infty} z^{-(k+2)} R^{k+1}\psi + \nu(\widetilde{\psi}_z)\psi = z\big(\widetilde{\psi}_z - z^{-1}\psi\big)\psi + \psi \;=\; z\widetilde{\psi}_z.$$

Thus z is an eigenvalue of P on $\mathcal{B}(\mathbb{C})$ since $\widetilde{\psi}_z \neq 0$ from $\nu(\widetilde{\psi}_z) = 1$. The claimed equivalence in Lemma 4.15 is proved. The last assertion follows from the first part of the proof, where we obtained that any $g \in \mathcal{B}(\mathbb{C})$ such that $Pg = zg$ with $|z| = 1$ satisfies $g = \nu(g)\widetilde{\psi}_z$. □

Lemma 4.16 *Let P satisfy Condition $(\boldsymbol{M}_{\nu,\psi})$ and $h_R^\infty = 0$. Let $z \in \mathbb{C}$ be such that $|z| = 1$. Then we have $\rho(z) = 1$ if, and only if, z is a $d-$th root of unity with d given in (41).*

Proof. Recall that $\mu_R(\psi) = \sum_{n=1}^{+\infty} \nu(R^{n-1}\psi) = 1$ from Theorem 4.1. Assume that $\rho(z) = 1$. Then

$$\sum_{n=1}^{+\infty} \nu(R^{n-1}\psi)\, z^n = 1 = \sum_{n=1}^{+\infty} \nu(R^{n-1}\psi).$$

Writing $z = e^{i\theta}$ with $\theta \in [0, 2\pi)$ we obtain that $\sum_{n=1}^{+\infty} \big(1 - \cos(n\theta)\big)\nu(R^{n-1}\psi) = 0$. Define the set $\mathcal{N} := \{n \geq 1 : \nu(R^{n-1}\psi) > 0\}$. Then $n \in \mathcal{N}$ implies that $\cos(n\theta) = 1$. Equivalently we have: $\forall n \in \mathcal{N}, z^n = 1$. Next from the definition of d, for p large enough there exists $\{n_j\}_{j=1}^p \in \mathcal{N}^p$ such that $d = \sum_{j=1}^p k_j n_j$ for some $\{k_j\}_{j=1}^p \in \mathbb{Z}^p$ (Bézout identity). Thus we have $z^d = \prod_{j=1}^p z^{k_j n_j} = 1$ since $z^{n_j} = 1$. Hence z is a $d-$th root of unity.

Conversely, let z be a $d-$th root of unity, i.e. $z^d = 1$. From the definition of d it then follows that $\rho(z) = \sum_{k=0}^{+\infty} \nu(R^{kd-1}\psi)\, z^{kd} = \mu_R(\psi) = 1$.

□

Now we prove Theorem 4.14.

Proof of Theorem 4.14. Assertion (a) is proved in Lemma 4.16, and the equivalence $(a) \Leftrightarrow (b)$ follows from Lemma 4.15. Now let us assume that P satisfies Assertion (b). Let $z_d = e^{2i\pi/d}$, $\widetilde{\psi}_d := \sum_{k=0}^{+\infty} z_d^{-(k+1)} R^k\psi$, and let $\widetilde{\psi}_{d,0}$ (resp. $\widetilde{\psi}_{d,1}$) denote the real (resp. imaginary) part of the function $\widetilde{\psi}_d$. Then it follows from (34) that

$$\widetilde{\psi}_{d,0} \leq |\widetilde{\psi}_d| \leq \sum_{k=0}^{+\infty} R^k\psi = \nu(1_{\mathbb{X}})^{-1}1_{\mathbb{X}}.$$

Since z_d is an eigenvalue of P on $\mathcal{B}(\mathbb{C})$ we have $\rho(z_d^{-1}) = 1$ from Lemma 4.15, thus $\nu(\widetilde{\psi}_d) = 1$ from (42). Then we have $\nu(\widetilde{\psi}_{d,0}) = 1 = \nu(\nu(1_{\mathbb{X}})^{-1}1_{\mathbb{X}})$, so that the following equalities hold $\nu-$a.e. on $\mathbb{X}$: $\widetilde{\psi}_{d,0} = \nu(1_{\mathbb{X}})^{-1}1_{\mathbb{X}}$ and $\widetilde{\psi}_{d,1} = 0$. Now define $g_d := \nu(1_{\mathbb{X}})\widetilde{\psi}_d$. From the above we know that $|g_d| \leq 1_{\mathbb{X}}$ and that the set $C_0 := \{g_d = 1\}$ is non-empty. Moreover we have $Pg_d = z_d g_d$ from Lemma 4.15. Let $x \in C_0$. Then

$$1 = g_d(x) = \frac{(Pg_d)(x)}{z_d} = \int_{\mathbb{X}} \frac{g_d(y)}{z_d} P(x, dy)$$

with $|g_d(y)/z_d| \leq 1$ for every $y \in \mathbb{X}$ since $|z_d| = 1$. It follows that $P(x, C_1) = 1$ where $C_1 := \{x \in \mathbb{X} : g_d(x) = z_d\}$. Replacing the set C_0 with C_1, we can similarly prove that, for every $x \in C_1$, we have $P(x, C_2) = 1$ where $C_2 := \{x \in \mathbb{X} : g_d(x) = z_d^2\}$. Repeating this arguments provides the existence of sets $C_0, \ldots, C_{d-1}$ in $\mathcal{X}$ satisfying the desired cycle property: $\forall \ell = 0, \ldots, d-1,\ \forall x \in C_\ell,\ P(x, C_{\ell+1}) = 1$. These sets are obviously disjoint. Finally define $E := \bigsqcup_{\ell=0}^{d-1} C_\ell$. This set is $P-$absorbing since, for every $x \in E$, there exists a (unique) $\ell \in \{0, \ldots, d-1\}$ such that $x \in C_\ell$, so that $1 = P(x, C_{\ell+1}) \leq P(x, E) \leq 1$, thus $P(x, E) = 1$. Since P is irreducible from Theorem 4.1, the set E is μ_R-full from Proposition 3.16. We have proved that (b) implies (c).

It remains to prove that (c) implies (a). Assume that P satisfies Assertion (c) and let P_E be the restriction of P to the μ_R-full $P-$absorbing set $E = \sqcup_{\ell=0}^{d-1} C_\ell$. Let z be any $d-$th root of unity and define $g_E : E \to \mathbb{C}$ by

$$\forall \ell = 0, \ldots, d-1,\ \forall x \in C_\ell, \quad g_E(x) = z^\ell.$$

Then we have for every $\ell = 0, \ldots, d-1$ and $x \in C_\ell$

$$(P_E g_E)(x) = \int_E g_E(y) P(x, dy) = \int_{C_{\ell+1}} g_E(y) P(x, dy) = z^{\ell+1} = z\, g_E(x)$$

since $P(x, C_{\ell+1}) = 1$ and $g_E(x) = z^\ell$, recalling moreover for the case $\ell = d-1$ that $C_d = C_0$ by convention and that $1 = z^d$. Thus $P_E g_E = z g_E$. Next recall that $\mu_R(\psi) = 1$ from Theorem 4.1. It then follows from Lemma 4.2 that P_E satisfies Condition $(\boldsymbol{M}_{\nu_E, \psi_E})$ on $(E, \mathcal{X}_E)$, that $h^\infty_{R_E} = 0$ on E from the assumption $h^\infty_R = 0$, and finally that

$$\forall z \in \overline{D}, \quad \rho_E(z) := \sum_{n=1}^{+\infty} \nu_E(R_E^{n-1} \psi_E)\, z^n = \rho(z).$$

We can now conclude. Since z is an eigenvalue of P_E, Lemma 4.15 applied to P_E ensures that $\rho_E(z) = 1$, so $\rho(z) = 1$. We have proved that, under Condition (c), any $d-$th root of unity satisfies Equation $\rho(z) = 1$. Moreover we know from Lemma 4.16 that any $z \in \mathbb{C}$ satisfying $|z| = 1$ and $\rho(z) = 1$ is a $d-$th root of unity. Thus (c) implies (a). □

4.4 Drift condition to obtain $h^\infty_R = 0$

Now, we introduce a drift condition to have the property $h^\infty_R := \lim_n R^n 1_{\mathbb{X}} = 0$, the relevance of which has been highlighted in Theorems 4.1, 4.3, 4.7, 4.12. Actually, under a drift inequality w.r.t. some measurable function $W : \mathbb{X} \to [0, +\infty)$, the property $h^\infty_R = 0$ is characterized in Proposition 4.17 by a control of h^∞_R or $\sum_{k=0}^{+\infty} R^k \psi$ on any level set $\mathcal{W}_r := \{x \in \mathbb{X} : W(x) \leq r\}$ of W. Finally, a condition ensuring this control is provided by Corollary 4.18.

Proposition 4.17 *Let P satisfy Condition $(\boldsymbol{M}_{\nu,\psi})$ and the following drift condition for some measurable function $W : \mathbb{X} \to [0, +\infty)$:*

$$\exists b > 0, \quad PW \leq W + b\,\psi. \tag{44}$$

For any $r>0$ let $\mathcal{W}_r$ denote the level set of order r defined by: $\mathcal{W}_r := \{x \in \mathbb{X} : W(x) \le r\}$. Then we have the following equivalences

$$h_R^\infty = 0 \iff \forall r>0,\ \sup_{x\in\mathcal{W}_r} h_R^\infty(x) < 1 \iff \forall r>0,\ \inf_{x\in\mathcal{W}_r} \sum_{k=0}^{+\infty}(R^k\psi)(x) > 0. \tag{45}$$

Proof. The second equivalence in (45) follows from (23). That $h_R^\infty = 0$ implies the second condition in (45) is obvious. It remains to prove that the second condition in (45), or equivalently the third one, implies that $h_R^\infty = 0$.

In the sequel, the third condition in (45) is assumed to hold. First prove that we have the following point-wise convergence on $\mathbb{X}$

$$\forall \rho > 0, \quad \lim_n R^n 1_{\mathcal{W}_\rho} = 0. \tag{46}$$

Let $\rho > 0$ and define $a \equiv a_\rho := \inf_{x\in\mathcal{W}_\rho} \sum_{k=0}^{+\infty}(R^k\psi)(x)$. By hypothesis we have $a > 0$ and $1_{\mathcal{W}_\rho} \le a^{-1}\sum_{k=0}^{+\infty} R^k\psi$, from which we deduce that

$$\forall n \ge 1, \quad 0 \le R^n 1_{\mathcal{W}_\rho} \le a^{-1} \sum_{k=n}^{+\infty} R^k\psi$$

from the monotone convergence theorem w.r.t. $R^n(x, dy)$ for each $x \in \mathbb{X}$. Property (46) then holds since the series $\sum_{k=0}^{+\infty} R^k\psi$ converges point-wise from Proposition 3.4.

Next note that $\nu(W)\psi \le PW$ everywhere on $\mathbb{X}$ from $(\boldsymbol{M}_{\nu,\psi})$, so that $\nu(W) < \infty$ and RW is well-defined. Let $d := \max(0\,, (b - \nu(W))/\nu(1_\mathbb{X}))$ and prove that

$$RW_d \le W_d \quad \text{where } W_d := W + d1_\mathbb{X}. \tag{47}$$

Note that $\nu(W_d) = \nu(W) + d\nu(1_\mathbb{X}) < \infty$ and that $PW_d = PW + d1_\mathbb{X}$. It then follows from $RW_d = PW_d - \nu(W_d)\psi$ and from the drift inequality (44) that

$$RW_d \le W + b\psi + d1_\mathbb{X} - \big(\nu(W) + d\,\nu(1_\mathbb{X})\big)\psi \le W_d + \big(b - \nu(W) - d\,\nu(1_\mathbb{X})\big)\psi$$

so that $RW_d \le W_d$ from the definition of d.

Now let us deduce from (46) and (47) that $h_R^\infty = 0$. Let $r > d$ with d given by (47). We have

$$1_\mathbb{X} = 1_{\{x\in\mathbb{X}:W_d(x)>r\}} + 1_{\{x\in\mathbb{X}:W_d(x)\le r\}} \le \frac{W_d}{r} + 1_{\mathcal{W}_{r-d}}.$$

Thus we get

$$\forall n \ge 1, \quad R^n 1_\mathbb{X} \le \frac{R^n W_d}{r} + R^n 1_{\mathcal{W}_{r-d}} \le \frac{W_d}{r} + R^n 1_{\mathcal{W}_{r-d}}$$

from the non-negativity of R and from $R^n W_d \le W_d$ using (47) and an immediate induction. Let $x \in \mathbb{X}$, $\varepsilon > 0$, and fix $r > d$ large enough so that $W_d(x)/r < \varepsilon/2$. From (46) applied to $\rho = r - d$, there exists $N \ge 1$ such that, for every $n \ge N$, we have $0 \le (R^n 1_{\mathcal{W}_{r-d}})(x) < \varepsilon/2$. Thus: $\forall n \ge N,\ 0 \le (R^n 1_\mathbb{X})(x) < \varepsilon$. This proves that $h_R^\infty = 0$.

□

We conclude this section providing an alternative sufficient condition for $h_R^\infty = 0$. Let us consider the Markov resolvent kernel Q defined in (30), i.e. $Q := \sum_{n=0}^{+\infty} 2^{-(n+1)} P^n$.

Corollary 4.18 *Let P satisfy Condition ($\boldsymbol{M}_{\nu,\psi}$) and the drift condition (44) for some measurable function $W : \mathbb{X} \to [0, +\infty)$. If the following condition holds*

$$\forall r > 0, \ \inf_{x \in \mathcal{W}_r} (Q\psi)(x) > 0, \tag{48}$$

then $h_R^\infty = 0$.

Proof. Below we prove that the third condition in (45) is fulfilled. The claimed conclusion then follows from Proposition 4.17. Recall that $\psi \in \mathcal{B}_+^*$, so that $Q\psi$ and the series $\sum_{n=0}^{+\infty} 2^{-(n+1)} R^n\psi$ are well-defined. Using (31) with ψ in place of 1_A,we obtain that

$$Q\psi = \sum_{n=0}^{+\infty} 2^{-(n+1)} R^n\psi + \alpha\, Q\psi$$

where $\alpha := \sum_{k=1}^{+\infty} 2^{-k}\nu(R^{k-1}\psi)$. Note that, either $\alpha = 0$, or $\alpha < \mu_R(\psi) \le 1$ from Proposition 3.4, so that

$$\sum_{n=0}^{+\infty} 2^{-(n+1)} R^n\psi = (1-\alpha)\, Q\psi \quad \text{with } 1-\alpha > 0.$$

Now let $r > 0$ and $a \equiv a_r := \inf_{x \in \mathcal{W}_r} (Q\psi)(x)$. We have $a > 0$ from (48), and

$$\forall x \in \mathcal{W}_r, \quad \sum_{k=0}^{+\infty} (R^k\psi)(x) \ge \sum_{k=0}^{+\infty} 2^{-(k+1)} (R^k\psi)(x) = (1-\alpha)\,(Q\psi)(x) \ge (1-\alpha)a > 0.$$

The third condition in (45) is proved. □

Condition (48) on Q is obviously satisfied under the following stronger condition

$$\forall r > 0, \ \exists q \equiv q(r) \ge 1, \ \inf_{x \in \mathcal{W}_r} (P^q\psi)(x) > 0. \tag{49}$$

Note that requiring Condition (49) means requiring that the irreducibility property for P (see (28)) holds uniformly on each level set $\mathcal{W}_r$. This condition is relevant only for unbounded function W. Indeed, otherwise, the set $\mathcal{W}_r$ is the whole space $\mathbb{X}$ for r large enough, and in this case Condition (49) is restrictive since it requires that $\inf_{x \in \mathbb{X}} (P^q\psi)(x) > 0$ for some $q \ge 1$. If $\mathbb{X}$ is discrete (say $\mathbb{X} = \mathbb{N}$) and $W = (W(n))_{n \in \mathbb{N}}$ is an unbounded increasing sequence, then the sets $\mathcal{W}_r$ are finite: In this case, Condition (49) holds if, and only if,

$$\forall s \in \mathbb{N}, \ \exists q \equiv q(s) \ge 1, \ \forall i \in \{0, \dots, s\}, \quad (P^q\psi)(i) > 0.$$

If X is a non-discrete topological space, then a natural assumption for Condition (49) to be fulfilled is that, for every $r > 0$, the set $\mathcal{W}_r$ is compact. However this is not sufficient. An additional natural assumption is that P is weakly Feller (i.e. if $g \in \mathcal{B}$ is continuous on $\mathbb{X}$, then so is Pg). Under these two assumptions, Condition (49) actually holds provided that there exists a bounded and continuous function ψ_0 such that $0 \le \psi_0 \le \psi$ and

$$\forall r > 0, \ \exists q \equiv q(r) \ge 1, \ \forall x \in \mathcal{W}_r, \quad (P^q\psi_0)(x) > 0.$$

Indeed the continuous function $P^q\psi_0$ then reaches its lower bound on the compact set $\mathcal{W}_r$, and this lower bound is thus positive under the previous condition.

4.5 Bibliographic comments

In the present bibliographic discussion we assume that P is irreducible. The uniqueness of $1_{\mathbb{X}}$ (up to a multiplicative constant) as $P-$harmonic functions is classically studied in link with the Harris-recurrence assumption. This is done in [Num84, Th. 3.8, p. 44], [MT09, Th. 17.1.5] and [DMPS18, Th. 10.2.11], essentially using the fact that, for a Markov chain $(X_n)_{n\geq 0}$ on $\mathbb{X}$ and for every $A \in \mathcal{X}$, the function $g_A^\infty : x \mapsto \mathbb{P}_x\{X_k \in A \text{ i.o.}\}$ is a $P-$harmonic function, where i.o. stands for infinitely often. Similarly, under the aperiodicity condition, the Harris-recurrence assumption is classically used to prove the convergence in total variation of the iterates of P to its (unique) invariant probability measure π (i.e. $\forall x \in \mathbb{X}$, $\lim_n \|\delta_x P^n - \pi\|_{TV} = 0$). This is proved in [MT09, Ths. 13.0.1, 13.3.5] and [DMPS18, Th. 11.3.1] via renewal theory and splitting construction, also see [RR04, Th. 4] for a proof based on coupling method.

In this section we choose a different approach, first focusing on function $h_R^\infty := \lim_n R^n 1_{\mathbb{X}}$ introduced in the previous section. Indeed the condition $h_R^\infty = 0$ enables us to prove the above conclusion on $P-$harmonic functions (Theorem 4.1), from which the Harris-recurrent property can be derived in Theorem 4.3 using the fact that for every $A \in \mathcal{X}$ the function $x \mapsto \mathbb{P}_x\{X_k \in A \text{ i.o.}\}$ is $P-$harmonic (no surprise there). In the case when measure μ_R is finite and P is aperiodic, the condition $h_R^\infty = 0$ is proved to be equivalent to the above mentioned iterate convergence in total variation (Theorem 4.7). So, to put it simply, the presentation in this section and the resulting statements focus on the condition $h_R^\infty = 0$ depending on the residual kernel R, rather than on the Harris-recurrence property. However note that the proof of Theorem 4.7 is original: Actually Property (23) and the power series formula (39) simply derived from the key equality (16) allow us to directly apply the renewal theorem proved in the seminal paper [EFP49] by Erdös, Feller and Pollard, to the power series $\rho(z)$ in (37) used to define the aperiodicity condition.

If P is recurrent, then the $P-$harmonic functions are still constant, but up to a negligible set w.r.t. to some maximal irreducibility measure, e.g. see [Num84, Prop. 3.13, p. 44]. In the same way, if P admits an invariant probability measure π, so that P is recurrent from a classical result (e.g. see [DMPS18, Th. 10.1.6]), then the property $\lim_n \|\delta_x P^n - \pi_R\|_{TV} = 0$ is known to hold for $\pi-$almost every $x \in \mathbb{X}$, e.g. see [DMPS18, Th. 11.3.1] and [RR04, pp. 32-33]. This is here highlighted using the explicit set $H = \{h_R^\infty = 0\}$ which is $P-$absorbing and μ_R-full under the recurrence condition (see Corollary 4.5 and the proof of Corollary 4.8). Complements using splitting construction can be found in [Num84, Cor. 5.1, p. 71].

Under the irreducibility condition the d-cycle property for P stated in Assertion (c) of Theorem 4.14 is the standard definition of the period of P, see [MT09, p. 114] and [DMPS18, Def. 9.3.5]. In our work, under Condition $(\boldsymbol{M}_{\nu,\psi})$ and irreducibility condition, Integer d is defined by $d := \text{g.c.d.}\,\{n \geq 1 : \nu(R^{n-1}\psi) > 0\}$. Then the alternative characterizations of this integer d, in particular the d-cycle property for P, are proved under the condition $h_R^\infty = 0$ in Theorem 4.14. The convergence in total variation norm stated in Theorem 4.12 corresponds to the standard statements [MT09, Th. 13.3.4] and [DMPS18, Cor. 11.3.2], except that the condition $h_R^\infty = 0$ is used here in Theorem 4.12 instead of the Harris-recurrence condition in [MT09, DMPS18]. In the same way the π_R-a.e. convergence in total variation norm obtained in Corollary 4.13 corresponds to the standard results in [MT09, Th. 13.3.4] and [DMPS18, Cor. 11.3.2]. Again the direct use of the μ_R-full $P-$absorbing set $H = \{h_R^\infty = 0\}$ provides a short proof of Corollary 4.13. The proofs in [MT09, Th. 13.3.4] and [DMPS18, Cor. 11.3.2] are based on the $d-$cycles property given in Assertion (c) of Theorem 4.14. However, since the set E of Theorem 4.14 is not the whole set $\mathbb{X}$ a priori (E is only μ_R-full), additional work is then required to obtain the conclusion of Theorem 4.14 (i.e. convergence for all $x \in \mathbb{X}$). The proof given in Appendix B does not rely on the $d-$cycles property: it adapts the arguments of the direct implication of Theorem 4.7 to the periodic case, thus directly giving the conclusion of Theorem 4.14.

The sufficient condition provided in Proposition 4.17 for the condition $h_R^\infty = 0$ to hold is the analogue of the standard statements ensuring that P is recurrent or Harris-recurrent under drift condition, e.g. see [Num84, Prop. 5.10, p. 77], [MT09, Th. 8.4.3] [DMPS18, Th. 10.2.13]. More precisely the drift inequality (44) in Proposition 4.17 is the same as in the previously cited works. Moreover Condition (45) in Proposition 4.17 replaces the classical assumption that W is unbounded off petite set (i.e. each level set $\mathcal{W}_r := \{W \le r\}$ is a petite set). This last condition means that, for every $r > 0$, there exists $a := (a_n)_n \in [0,1]^{\mathbb{N}}$ with $\sum_{n=0}^{+\infty} a_n = 1$ and a positive measure $\nu_{r,a}$ such that $Q_a \ge 1_{\mathcal{W}_r} \otimes \nu_{r,a}$ where $Q_a := \sum_{n=0}^{+\infty} a_n P^n$. Expressed with $a_n = 2^{-(n+1)}$, this assumption is clearly stronger than Condition (48) in Corollary 4.18, which only focusses on the lower bound of the function $Q\psi$ on $\mathcal{W}_r$ (no minorizing measure is involved in (48)).

Before diving into the details of the modulated drift condition used in the next sections, let us present some comment on the probabilistic meaning of the simpler drift condition (44). Let $(X_n)_{n\ge0}$ be a Markov chain with state space $\mathbb{X}$ and transition kernel P. Let $W : \mathbb{X} \to [0,+\infty)$ be measurable. For any $r > 0$ the set $\mathcal{W}_r = \{x \in \mathbb{X} : W(x) \le r\}$ must be thought of as the level set of order r in $\mathbb{X}$ w.r.t. the function W. Since $(PW)(x) = \mathbb{E}_x\big[W(X_1)\big]$ for any $x \in \mathbb{X}$, the Markov kernel P satisfies Condition (44) with $\psi = 1_{\mathcal{W}_s}$ for some $s > 0$ if, and only if,

$$\sup_{x\in\mathcal{W}_s} \mathbb{E}_x\big[W(X_1)\big] < \infty \quad \text{and} \quad \forall x \in \mathbb{X} \setminus \mathcal{W}_s, \quad \mathbb{E}_x\big[W(X_1)\big] \le W(x). \tag{50}$$

The second condition in (50) means that, for any $r > s$, each point $x \in \mathbb{X}$ such that $W(x) = r$ transits in mean in $\mathcal{W}_r$. If $\mathbb{X} = \mathbb{R}^d$ is equipped with some norm $\|\cdot\|$, then W may be of the form $W = v(\|\cdot\|)$ with unbounded increasing function $v : [0,+\infty) \to [0,+\infty)$. In particular, if $W = \|\cdot\|$, then the second condition in (50) means that, starting from $x \in \mathbb{R}^d$ far enough from the origin, the state visited after a first transition of the Markov chain admits in mean a norm less than $\|x\|$, namely is closer to the origin. For a random walk on $\mathbb{N}$, it means that, for i large enough, the steps of the walker starting from i are in mean more to the left than to the right, namely it tends to go back towards 0. In case $\mathbb{X} = \mathbb{Z}$ and $W(x) = |x|$, a typical illustration of the explicit computations needed for obtaining the drift inequality (44) can be found in [MT09, Sect. 8.4.3] for random walks with bounded range and zero mean increment. If $(\mathbb{X}, d)$ is a metric space and $W(x) = d(x, x_0)$, level sets are the balls centred at x_0. However the possibility of considering other level functions more suited to the transition kernel (i.e. possibly considering level sets other than balls) offers flexibility for the validity of Conditions (50) or of the modulated drift condition involved in the next sections.

5 Modulated drift condition and Poisson's equation

Throughout this section, the Markov kernel P is assumed to satisfy the minorization Condition $(\boldsymbol{M}_{\nu,\psi})$. Then, the following V_1-modulated drift condition is introduced: $PV_0 \le V_0 - V_1 + b\psi$ with some measurable function $V_0 : \mathbb{X} \to [1,+\infty)$ and the so-called modulated measurable function $V_1 : \mathbb{X} \to [1,+\infty)$. The minorization condition is the first pillar in this work, this modulated drift condition is the second one. Note that the modulated drift condition is a re-enforcement of the drift inequality (44) of Proposition 4.17.

Under the minorization Condition ($\boldsymbol{M}_{\nu,\psi}$) and the V_1-modulated drift condition, the convergence of the series $\sum_{k=0}^{+\infty} R^k V_1$ is proved in Theorem 5.4 thanks to an auxiliary V_1-modulated residual drift inequality following the same lines as for (47). Then the series $\sum_{k=0}^{+\infty} R^k 1_{\mathbb{X}}$ converges point-wise since $1_{\mathbb{X}} \le V_1$, so that the function $h_R^\infty := \lim_n R^n 1_{\mathbb{X}}$ (see (19)) is zero on $\mathbb{X}$. Under the same assumptions it is also shown in Theorem 5.4 that the positive measure μ_R given in (20) is finite (i.e. $\mu_R(1_{\mathbb{X}}) < \infty$). Accordingly, when Condition ($\boldsymbol{M}_{\nu,\psi}$) and the V_1-modulated drift condition are assumed to hold, all the conclusions of Theorems 4.1, 4.3, and Theorem 4.7 or 4.12 hold true, that is:

(i) The $P-$harmonic functions are constant on $\mathbb{X}$.

(ii) P is irreducible (see (28)) and recurrent (see (26)).

(iii) The positive measure μ_R (see (20)) satisfies $\mu_R(\psi) = 1$, and is the unique $P-$invariant positive measure η such that $\eta(\psi) < \infty$.

(iv) $\pi_R := \mu_R(1_{\mathbb{X}})^{-1}\mu_R$ (see (25)) is the unique $P-$invariant probability measure on $(\mathbb{X}, \mathcal{X})$, we have $\pi_R(\psi) > 0$, and P is Harris-recurrent (see (35)).

(v) The convergence in total variation of Theorem 4.7 or Theorem 4.12, depending on whether P is aperiodic or periodic, holds.

However the convergence of the series $\sum_{k=1}^{+\infty} R^k V_1$ actually gives more, in particular it naturally provides solutions to the so-called Poisson's equation (Theorem 5.6). This is the main motivation of this section.

5.1 Modulated drift condition $\boldsymbol{D}_\psi(V_0, V_1)$

Let us introduce the following condition for any couple (V_0, V_1) of measurable functions from $\mathbb{X}$ to $[1, +\infty)$:

$$\exists \psi \in \mathcal{B}_+^*,\ \exists b_0 \equiv b_0(V_0, V_1, \psi) > 0 : \quad PV_0 \le V_0 - V_1 + b_0\psi. \qquad (\boldsymbol{D}_\psi(V_0, V_1))$$

This condition is said to be a V_1-*modulated drift condition* for P, and V_0 and V_1 in $\boldsymbol{D}_\psi(V_0, V_1)$ are called *Lyapunov functions* for P. The functions V_0, V_1, ψ are assumed to be everywhere finite, so the function PV_0 is too. It is worth noticing that the modulated function V_1 must be larger than one for the results of this section to hold. In fact, it is only required that V_0 is non-negative and V_1 is uniformly bounded from below by a positive constant. Indeed, if $PV_0' \le V_0' - V_1' + b'\psi$ for some positive constant b' and some measurable functions $V_0' \ge 0$ and $V_1' \ge c1_{\mathbb{X}}$ with $c > 0$, then Condition $\boldsymbol{D}_\psi(V_0, V_1)$ holds with $V_1 := V_1'/c \ge 1_{\mathbb{X}}$, $V_0 := 1_{\mathbb{X}} + V_0'/c \ge 1_{\mathbb{X}}$ and $b_0 := b'/c > 0$. Moreover observe that if Conditions $\boldsymbol{D}_\phi(V_0, V_1)$ for some $\phi \in \mathcal{B}_+^*$ is satisfied then $\boldsymbol{D}_\psi(V_0, V_1)$ holds for any $\psi \in \mathcal{B}_+^*$ such that $\psi \ge \phi$ (using any constant $b_0(V_0, V_1, \psi)$ larger than $b_0(V_0, V_1, \phi)$).

In the special case $\psi := 1_S$ for some $S \in \mathcal{X}^*$, the above condition writes as

$$\exists S \in \mathcal{X}^*,\ \exists b_0 \equiv b_0(V_0, V_1, 1_S) > 0 : \quad PV_0 \le V_0 - V_1 + b_0 1_S. \qquad (\boldsymbol{D}_{1_S}(V_0, V_1))$$

Note that Condition $\boldsymbol{D}_{1_S}(V_0, V_1)$ implies that $V_0 \geq V_1$ on S^c. In fact Condition $\boldsymbol{D}_{1_S}(V_0, V_1)$ is equivalent to : There exists $S \in \mathcal{X}^*$ such that $\sup_{x \in S^c} \Gamma(x) \leq 0$ and $\sup_{x \in S} \Gamma(x) < \infty$ with the measurable finite function $\Gamma(x) := (PV_0)(x) - V_0(x) + V_1(x)$. Thus if Condition $\boldsymbol{D}_{1_S}(V_0, V_1)$ holds, then any constant $b_0(V_0, V_1, 1_S) \geq \sup_{x \in S} \Gamma(x)$ may be chosen. Finally recall that Conditions $(\boldsymbol{M}_{\nu,1_S})$ and $\boldsymbol{D}_{1_S}(V_0, V_1)$ are the most classical minorization/drift assumptions in the literature.

Let us return to Markov kernel P satisfying the assumptions of Proposition 3.1. Then both Conditions $(\boldsymbol{M}_{\nu,1_S})$ and $(\boldsymbol{M}_{\nu,\psi_S})$ hold with $\nu \in \mathcal{M}_{+,b}^*$ and $\psi_S \geq 1_S$ given in (14). Moreover, if P satisfies $\boldsymbol{D}_{1_S}(V_0, V_1)$, then Condition $\boldsymbol{D}_{\psi_S}(V_0, V_1)$ holds since $\psi_S \geq 1_S$. The next statement ensures that the constant $b_0(V_0, V_1, \psi_S)$ may be chosen smaller than $b_0(V_0, V_1, 1_S)$.

Proposition 5.1 *Let P satisfy the assumptions of Proposition 3.1 and Condition $\boldsymbol{D}_{1_S}(V_0, V_1)$ for some couple (V_0, V_1) of Lyapunov functions on $\mathbb{X}$. Then P satisfies Condition $\boldsymbol{D}_{\psi_S}(V_0, V_1)$ with $\psi_S \geq 1_S$ given in (14), and we can choose*

$$b_0(V_0, V_1, \psi_S) \leq b_0(V_0, V_1, 1_S). \tag{51}$$

Proof. Since ψ_S defined in (14) is such that $\psi_S \geq 1_S$ we already quoted that P also satisfies Condition $\boldsymbol{D}_{\psi_S}(V_0, V_1)$. Next, set

$$b_0(V_0, V_1, \psi_S) := \sup_{x \in S} \frac{\Gamma(x)}{\psi_S(x)} \quad \text{with } \Gamma(x) := (PV_0)(x) - V_0(x) + V_1(x).$$

Since $\psi_S \geq 1_S$, we have $b_0(V_0, V_1, \psi_S) \leq \sup_{x \in S} \Gamma(x) \leq b_0(V_0, V_1, 1_S)$. □

Example 5.2 (Geometric drift condition) *Let us introduce the following so-called $V-$geometric drift condition (to be discussed in Section 6)*

$$\exists \psi \in \mathcal{B}_+^*,\ \exists \delta \in (0,1),\ \exists b \in (0,+\infty):\quad PV \leq \delta V + b\,\psi \tag{$\boldsymbol{G}_\psi(\delta, V)$}$$

where $V : \mathbb{X} \to [1,+\infty)$ is a measurable function. Again recall that the most classical case is when $\psi := 1_S$ for some $S \in \mathcal{X}^*$, that is

$$\exists S \in \mathcal{X}^*,\ \exists \delta \in (0,1),\ \exists b \in (0,+\infty):\quad PV \leq \delta V + b\,1_S. \tag{$\boldsymbol{G}_{1_S}(\delta, V)$}$$

Observe that $\boldsymbol{G}_\psi(\delta, V)$ implies that $PV \leq V - (1-\delta)V + b\,\psi$, so that P satisfies the V_1-modulated drift Condition $\boldsymbol{D}_\psi(V_0, V_1)$ with $V_0 := V/(1-\delta)$, $V_1 := V$ and $b_0 := b/(1-\delta)$.

5.2 Residual-type modulated drift condition

Under Conditions $(\boldsymbol{M}_{\nu,\psi})$ and for any couple (V, W) of measurable functions from $\mathbb{X}$ to $[1, +\infty)$ such that $\nu(V) < \infty$, let us introduce the following residual-type modulated drift condition involving the residual kernel $R \equiv R_{\nu,\psi}$ given in (12):

$$RV \leq V - W. \tag{$\boldsymbol{R}_{\nu,\psi}(V, W)$}$$

Note that Condition $\boldsymbol{R}_{\nu,\psi}(V, W)$ rewrites as $PV \leq V - W + \nu(V)\psi$, which is a specific instance of Condition $\boldsymbol{D}_\psi(V, W)$ with $b_0 = \nu(V)$. The next simple lemma shows that $\boldsymbol{D}_\psi(V_0, V_1)$ generates a residual-type modulated drift condition up to slightly modify V_0. Recall that the kernel identity (15) used throughout Sections 3-4 and only based on the minorization condition $(\boldsymbol{M}_{\nu,\psi})$ is the first key point of this work. Lemma 5.3 based on the modulated drift condition $\boldsymbol{D}_\psi(V_0, V_1)$ is the second key point (already used in the proof of Proposition 4.17 under the weaker drift condition (44)).

Lemma 5.3 *If P satisfies Conditions ($\boldsymbol{M}_{\nu,\psi}$) and $\boldsymbol{D}_\psi(V_0, V_1)$, then $\nu(V_0) < \infty$ and for any constant c satisfying $c \geq (b_0 - \nu(V_0))/\nu(1_{\mathbb{X}})$ the residual kernel $R \equiv R_{\nu,\psi}$ given in (12) satisfies Condition $\boldsymbol{R}_{\nu,\psi}(V_{0,d}, V_1)$ with $V_{0,d} := V_0 + d1_{\mathbb{X}} \geq V_0$ where $d = \max(0, c)$.*

Proof. We already quoted that PV_0 is everywhere finite under Condition $\boldsymbol{D}_\psi(V_0, V_1)$, so that $0 \leq \nu(V_0)\psi(x) \leq (PV_0)(x)$ for every $x \in \mathbb{X}$ from ($\boldsymbol{M}_{\nu,\psi}$). Then it follows that the function RV_0 is well-defined and is everywhere finite. Note that $\nu(V_{0,d}) = \nu(V_0) + d\nu(1_{\mathbb{X}}) < \infty$ and that $PV_{0,d} = PV_0 + d1_{\mathbb{X}}$. We get from the definitions of R and $V_{0,d}$

$$\begin{aligned} RV_{0,d} &= PV_{0,d} - \nu(V_{0,d})\psi = PV_0 + d1_{\mathbb{X}} - \big(\nu(V_0) + d\,\nu(1_{\mathbb{X}})\big)\psi \\ &\leq V_0 - V_1 + b_0\psi + d1_{\mathbb{X}} - \big(\nu(V_0) + d\,\nu(1_{\mathbb{X}})\big)\psi \quad \text{(from Assumption } \boldsymbol{D}_\psi(V_0, V_1)) \\ &= V_{0,d} - V_1 + \big(b_0 - \nu(V_0) - d\,\nu(1_{\mathbb{X}})\big)\psi \\ &\leq V_{0,d} - V_1 \quad \text{(from the definitions of } c \text{ and } d). \end{aligned}$$

Hence the proof is complete. □

Under Conditions ($\boldsymbol{M}_{\nu,\psi}$)–$\boldsymbol{D}_\psi(V_0, V_1)$ the following theorem provides relevant properties on the non-negative kernel $\sum_{k=0}^{+\infty} R^k$ involving the residual kernel R, from which further statements on P and π_R are obtained. Moreover the bounds (52a)-(52b) below are crucial for the study of Poisson's equation in the next subsection.

Theorem 5.4 *Assume that P satisfies Conditions ($\boldsymbol{M}_{\nu,\psi}$)–$\boldsymbol{D}_\psi(V_0, V_1)$. Then*

$$0 \;\leq\; \sum_{k=0}^{+\infty} R^k 1_{\mathbb{X}} \;\leq\; \sum_{k=0}^{+\infty} R^k V_1 \leq (1 + d_0)V_0 \quad \textit{with} \quad d_0 := \max\left(0, \frac{b_0 - \nu(V_0))}{\nu(1_{\mathbb{X}})}\right) \tag{52a}$$

$$0 \;\leq\; \sum_{k=0}^{+\infty} \nu\big(R^k 1_{\mathbb{X}}\big) \;\leq\; \sum_{k=0}^{+\infty} \nu\big(R^k V_1\big) \leq (1 + d_0)\,\nu(V_0) < \infty. \tag{52b}$$

Moreover the conclusions (i)-(v) *provided at the beginning of this section hold true, as well as the following additional assertions:*

(vi) *The unique $P-$invariant probability measure π_R is such that $\pi_R(V_1) < \infty$.*

(vii) *If $\pi_R(V_0) < \infty$, then $\pi_R(V_1) \leq b_0\,\pi_R(\psi) \leq b_0$ where b_0 is the constant in $\boldsymbol{D}_\psi(V_0, V_1)$.*

(viii) *if PV_1/V_1 is bounded on $\mathbb{X}$, i.e. $P\mathcal{B}_{V_1} \subset \mathcal{B}_{V_1}$, then the $P-$harmonic functions in $\mathcal{B}_{V_1}$ (i.e. $g \in \mathcal{B}_{V_1}$ such that $Pg = g$) are constant on $\mathbb{X}$.*

Inequalities (52a)-(52b), thus the constant d_0, will play a crucial role for the bounds of solutions to Poisson equation in Subsection 5.3 and for the rates of convergence in Section 6. Recall that the constant d_0 depends on the minorizing measure ν in ($\boldsymbol{M}_{\nu,\psi}$) and on the constant $b_0(V_0, V_1, \psi)$ in $\boldsymbol{D}_\psi(V_0, V_1)$. First prove the following.

Lemma 5.5 *Assume that P satisfies Condition ($\boldsymbol{M}_{\nu,\psi}$) and that the associated residual kernel $R \equiv R_{\nu,\psi}$ given in (12) satisfies Condition $\boldsymbol{R}_{\nu,\psi}(V, W)$ for some couple of Lyapunov functions (V, W) such that $\nu(V) < \infty$. Then we have*

$$0 \leq \sum_{k=0}^{+\infty} R^k 1_{\mathbb{X}} \;\leq\; \sum_{k=0}^{+\infty} R^k W \leq V \tag{53a}$$

$$0 \leq \sum_{k=0}^{+\infty} \nu\big(R^k 1_{\mathbb{X}}\big) \;\leq\; \sum_{k=0}^{+\infty} \nu\big(R^k W\big) \leq \nu(V) < \infty. \tag{53b}$$

Proof. From $\boldsymbol{R}_{\nu,\psi}(V,W)$, we derive that $0 \le W \le V - RV$, so that

$$\forall n \geq 1, \quad 0 \leq \sum_{k=0}^{n} R^k W \leq \sum_{k=0}^{n} R^k V - \sum_{k=1}^{n+1} R^k V \leq V \tag{54}$$

since $R^{n+1}V \geq 0$. This proves (53a). Next (53b) is obtained using Lebesgue's theorem. □

Proof of Theorem 5.4. Inequalities (52a)-(52b) directly follow from Lemma 5.3 and from Lemma 5.5 applied to $W = V_1$ and $V := V_0 + d_0 1_{\mathbb{X}}$ with $d_0 = \max(0, (b_0 - \nu(V_0))/\nu(1_{\mathbb{X}}))$ observing that $V \leq (1+d_0)V_0$. Next, the point-wise convergence of the first series in (52a) proves that $h_R^\infty := \lim_n R^n 1_{\mathbb{X}} = 0$ (see (19)), while the convergence of the first series in (52b) reads as $\mu_R(1_{\mathbb{X}}) = \sum_{k=0}^{+\infty} \nu(R^k 1_{\mathbb{X}}) < \infty$ (see (20)). Recall that the conclusions (i)-(v) provided at the beginning of this section then follows from Theorems 4.1, 4.3 and 4.7. Now prove the additional assertions (vi)-(viii). That $\pi_R(V_1) < \infty$ follows from the definition of π_R and from the second inequality in (52b) which provides $\mu_R(V_1) < \infty$. To prove (vii), note that

$$\pi_R(PV_0) = \pi_R(V_0) \leq \pi_R(V_0) - \pi_R(V_1) + b_0 \pi_R(\psi)$$

from the $P-$invariance of π_R and $\boldsymbol{D}_\psi(V_0, V_1)$. Finally the proof of (viii) follows the same lines as for Assertion 1. of Theorem 4.1, replacing the function 1_X with V_1 and observing that $P(\mathcal{B}_{V_1}) \subset \mathcal{B}_{V_1}$, thus $R(\mathcal{B}_{V_1}) \subset \mathcal{B}_{V_1}$, when PV_1/V_1 is bounded on $\mathbb{X}$. Indeed, first recall that $\widetilde{\psi} := \sum_{k=0}^{+\infty} R^k \psi = \nu(1_{\mathbb{X}})^{-1} 1_{\mathbb{X}}$ from (34) since $h_R^\infty = 0$. Now let $g \in \mathcal{B}_{V_1}$ be such that $Pg = g$. Using $R(\mathcal{B}_{V_1}) \subset \mathcal{B}_{V_1}$ and proceeding as in Lemma 3.3, we obtained that $\nu(g) \sum_{k=0}^{n} R^k \psi = g - R^{n+1} g$ for every $n \geq 1$. Moreover we have $\lim_n R^n g = 0$ since $|R^n g| \leq R^n |g| \leq \|g\|_{V_1} R^n V_1$ and $\lim_n R^n V_1 = 0$ from (52a). Thus $g = \nu(g)\widetilde{\psi}$, from which it follows that g is constant. □

5.3 Poisson's equation

When P satisfies Conditions $(\boldsymbol{M}_{\nu,\psi})$ and $\boldsymbol{D}_\psi(V_0, V_1)$, recall that π_R given in (25) is the unique $P-$invariant probability measure on $(\mathbb{X}, \mathcal{X})$.

Theorem 5.6 *Assume that P satisfies Conditions $(\boldsymbol{M}_{\nu,\psi})$–$\boldsymbol{D}_\psi(V_0, V_1)$. Let $R \equiv R_{\nu,\psi}$ be the associated residual kernel given in (12). Then the following assertions hold.*

1. *For any $g \in \mathcal{B}_{V_1}$, the function series $\widetilde{g} := \sum_{k=0}^{+\infty} R^k g$ absolutely converges on $\mathbb{X}$ (point-wise convergence). Moreover we have $\widetilde{g} \in \mathcal{B}_{V_0}$ and*

$$\forall g \in \mathcal{B}_{V_1}, \quad \|\widetilde{g}\|_{V_0} \leq (1+d_0)\|g\|_{V_1} \quad \text{with} \quad d_0 := \max\left(0\,, \frac{b_0 - \nu(V_0)}{\nu(1_{\mathbb{X}})}\right) \tag{55}$$

where b_0 is the positive constant given in $\boldsymbol{D}_\psi(V_0, V_1)$.

2. *For any $g \in \mathcal{B}_{V_1}$ such that $\pi_R(g) = 0$, the function $\widetilde{g}$ satisfies Poisson's equation*

$$(I - P)\widetilde{g} = g. \tag{56}$$

Proof. Let $g \in \mathcal{B}_{V_1}$. Using $|g| \leq \|g\|_{V_1} V_1$ and $|R^k g| \leq R^k|g| \leq \|g\|_{V_1} R^k V_1$, Assertion 1. follows from (52a). Next, note that $\pi_R(|g|) < \infty$ since $\pi_R(V_1) < \infty$ from Assertion (vi) of Theorem 5.4. Now define

$$\forall n \geq 1, \quad \widetilde{g}_n := \sum_{k=0}^{n} R^k g.$$

Then, using $P = R + \psi \otimes \nu$ we have

$$\widetilde{g}_n - P\widetilde{g}_n = \widetilde{g}_n - R\widetilde{g}_n - \nu(\widetilde{g}_n)\psi = g - R^{n+1}g - \nu(\widetilde{g}_n)\psi. \tag{57}$$

We know that $\lim_n R^{n+1}g = 0$ (pointwise convergence) from the convergence of the series $\sum_{k=0}^{+\infty} R^k g$. Moreover, using $\nu(\widetilde{g}_n) = \sum_{k=0}^{n} \nu(R^k g)$ and $\mu_R(V_1) < \infty$, we obtain that $\lim_{n\to+\infty} \nu(\widetilde{g}_n) = \mu_R(g)$ from Lebesgue's theorem w.r.t. the measure ν. Finally, for every $x \in \mathbb{X}$, we have $\lim_n (P\widetilde{g}_n)(x) = (P\widetilde{g})(x)$ from Lebesgue's theorem applied to the sequence $(\widetilde{g}_n)_n$ w.r.t. the probability measure $P(x, dy)$ since $\lim_n \widetilde{g}_n = \widetilde{g}$, $|\widetilde{g}_n| \le \|g\|_{V_1} V_0$ (from Assertion 1.) and $(PV_0)(x) < \infty$. Taking the limit when n goes to infinity in (57), we obtain that

$$(I - P)\widetilde{g} = g - \mu_R(g)\psi. \tag{58}$$

Next, if we assume that $\pi_R(g) = 0$, then Equality (58) rewrites as $(I - P)\widetilde{g} = g$ since $\mu_R(g) = \pi_R(g)/\pi_R(\psi) = 0$ from (25). Theorem 5.6 is proved. □

For $g \in \mathcal{B}_{V_1}$ such that $\pi_R(g) = 0$, the solution $\widetilde{g} := \sum_{k=0}^{+\infty} R^k g$ in $\mathcal{B}_{V_0}$ to Poisson's equation $(I - P)\widetilde{g} = g$ in Theorem 5.6 is not π_R-centred a priori, i.e. $\pi_R(\widetilde{g}) \neq 0$. The natural way to get a π_R-centred solution is to define $\widehat{g} = \widetilde{g} - \pi_R(\widetilde{g})1_{\mathbb{X}}$, but we then need to assume that $\widetilde{g}$ is π_R-integrable. Accordingly, to obtain such a π_R-centred solution to Poisson's equation in general terms, the assumption $\pi_R(V_0) < \infty$ must be made.

Corollary 5.7 *Let P satisfy Conditions $(\boldsymbol{M}_{\nu,\psi})$–$\boldsymbol{D}_\psi(V_0, V_1)$ with $\pi_R(V_0) < \infty$. For any $g \in \mathcal{B}_{V_1}$ such that $\pi_R(g) = 0$, let $\widetilde{g} := \sum_{k=0}^{+\infty} R^k g$. Then the function $\widehat{g} = \widetilde{g} - \pi_R(\widetilde{g})1_{\mathbb{X}}$ is a π_R-centered solution on $\mathcal{B}_{V_0}$ to Poisson's equation $(I - P)\widehat{g} = g$. Moreover we have*

$$\|\widehat{g}\|_{V_0} \le (1 + d_0)\big(1 + \pi_R(V_0)\big)\|g\|_{V_1} \tag{59}$$

where the positive constant d_0 is given in (55).

Proof. Let $g \in \mathcal{B}_{V_1}$ be such that $\pi_R(g) = 0$. Obviously we have $\widehat{g} \in \mathcal{B}_{V_0}$ and $\pi_R(\widehat{g}) = 0$. Moreover we obtain that $(I - P)\widehat{g} = (I - P)\widetilde{g} = g$ from Theorem 5.6 and $(I - P)1_{\mathbb{X}} = 0$. Finally we have

$$\|\widehat{g}\|_{V_0} \le \big(1 + \pi_R(V_0)\,\|1_{\mathbb{X}}\|_{V_0}\big)\|\widetilde{g}\|_{V_0} \le (1 + d_0)\big(1 + \pi_R(V_0)\big)\|g\|_{V_1} \tag{60}$$

using the definition of $\widehat{g}$, the triangular inequality and $|\widetilde{g}| \le \|\widetilde{g}\|_{V_0} V_0$ for the first inequality, and the bound (55) applied to $\widetilde{g}$ for the second one. □

Let $g \in \mathcal{B}_{V_1}$ be such that $\pi_R(g) = 0$. Under the assumptions of Corollary 5.7, when a π_R-centred solution $\mathfrak{g} \in \mathcal{B}_{V_0}$ to Poisson's equation $(I - P)\mathfrak{g} = g$ is known, and when two solutions to Poisson's equation in $\mathcal{B}_{V_0}$ differ from an additive constant, then we have $\mathfrak{g} = \widehat{g}$, so that the bound (59) applies to $\mathfrak{g}$. Of course such a solution $\mathfrak{g}$ may be obtained independently of the function $\widetilde{g}$. For instance it can be given by $\mathfrak{g} = \sum_{k=0}^{+\infty} P^k g$ provided that this series point-wise converges and defines a function of $\mathcal{B}_{V_0}$. Note that the choice of the minorizing measure ν and of the function ψ used in Conditions $(\boldsymbol{M}_{\nu,\psi})$ and $\boldsymbol{D}_\psi(V_0, V_1)$ of Corollary 5.7 naturally has an impact on the constant d_0 in (59).

Remark 5.8 *Recall that, under Conditions $(\boldsymbol{M}_{\nu,\psi})$–$\boldsymbol{D}_\psi(V_0, V_1)$, the function $h_R^\infty := \lim_n R^n 1_{\mathbb{X}}$ (see (19)) is zero from the convergence of the first series in (52a), so that $\widetilde{\psi} := \sum_{k=0}^{+\infty} R^k\psi = \nu(1_{\mathbb{X}})^{-1}1_{\mathbb{X}}$ from* (34). *So the presence of the term $\nu(1_{\mathbb{X}})^{-1}$ in the general bound* (55) *is quite* *natural (it is not due to the proof of Theorem 5.6). This does not mean that the bound of the V_0- norm of solutions to Poisson's equation could not be improved. But in fact this last question is not well formulated since solutions to Poisson's equation are not unique, and the solutions given in Theorem 5.6 are very specific: they are defined from the residual kernel R, in particular they are not π_R-centred (see Corollary 5.7).*

Remark 5.9 *Assume that P satisfies Conditions $(\boldsymbol{M}_{\nu,1_S})$–$\boldsymbol{D}_{1_S}(V_0,V_1)$ with $V_0 \geq V_1$ and $\inf V_0 = 1$. Then we have $d_0 = 0$ in the bound (55) of Theorem 5.6 if, and only if, S is an atom, i.e. $\forall a \in S,\ \nu(dy) = P(a,dy)$. Indeed, if S is an atom, then P satisfies $\boldsymbol{D}_{1_S}(V_0,V_1)$ with $b_0 = \nu(V_0)$ since $V_0 \geq V_1$. Thus $d_0 = 0$. To prove the converse implication, note that*

$$\nu(1_{\mathbb{X}})^{-1} = \nu(1_{\mathbb{X}})^{-1}\|1_{\mathbb{X}}\|_{V_0} \leq (1+d_0)\|1_S\|_{V_1} \leq (1+d_0)$$

from (55) applied to $g := 1_S$ and (34) with here $\psi = 1_S$. Hence, if $d_0 = 0$, then $\nu(1_{\mathbb{X}}) \geq 1$. Thus S is an atom since, for every $a \in S$, the non-negative measure $\eta_a(dy) = P(a,dy) - \nu(dy)$ satisfies $\eta_a(1_{\mathbb{X}}) \leq 0$, so that $\eta_a = 0$.

5.4 Further statements

Under Conditions $(\boldsymbol{M}_{\nu,\psi})$-$\boldsymbol{D}_\psi(V_0,V_1)$ and the additional condition $\pi_R(V_0) < \infty$, the sequence $(P^nV_0)_n$ is shown to be bounded in $(\mathcal{B}_{V_0}, \|\cdot\|_{V_0})$ in the following lemma.

Lemma 5.10 *Let P satisfy Conditions $(\boldsymbol{M}_{\nu,\psi})$–$\boldsymbol{D}_\psi(V_0,V_1)$ with $\pi_R(V_0) < \infty$. Then we have for every $n \geq 1$:*

$$P^nV_0 \leq V_0 + \frac{\|\psi\|_{1_{\mathbb{X}}}\big(\pi_R(V_0)+d_0\big)}{\pi_R(\psi)}1_{\mathbb{X}} \quad \textit{with} \quad \|\psi\|_{1_{\mathbb{X}}} := \sup_{x\in\mathbb{X}} \psi(x),\ d_0 := \max\left(0, \frac{b_0 - \nu(V_0))}{\nu(1_{\mathbb{X}})}\right).$$

Proof. It follows from $(\boldsymbol{M}_{\nu,\psi})$ and Lemma 5.3 that $RV_{0,d_0} \leq V_{0,d_0}$ with $V_{0,d_0} := V_0 + d_0 1_{\mathbb{X}}$ and $R \equiv R_{\nu,\psi}$ in (12). Using the non-negativity of R and iterating this inequality gives: $\forall n \geq 1,\ R^nV_{0,d} \leq V_{0,d}$. From Formula (16) and $0 \leq P^k\psi \leq \|\psi\|_{1_{\mathbb{X}}} 1_{\mathbb{X}}$, we obtain that

$$\forall n \geq 1, \quad P^nV_{0,d} = R^nV_{0,d} + \sum_{k=1}^{n} \nu(R^{k-1}V_{0,d})\, P^{n-k}\psi \leq V_{0,d} + \|\psi\|_{1_{\mathbb{X}}}\, \mu_R(V_{0,d})1_{\mathbb{X}}.$$

with $\mu_R = \pi_R/\pi_R(\psi)$ given in (25). This provides the desired inequality using the definition of $V_{0,d}, P1_{\mathbb{X}} = 1_{\mathbb{X}}$ and $\pi_R(V_0) < \infty$. □

Now, given any measurable function $V_1 : \mathbb{X} \to [1,+\infty)$, we present a necessary and sufficient condition for P to satisfy a V_1-modulated drift condition.

Proposition 5.11 *Assume that P satisfies Condition $(\boldsymbol{M}_{\nu,\psi})$. Let $V_1 : \mathbb{X} \to [1,+\infty)$ be any measurable function. Then there exists a measurable function $V_0 : \mathbb{X} \to [1,+\infty)$ such that P sastifies $\boldsymbol{D}_\psi(V_0,V_1)$ if and only if*

$$\forall x \in \mathbb{X}, \quad \widetilde{V_1}(x) := \sum_{k=0}^{+\infty}(R^kV_1)(x) < \infty \quad \textit{and} \quad \nu(\widetilde{V_1}) < \infty \tag{61}$$

where $R \equiv R_{\nu,\psi}$ is the residual kernel in (12).

Proof. If P satisfies Condition $\boldsymbol{D}_\psi(V_0, V_1)$ for some Lyapunov function V_0, then (61) holds true from Theorem 5.4 (in fact we know that $\widetilde{V_1} \le c\,V_0$ for some positive constant c). Conversely, if V_1 satisfies (61) with $R \equiv R_{\nu,\psi}$ in (12), then we have $(R\widetilde{V_1})(x) = \widetilde{V_1}(x) - V_1(x)$ for every $x \in \mathbb{X}$ from the monotone convergence theorem w.r.t. the measure $R(x, dy)$. Hence Condition $\boldsymbol{R}_{\nu,\psi}(\widetilde{V_1}, V_1)$ holds. Then Condition $\boldsymbol{D}_\psi(\widetilde{V_1}, V_1)$ holds with $b_0 = \nu(\widetilde{V_1})$.

□

The next statement completes Theorem 3.6.

Proposition 5.12 *Assume that P satisfies Condition $(\boldsymbol{M}_{\nu,\psi})$ and is irreducible. Then the two equivalent conditions 1. and 2. of Theorem 3.6 are also equivalent to the following one: There exists a $P-$absorbing and μ_R-full set $A \in \mathcal{X}$ such that the restriction of P to A satisfies the modulated drift condition $\boldsymbol{D}_{\psi_A}(V_A, 1_A)$ for some measurable function $V_A : A \to [1, +\infty)$, where ψ_A is the restriction of ψ to A.*

Using Conditions 1. of Theorem 3.6 it follows from Proposition 5.12 that a Markov kernel P satisfying the minorization condition $(\boldsymbol{M}_{\nu,\psi})$, irreducible and admitting an invariant probability measure π such that $\pi(\psi) > 0$ actually satisfies all the conclusions of Theorem 5.4 on some $P-$absorbing and $\pi-$full set. Note that the irreducibility assumption on P is only used to obtain that the $P-$absorbing set A of Proposition 5.12 is $\pi-$full.

Proof. Under Condition $\boldsymbol{M}_{\nu,\psi}$, let $R \equiv R_{\nu,R}$ be the residual kernel defined in (12). Assume that Condition 2. of Theorem 3.6 holds, i.e. $\mu_R(1_\mathbb{X}) < \infty$. Define on $\mathbb{X}$ the function $V := \sum_{k=0}^{+\infty} R^k 1_\mathbb{X}$ taking its value in $[0, +\infty]$ a priori. Since $\nu(V) = \mu_R(1_\mathbb{X}) < \infty$, the set

$$A := \big\{x \in \mathbb{X} : V(x) < \infty\big\}$$

is non-empty. Moreover, if $x \in A$, then we have $(RV)(x) < \infty$ since $(RV)(x) = V(x) - 1$ from the monotone convergence theorem w.r.t. the measure $R(x, dy)$. We then obtain that $(PV)(x) = (RV)(x) + \nu(V)\psi(x) = V(x) - 1 + \nu(V)\psi(x) < \infty$. This proves that A is $P-$absorbing. Since P is irreducible, A is μ_R-full from Proposition 3.16. Furthermore, the previous equality proves that the restriction of P to A satisfies the modulated drift condition $\boldsymbol{D}_{\psi_A}(V_A, 1_A)$ where V_A is the restriction of V to the set A.

Conversely assume that the condition provided in Proposition 5.12 holds. Using the fact that A is $P-$absorbing and proceeding as in the proof of Corollary 4.5, it can be proved that the restriction P_A of P to A satisfies on A the minorization condition $(\boldsymbol{M}_{\nu_A,\psi_A})$ with small-function ψ_A and minorizing measure ν_A defined as the restriction of ν to A. Then it follows from Theorem 5.4 applied to the Markov kernel P_A that there exists a unique P_A-invariant probability measure η_A on A and that $\eta_A(\psi_A) > 0$ (apply Assertion (iv) to P_A). Next let us define the following positive measure on $(\mathbb{X}, \mathcal{X})$: $\forall B \in \mathcal{X},\ \eta(1_B) := \eta_A(1_{A\cap B})$. Since A is $P-$absorbing, η is a P-invariant probability measure, and we have $\eta(\psi) = \eta_A(\psi_A) > 0$. Consequently Condition 1. of Theorem 3.6 holds for P and Proposition 5.12 is proved.

□

Finally, under Conditions $(\boldsymbol{M}_{\nu,\psi})$–$\boldsymbol{D}_\psi(V_0, V_1)$, the next statement provides a necessary and sufficient condition for the (unique) $P-$invariant probability measure π_R given in (25) to satisfy $\pi_R(V_0) < \infty$.

Proposition 5.13 *Let P satisfy Conditions $(\boldsymbol{M}_{\nu,\psi})$–$\boldsymbol{D}_\psi(V_0, V_1)$. Then the two following conditions are equivalent:*

1. $\pi_R(V_0) < \infty$.

2. There exists a $P-$*absorbing and* π_R-*full set* $A \in \mathcal{X}$ *and a measurable function* $L \geq V_0$ *on* A *such that the restriction* P_A *of* P *to* A *satisfies the modulated drift condition* $\boldsymbol{D}_{\psi_A}(L, V_{0_{|A}})$*, where* $V_{0_{|A}}$ *(resp.* ψ_A*) is the restriction of* V_0 *(resp. of* ψ*) to* A*.*

Proof. The proof follows the same limes as for Proposition 5.12. Let $R \equiv R_{\nu,R}$ be the residual kernel given in (12). Assume that $\pi_R(V_0) < \infty$ and define on $\mathbb{X}$ the function $\widetilde{V_0} = \sum_{k=0}^{+\infty} R^k V_0$ taking its value in $[0, +\infty]$ a priori. Then $\widetilde{V_0} \geq V_0$, and the following equality holds in $[0, +\infty]$: $R\widetilde{V_0} = \widetilde{V_0} - V_0$. Note that there exists $x \in \mathbb{X}$ such that $\widetilde{V_0}(x) < \infty$ since $\nu(\widetilde{V_0}) = \mu_R(V_0) < \infty$ from $\pi_R(V_0) < \infty$, where $\mu_R := \sum_{k=0}^{+\infty} \nu R^k$ (see (25)). Now define the non-empty set $A := \{x \in \mathbb{X} : \widetilde{V_0}(x) < \infty\} \in \mathcal{X}$. Let $x \in A$. Then we have $(R\widetilde{V_0})(x) < \infty$ from $(R\widetilde{V_0})(x) = \widetilde{V_0}(x) - V_0(x)$, so that $(P\widetilde{V_0})(x) = (R\widetilde{V_0})(x) + \nu(\widetilde{V_0})\psi(x) < \infty$. Thus $P(x, A) = 1$. This proves that A is $P-$absorbing. Since P is irreducible from Theorem 5.4, A is π_R-full from Proposition 3.16. Moreover the restriction $L := \widetilde{V_0}_{|A}$ of $\widetilde{V_0}$ to A is a measurable function on A satisfying $RL = L - V_0$ on A, so that the restriction P_A of P to A satisfies the modulated drift condition $\boldsymbol{D}_{\psi_A}(L, V_{0_{|A}})$ as stated in Assertion 2 of Proposition 5.13.

Conversely assume that P satisfies Assertion 2. Then, proceeding as in the proof of Corollary 4.5, we know that P_A satisfies on A the minorization condition $(\boldsymbol{M}_{\nu_A,\psi_A})$ where ν_A is the restriction of the minorizing measure ν to A. Thus it follows from Assertion (vi) of Theorem 5.4 applied to P_A under Condition $(\boldsymbol{M}_{\nu_A,\psi_A})$ and $\boldsymbol{D}_{\psi_A}(L, V_{0_{|A}})$ that the unique P_A-invariant probability measure, say π_A, is such that $\pi_A(V_{0_{|A}}) < \infty$. Using the fact that π_R is the unique $P-$ invariant probability measure, we then obtained that π_A is the restriction of π_R to A and that $\pi_R(V_0) = \pi_A(V_{0_{|A}}) < \infty$ since A is $P-$absorbing and π_R-full. □

5.5 Bibliographic comments

Condition $\boldsymbol{D}_\psi(V_0, V_1)$ (or $\boldsymbol{D}_{1_S}(V_0, V_1)$) is the so-called V_1-modulated drift condition, e.g. see Condition (V3) in [MT09, p. 343]. Although the functions V_0, V_1 in $\boldsymbol{D}_\psi(V_0, V_1)$ satisfy $V_0 \geq V_1$ in general, this condition is not useful in this section. Such drift conditions was first introduced for infinite stochastic matrices in [Fos53] to study the return times to a set, see [MT09, p. 198] and [DMPS18, p. 96, 164, 337] for an historical background on this subject. Lemma 5.3 and its direct use to obtain Theorem 5.4 (via Lemme 5.5) were presented in [HL24a]. Again note that the non-negativity of the residual kernel R plays a crucial role in Theorem 5.4 since the point-wise convergence of the series in (52a) is simply obtained bounding the partial sums (see (54)).

Under the V_1-modulated drift condition $\boldsymbol{D}_{1_S}(V_0, V_1)$ w.r.t. some petite set $S \in \mathcal{X}$, the existence of a solution $\xi \in \mathcal{B}_{V_0}$ to Poisson's equation $(I - P)\xi = g$ was proved in [GM96, Th. 2.3] for every π_R-centred function $g \in \mathcal{B}_{V_1}$, together with the bound $\|\xi\|_{V_0} \leq c_0 \|g\|_{V_1}$ for some positive constant c_0 (independent of g). When S is an atom, the solution ξ in [GM96, Th. 2.3] can be expressed in terms of the first hitting time in S, and the non-atomic case is solved via the splitting method. Under the irreducibility and aperiodicity conditions, Glynn-Meyn's theorem is related to point-wise convergence of the series $\sum_{k=0}^{+\infty} P^k g$, see [MT09, Th. 14.0.1]. With regard to the above two representations of solutions to Poisson's equation, the reader may consult the recent article [GI24]. We point out that the constant c_0 in [GM96, Th. 2.3] is unknown in general, excepted in atomic case: see [LL18, Prop. 1] for a discrete state-space $\mathbb{X}$. Thus, the novelty of Theorem 5.6 and Corollary 5.7 already proved in [HL24a] is to provide a simple and explicit bound in Poisson's equation in the non-atomic case.

Let us briefly discuss the Central Limit Theorem (C.L.T.), which is a standard topic where Poisson's equation is useful. If $(X_n)_{n\in\mathbb{N}}$ is a Markov chain with state space $\mathbb{X}$ and invariant distribution π, then a measurable $\pi-$centred real-valued function g on $\mathbb{X}$ is said to satisfy the C.L.T. under $\mathbb{P}_\eta$ for some initial probability measure η (i.e. η is the probability distribution of X_0) when the asymptotic distribution of $n^{-1/2}S_n(g)$ with $S_n(g) = \sum_{k=0}^{n-1} g(X_k)$ is the Gaussian distribution $\mathcal{N}(0, \sigma_g^2)$ for some positive constant σ_g^2, called the asymptotic variance of g. We refer to [DMPS18, Chap. 21] for a nice and comprehensive account on the Markovian C.L.T. and the classical approach via Poisson's equation. Here, in link with Corollary 5.7, we just recall the following classical C.L.T. proved in [GM96] for Markov chains satisfying a modulated drift condition:

Glynn-Meyn's C.L.T. [GM96]: *If the transition kernel P of the Markov chain $(X_n)_{n\in\mathbb{N}}$ satisfies Conditions $(\boldsymbol{M}_{\nu,\psi})$–$\boldsymbol{D}_\psi(V_0, V_1)$ with $V_1 \le V_0$, $\pi_R(V_0^2) < \infty$, and if η is any initial probability measure, then every π_R-centred function $g \in \mathcal{B}_{V_1}$ satisfies the C.L.T. under $\mathbb{P}_\eta$ with asymptotic variance given by $\sigma_g^2 = 2\pi_R(g\widehat{g}) - \pi_R(g^2)$, where $\widehat{g} \in \mathcal{B}_{V_0}$ is the solution to Poisson's equation $(I-P)\widehat{g} = g$ provided by Corollary 5.7.*

The condition $\pi_R(V_0^2) < \infty$ is required for the function $\widehat{g}$ to be square π_R-integrable in order to apply the Markovian C.L.T. [DMPS18, Th. 21.2.5] under $\mathbb{P}_{\pi_R}$, where π_R is the unique $P-$invariant probability measure from Theorem 5.4. The extension to any initial probability measure follows from [DMPS18, Cor. 21.1.6] since P is Harris recurrent under the assumptions of Corollary 5.7 from Theorem 5.4. Note that the asymptotic variance σ_g^2 can be upper bounded using the bound (59) (see [HL24a]).

To conclude this section let us make a few additional comments on the modulated drift condition, which is the main assumption of this work together with the minorization condition. If $(X_n)_{n\ge 0}$ is a Markov chain with state space $\mathbb{X}$ and transition kernel P, then the modulated drift condition has the following form when the modulated function V_1 is constant and $\psi = 1_{\mathcal{V}_s}$ for some $s > 0$ where $\mathcal{V}_s = \{x \in \mathbb{X} : V_0(x) \le s\}$ is the level set of order s w.r.t. the function V_0:

$$\sup_{x\in\mathcal{V}_s} \mathbb{E}_x\big[V_0(X_1)\big] < \infty \quad \text{and} \quad \exists a > 0,\ \forall x \in \mathbb{X}\setminus\mathcal{V}_s, \quad \mathbb{E}_x\big[V_0(X_1)\big] \le V_0(x) - a. \tag{62}$$

The second condition in (62) means that, for any $r > s$, each point $x \in \mathbb{X}$ such that $V_0(x) = r$ transits in mean to a point of the level set $\mathcal{V}_{r-a}$. For a random walk on $\mathbb{N}$, it means that, for i large enough, the steps of the walker starting from i are in mean strictly more to the left than to the right, the gap being controlled by a fixed additive constant $a > 0$. Recall that the weaker drift condition (50) was introduced in Proposition 4.17 to obtain $\lim_k R^k 1_{\mathbb{X}} = 0$. The additive reduction by the positive constant a in (62) is the sole difference with (50), but it is crucial for obtaining the convergence of the series $\sum_{k=0}^{+\infty} R^k 1_{\mathbb{X}}$ in Theorem 5.4. The general modulated drift condition $\boldsymbol{D}_\psi(V_0, V_1)$ corresponds to (62) with a positive term $V_1(x)$ depending on x instead of the positive constant a.

Proposition 5.12 shows that, in the context of Theorem 3.6, an irreducible Markov kernel P always satisfies a modulated drift condition with $V_1(x) = 1$, up to restrict P to some absorbing and π_R-full set. Hence modulated drift condition is a perfectly natural assumption. This explains why the minorization and drift conditions are so popular for studying Markov models. In particular, it follows from Proposition 5.12 that an irreducible discrete Markov kernel P admitting an invariant probability measure π actually satisfies all the conclusions of Theorem 5.4 on a $P-$absorbing and $\pi-$full set: Indeed $S = \{x\}$ for some state x may be chosen such that $\pi(1_{\{x\}}) > 0$, and $S = \{x\}$ is obviously a first order small-set. In the same way, for the Markov chain Monte Carlo (MCMC) algorithms on any state space $\mathbb{X}$, the so-called target probability measure π is a data. Moreover, by construction, the associated MCMC kernel satisfies Assumptions (13), is irreducible and it admits π as invariant probability measure. Then, choosing the small-set S in Proposition 3.1 such that $\pi(1_S) > 0$, it follows from Proposition 5.12 that the MCMC kernel satisfies all the conclusions of Theorem 5.4 on some $P-$absorbing and $\pi-$full set. Note, however, that Proposition 5.12, as well as Proposition 5.11, are only of theoretical interest. In practice the form of the Markov kernel P is directly taken into account to find explicit functions V_0 and V_1 satisfying Condition $\boldsymbol{D}_\psi(V_0, V_1)$. Finally, as shown for instance for random walks on the half line in [JT03], recall that the condition $\pi_R(V_0) < \infty$ is not automatically fulfilled under Condition $\boldsymbol{D}_\psi(V_0, V_1)$. In fact, as proved in Proposition 5.13, this additional condition $\pi_R(V_0) < \infty$ is closely related to an extra V_0-modulated drift condition.

6 $V-$geometric ergodicity

Let $V : \mathbb{X} \to [1, +\infty)$ be measurable. Recall that the $V-$geometric drift condition is

$$\exists \psi \in \mathcal{B}^*_+,\ \exists \delta \in (0,1),\ \exists b \in (0,+\infty): \quad PV \le \delta V + b\,\psi \qquad (\boldsymbol{G}_\psi(\delta, V))$$

and that this condition provides the modulated drift Condition $\boldsymbol{D}_\psi(V_0, V_1)$ with

$$V_0 := V/(1-\delta), \quad V_1 := V \quad \text{and} \quad b_0 := b/(1-\delta) \tag{63}$$

(see Example 5.2). Consequently, when P satisfies Conditions $(\boldsymbol{M}_{\nu,\psi})$-$\boldsymbol{G}_\psi(\delta, V)$, it follows from Theorem 5.4 and Condition $\boldsymbol{D}_\psi(V_0, V_1)$ with V_0, V_1 and b_0 given in (63) that the residual kernel $R \equiv R_{\nu,\psi}$ given in (12) fulfils the following properties

$$0 \le \sum_{k=0}^{+\infty} R^k 1_{\mathbb{X}} \ \le\ \sum_{k=0}^{+\infty} R^k V \le \frac{1+d_0}{1-\delta}\, V \quad \text{with} \quad d_0 := \max\left(0, \frac{b-\nu(V))}{\nu(1_{\mathbb{X}})(1-\delta)}\right) \tag{64a}$$

$$0 \le \sum_{k=0}^{+\infty} \nu\big(R^k 1_{\mathbb{X}}\big) \ \le\ \sum_{k=0}^{+\infty} \nu\big(R^k V\big) \le \frac{(1+d_0)\nu(V)}{1-\delta} < \infty, \tag{64b}$$

and that $\pi_R := \mu_R(1_{\mathbb{X}})^{-1}\mu_R$ (see (25)) is the unique $P-$invariant probability measure on $(\mathbb{X}, \mathcal{X})$. Moreover, again from Theorem 5.4 (Conclusions (iii) and (vi)), we have

$$\mu_R(\psi) = 1 \quad \text{and} \quad \pi_R(V) = \pi_R(V_1) < \infty. \tag{65}$$

Corollary 6.1 is the direct application of Theorem 5.6 and Corollary 5.7 for Poisson's equation under Conditions $(\boldsymbol{M}_{\nu,\psi})$–$\boldsymbol{G}_\psi(\delta, V)$. Then the so-called $V-$geometric ergodicity is obtained in Subsection 6.2 using elementary spectral theory under Conditions $(\boldsymbol{M}_{\nu,\psi})$-$\boldsymbol{G}_\psi(\delta, V)$ and the aperiodicity condition (38).

6.1 Poisson's equation under the geometric drift condition

Corollary 6.1 *Assume that P satisfies Conditions $(\boldsymbol{M}_{\nu,\psi})$–$\boldsymbol{G}_\psi(\delta, V)$. Let $R \equiv R_{\nu,\psi}$ be the associated residual kernel given in* (12)*. Then*

1. *For any $g \in \mathcal{B}_V$, the function series $\widetilde{g} := \sum_{k=0}^{+\infty} R^k g$ absolutely converges on $\mathbb{X}$ (point-wise convergence). Moreover we have $\widetilde{g} \in \mathcal{B}_V$ and*

$$\forall g \in \mathcal{B}_V, \quad \|\widetilde{g}\|_V \leq \frac{1+d_0}{1-\delta}\,\|g\|_V \quad \textit{with} \quad d_0 := \max\left(0\,, \frac{b-\nu(V)}{\nu(1_{\mathbb{X}})(1-\delta)}\right) \tag{66}$$

   where δ, b are the constants given in $\boldsymbol{G}_\psi(\delta, V)$.

2. *For every $g \in \mathcal{B}_V$ such that $\pi_R(g) = 0$, the function $\widehat{g} = \widetilde{g} - \pi_R(\widetilde{g})1_{\mathbb{X}}$ is the unique π_R-centered function in $\mathcal{B}_V$ solution to Poisson's equation $(I-P)\widehat{g} = g$, and we have*

$$\|\widehat{g}\|_V \quad \leq \quad \frac{(1+d_0)\,(1+\pi_R(V))}{1-\delta}\,\|g\|_V. \tag{67}$$

For the sake of simplicity this statement is directly deduced below from Theorem 5.6 and Corollary 5.7. A self-contained proof of Corollary 6.1 could be also developed starting from (64a) and mimicking the proofs of Theorem 5.6 and Corollary 5.7.

Proof. Using the modulated drift condition $\boldsymbol{D}_\psi(V_0, V_1)$ with V_0, V_1, b_0 given in (63), it follows from Assertion 1. of Theorem 5.6 that

$$\forall g \in \mathcal{B}_V, \quad \|\widetilde{g}\|_{V_0} \leq (1+d_0)\|g\|_V \quad \text{with} \quad d_0 := \max\left(0, \frac{b-\nu(V))}{\nu(1_{\mathbb{X}})(1-\delta)}\right)$$

from which we deduce (66) since $\|\cdot\|_{V_0} = (1-\delta)\|\cdot\|_V$. To prove Assertion 2., we apply Corollary 5.7. First note that $\pi_R(V_0) < \infty$ since $V_0 = V/(1-\delta)$ and $\pi_R(V) < \infty$ (see (65)). Next we know from Corollary 5.7 that $\widehat{g} = \widetilde{g} - \pi_R(\widetilde{g})1_{\mathbb{X}}$ is a π_R-centered function in $\mathcal{B}_V$ solution to Poisson's equation $(I-P)\widehat{g} = g$. Moreover observe that $\pi_R(V_0)\,\|1_{\mathbb{X}}\|_{V_0} \leq \pi_R(V)$. From the first inequality in (60) and again $\|\cdot\|_{V_0} = (1-\delta)\|\cdot\|_V$, we obtained that

$$\|\widehat{g}\|_V \leq \big(1+\pi_R(V)\|_{V_0}\big)\|\widetilde{g}\|_V$$

from which we deduce (67) using (66).

Finally it follows from Condition $\boldsymbol{G}_\psi(\delta, V)$ that PV/V is bounded on $\mathbb{X}$, i.e. $P\mathcal{B}_V \subset \mathcal{B}_V$, since the small-function ψ is bounded and $1_{\mathbb{X}} \leq V$. Then Assertion (viii) of Theorem 5.4 ensures that $E_1 := \{g \in \mathcal{B}_V : Pg = g\} = \mathbb{R}\cdot 1_{\mathbb{X}}$. Hence two solutions to Poisson's equation in $\mathcal{B}_V$ differ from an additive constant. Consequently $\widehat{g}$ is the unique π_R-centered function in $\mathcal{B}_V$ solution to Poisson's equation $(I-P)\widehat{g} = g$. □

6.2 $V-$geometric ergodicity

Note that, under Conditions $(\boldsymbol{M}_{\nu,\psi})$-$\boldsymbol{G}_\psi(\delta, V)$, we have $h_R^\infty = 0$ from (64a), so that the aperiodicity condition (38) corresponds to the case $d = 1$ in Theorem 4.14. Now, under Conditions $(\boldsymbol{M}_{\nu,\psi})$-$\boldsymbol{G}_\psi(\delta, V)$ and (38), we prove the so-called $V-$geometric ergodicity of P. The proof is based on Inequalities (64a)-(64b), Corollary 6.1 and elementary spectral theory.

Theorem 6.2 *Assume that P satisfies Conditions* $(\boldsymbol{M}_{\nu,\psi})$-$\boldsymbol{G}_\psi(\delta, V)$ and is aperiodic (see (38)). Then P is $V-$geometrically ergodic, that is

$$\exists c > 0,\ \exists \rho \in (0,1),\ \forall g \in \mathcal{B}_V,\ \forall n \geq 1, \quad \|P^n g - \pi_R(g)1_{\mathbb{X}}\|_V \leq c\rho^n \|g\|_V. \tag{68}$$

Let $g \in \mathcal{B}_V$ be such that $\pi_R(g) = 0$. It follows from Property (68) that

$$\sum_{k=0}^{+\infty} \|P^k g\|_V \leq c(1-\rho)^{-1}\|g\|_V < \infty.$$

Consequently the function series $\mathfrak{g} := \sum_{k=0}^{+\infty} P^k g$ absolutely converges in $(\mathcal{B}_V, \|\cdot\|_V)$ and

$$\|\mathfrak{g}\|_V \leq c(1-\rho)^{-1}\|g\|_V.$$

Note that $\mathfrak{g}$ is π_R-centred and satisfies Poisson's equation $(I-P)\mathfrak{g} = g$, so that $\mathfrak{g}$ equals to the function $\widehat{g}$ of Corollary 6.1. Inequality (67) then provides the following alternative bound:

$$\|\mathfrak{g}\|_V \leq \frac{(1+d_0)\,(1+\pi_R(V))}{1-\delta}\,\|g\|_V.$$

Finally note that the geometric rate of convergence in the case of uniform ergodicity (see Example 3.7) corresponds to the $V-$geometric ergodicity property (68) in the special case $V = 1_{\mathbb{X}}$.

Let $V : \mathbb{X} \to [1, +\infty)$ be measurable. In the next lemmas using the spectral theory, the definition of $\mathcal{B}_V$ is extended to complex-valued functions, that is: For every measurable function $g : \mathbb{X} \to \mathbb{C}$, we set $\|g\|_V := \sup_{x\in\mathbb{X}} |g(x)|/V(x) \in [0, +\infty]$ where $|\cdot|$ stands here for the modulus in $\mathbb{C}$, and we define

$$\mathcal{B}_V(\mathbb{C}) := \big\{g : \mathbb{X} \to \mathbb{C}, \text{measurable such that } \|g\|_V < \infty\big\}.$$

Recall that PV/V is bounded on $\mathbb{X}$ from Condition $\boldsymbol{G}_\psi(\delta, V)$ since $\psi \leq c1_{\mathbb{X}} \leq cV$ for some $c > 0$. Thus P defines a bounded linear operator on the Banach space $(\mathcal{B}_V(\mathbb{C}), \|\cdot\|_V)$. Below the only prerequisites in spectral theory are the following points. Let L be a bounded linear operator on a Banach space $(\mathcal{L}, \|\cdot\|)$:

- The spectrum $\sigma(L)$ of L: the subset of $\mathbb{C}$ composed of all the complex numbers z such that $zI - L$ is not invertible, where I denotes the identity map on $\mathcal{L}$. Recall that $\sigma(L)$ is a compact subset of $\mathbb{C}$.

- The operator-norm of L, still denoted by $\|L\|$: $\|L\| := \sup\{\|Lf\| : f \in \mathcal{L}, \|f\| \leq 1\}$.

- The spectral radius $r(L)$ of L: $r(L) := \max\{|z| : z \in \sigma(L)\}$, and Gelfand's formula: $r(L) = \lim_n \|L^n\|^{1/n}$.

Lemmas 6.3–6.4 below show that, for every $z \in \mathbb{C}$ such that $|z| = 1$ and $z \neq 1$, the bounded linear operator $zI - P$ on $\mathcal{B}_V(\mathbb{C})$ is invertible under the assumptions of Theorem 6.2.

Lemma 6.3 *If P satisfies Conditions $(\boldsymbol{M}_{\nu,\psi})$-$\boldsymbol{G}_\psi(\delta, V)$ and is aperiodic (see (38)), then for every $z \in \mathbb{C}$ such that $|z| = 1$ and $z \neq 1$ the bounded linear operator $zI - P$ on $\mathcal{B}_V(\mathbb{C})$ is one-to-one.*

Proof. Let $z \in \mathbb{C}$ be such that $|z| = 1$ and assume that $zI - P$ on $\mathcal{B}_V(\mathbb{C})$ is not one-to-one, that is: there exists $g \in \mathcal{B}_V(\mathbb{C})$, $g \neq 0$, such that $(zI - P)g = 0$. Below we prove that this is only possible for $z = 1$, which provides the desired result. Let $g \in \mathcal{B}_V(\mathbb{C})$, $g \neq 0$, be such that $(zI - P)g = 0$. Since P, thus R, defines a bounded linear operator on the Banach space $(\mathcal{B}_V(\mathbb{C}), \|\cdot\|_V)$, Equality (43) of Lemma 4.15 can be proved similarly, that is we have:

$$\forall n \geq 0, \quad \nu(g) \sum_{k=0}^{n} z^{-(k+1)} R^k \psi = g - z^{-(n+1)} R^{n+1} g.$$

Moreover we know from Assertion 1. of Corollary 6.1 that the series $\widetilde{g} := \sum_{k=0}^{+\infty} R^k g$ point-wise converges on $\mathbb{X}$, thus: $\lim_k R^k g = 0$ (point-wise convergence). Hence we have $g = \nu(g)\widetilde{\psi}_z$, with $\widetilde{\psi}_z := \sum_{k=0}^{+\infty} z^{-(k+1)} R^k \psi$. Recall that $\widetilde{\psi}_z$ is bounded on $\mathbb{X}$ from Proposition 3.4. Thus g is bounded on $\mathbb{X}$, so that z is an eigenvalue of P on $\mathcal{B}(\mathbb{C})$ and $\rho(z) = 1$ from Lemma 4.15, where $\rho(\cdot)$ is defined (37). Since the aperiodicity condition corresponds to the case $d = 1$ in Theorem 4.14, it follows that $z = 1$ from Assertion (a) of Theorem 4.14. □

Lemma 6.4 *If P satisfies Conditions $(\boldsymbol{M}_{\nu,\psi})$-$\boldsymbol{G}_\psi(\delta, V)$ and is aperiodic (see (38)), then for every $z \in \mathbb{C}$ such that $|z| = 1$ and $z \neq 1$ the bounded linear operator $zI - P$ on $\mathcal{B}_V(\mathbb{C})$ is surjective.*

Proof. Let $z \in \mathbb{C}$ be such that $|z| = 1$ and $g \in \mathcal{B}_V$. Define

$$\forall n \geq 1, \quad \widetilde{g}_{n,z} := \sum_{k=0}^{n} z^{-(k+1)} R^k g.$$

Using $P = R + \psi \otimes \nu$ we obtain that

$$z\widetilde{g}_{n,z} - P\widetilde{g}_{n,z} = z\,\widetilde{g}_{n,z} - R\widetilde{g}_{n,z} - \nu(\widetilde{g}_{n,z})\psi = g - z^{-(n+1)} R^{n+1} g - \nu(\widetilde{g}_{n,z})\psi. \tag{69}$$

Moreover we have

$$\lim_{n \to +\infty} \widetilde{g}_{n,z} = \widetilde{g}_z := \sum_{k=0}^{+\infty} z^{-(k+1)} R^k g \quad \text{(point-wise convergence)} \quad \text{with} \quad \widetilde{g}_z \in \mathcal{B}_V(\mathbb{C})$$

since

$$\sum_{k=0}^{+\infty} |z^{-(k+1)} R^k g| \leq \|g\|_V \sum_{k=0}^{+\infty} R^k V \leq c\,V \quad \text{with} \quad c = (1 + d_0)(1 - \delta)^{-1}$$

from the second inequality in (64a). Also note that, for any $x \in \mathbb{X}$, we have $(PV)(x) < \infty$ from Condition $\boldsymbol{D}_\psi(V_0, V_1)$, and that $|\widetilde{g}_{n,z}| \leq c\,V$. It then follows from Lebesgue's theorem w.r.t. the probability measure $P(x, dy)$ that $\lim_n (P\widetilde{g}_{n,z})(x) = (P\widetilde{g}_z)(x)$. Finally we have

$$\lim_{n \to +\infty} \nu(\widetilde{g}_{n,z}) = \lim_{n \to +\infty} \sum_{k=0}^{n} z^{-(k+1)} \nu(R^k g) = \mu_z(g) := \sum_{k=0}^{+\infty} z^{-(k+1)} \nu(R^k g)$$

since the last series converges from $|z^{-(k+1)} \nu(R^k g)| \leq \|g\|_V \nu(R^k V)$ and (64b). Then, passing to the limit (point-wise convergence on $\mathbb{X}$) when $n \to +\infty$ in Equality (69), we obtain that $(zI - P)\widetilde{g}_z = g - \mu_z(g)\psi$. With $g = \psi$ this provides $(zI - P)\widetilde{\psi}_z = \big(1 - \mu_z(\psi)\big)\psi$ with

$$\widetilde{\psi}_z := \sum_{k=0}^{+\infty} z^{-(k+1)} R^k \psi \;\in \mathcal{B}_V(\mathbb{C}) \quad \text{and} \quad \mu_z(\psi) = \sum_{k=0}^{+\infty} z^{-(k+1)} \nu(R^k \psi) = \rho(z^{-1})$$

where $\rho(\cdot)$ is defined (37). Since $z \neq 1$ and $d = 1$ (aperiodicity condition), we know from Assertion (a) of Theorem 4.14 that $\rho(z^{-1}) \neq 1$. Thus

$$(zI - P)\bigg(\widetilde{g}_z + \frac{\mu_z(g)}{1-\mu_z(\psi)}\widetilde{\psi}_z\bigg) = g,$$

from which we deduce that $zI - P$ is surjective. □

Proof of Theorem 6.2. Recall that $\pi_R(V) < \infty$ under the assumptions of Theorem 6.2 (see (65)). Thus π_R defines a bounded linear form on $\mathcal{B}_V(\mathbb{C})$, so that $\mathcal{B}_0 := \{g \in \mathcal{B}_V(\mathbb{C}) : \pi_R(g) = 0\}$ is a closed subspace of $\mathcal{B}_V(\mathbb{C})$. Note that $\mathcal{B}_0$ is $P-$stable (i.e. $P(\mathcal{B}_0) \subset \mathcal{B}_0$) from the $P-$invariance of π_R. Let P_0 be the restriction of P to $\mathcal{B}_0$. Assertion 2. of Corollary 6.1 shows that $I - P_0$ is invertible on $\mathcal{B}_0$. Next let $z \in \mathbb{C}$ be such that $|z| = 1$, $z \neq 1$. It follows from Lemma 6.3 that $zI - P_0$ is one-to-one. Now, let $g \in \mathcal{B}_0$. From Lemma 6.4 there exists $h \in \mathcal{B}_V(\mathbb{C})$ such that $(zI - P)h = g$. We have $(z-1)\pi_R(h) = \pi_R(g) = 0$, thus $\pi_R(h) = 0$ (i.e. $h \in \mathcal{B}_0$) since $z \neq 1$. Hence $zI - P_0$ is surjective.

We have proved that, for every $z \in \mathbb{C}$ such that $|z| = 1$, the bounded linear operator $zI - P_0$ is invertible on $\mathcal{B}_0$. Let r denote the spectral radius of P on $\mathcal{B}_V(\mathbb{C})$. Recall that $r = \lim_n(\|P^n\|_V)^{1/n}$ from Gelfand's formula, where $\|\cdot\|_V$ denotes here the operator norm on $\mathcal{B}_V(\mathbb{C})$. We know that $r \le 1$ from Lemma 5.10 (in fact we have $r = 1$ since $P1_{\mathbb{X}} = 1_{\mathbb{X}}$). Hence the spectral radius r_0 of P_0 on $\mathcal{B}_0$ is less than one too. In fact we have $r_0 < 1$ since the spectrum $\sigma(P_0)$ of P_0 is a compact subset of $\mathbb{C}$ which, according to the above, is contained in the unit disk of $\mathbb{C}$ and does not contain any complex number of modulus one.

Let $\rho \in (r_0, 1)$. Since $r_0 = \lim_n(\|P_0^n\|_0)^{1/n}$ from Gelfand's formula where $\|\cdot\|_0$ denotes the operator norm on $\mathcal{B}_0$, there exists a positive constant c_ρ such that: $\|P_0^n\|_0 \le c_\rho \rho^n$. Thus

$$\begin{aligned}
\forall n \ge 1,\ \forall g \in \mathcal{B}_V(\mathbb{C}), \quad \|P^n g - \pi_R(g)1_{\mathbb{X}}\|_V &= \|P^n(g - \pi_R(g)1_{\mathbb{X}})\|_V \text{ (from } P^n 1_{\mathbb{X}} = 1_{\mathbb{X}}) \\
&= \|P_0^n(g - \pi_R(g)1_{\mathbb{X}})\|_V \text{ (since } g - \pi_R(g)1_{\mathbb{X}} \in \mathcal{B}_0) \\
&\le c_\rho \rho^n \|g - \pi_R(g)1_{\mathbb{X}}\|_V \\
&\le c_\rho(1 + \pi_R(V))\rho^n \|g\|_V
\end{aligned}$$

from triangular inequality and $\pi_R(|g|) \le \pi_R(V)\|g\|_V$. This proves (68). □

6.3 Bibliographic comments

A detailed and comprehensive history of geometric ergodicity, from the pioneering papers [Mar06, Doe37, Ken59] to modern works, can be found in [MT09, Sec. 15.6, 16.6], see also [DMPS18, Sec. 15.5]. Theorem 6.2 corresponds to the statement [MT09, Th. 16.1.2] and [DMPS18, Th. 15.2.4], except that it is stated here with a first-order small-function instead of a petite set. The proof in [MT09, DMPS18] is based on renewal theory and Nummelin's splitting construction. Alternative proofs of $V-$geometric ergodicity can be found in [RR04] based on coupling arguments, in [Bax05] based on renewal theory, in [HM11] based on an elegant idea using Wasserstein distance, and finally in [Hen06, HL14a, Del17, HL20] based on spectral theory (quasi-compactness) whose first founding ideas are already present in [DF37]. We refer to the recent paper [GHLR24] where 27 conditions for geometric ergodicity are discussed.

Since the pioneer work [MT94] much effort has been made to find explicit constant c and rate of convergence ρ in Inequality (68). Under Assumptions $(\boldsymbol{M}_{\nu,\psi})$-$\boldsymbol{G}_\psi(\delta, V)$ and the strong aperiodicity condition, such an issue is fully addressed in [Bax05] via renewal theory. Alternative computable upper bounds of the rate of convergence ρ can be found in [LT96, RT99, RT00, Ros02] using splitting or coupling methods, and in [HL14b, HL24b] using spectral theory. Recall that the $\mathbb{L}^2(\pi)-$rate of convergence can be also addressed for reversible Markov kernels satisfying Conditions ($\boldsymbol{M}_{\nu,\psi}$)-$\boldsymbol{G}_\psi(\delta, V)$, see [RR97, RT01, Bax05]. These issues are not addressed in our work.

Poisson's equation for V-geometrically ergodic Markov models is classically studied starting from Inequality (68), which ensures that, for every $g \in \mathcal{B}_V$ such that $\pi_R(g) = 0$, the function $\mathfrak{g} := \sum_{k=0}^{+\infty} P^k g$ in $\mathcal{B}_V$ is the unique π_R-centred solution to Poisson's equation $(I - P)\mathfrak{g} = g$. A quite different development is proposed in this section: Indeed Poisson's equation is first solved in Corollary 6.1 as a by-product of the modulated drift Condition $\boldsymbol{D}_\psi(V_0, V_1)$ (see (63)). Next this study is used for proving the $V-$geometric ergodicity: Indeed note that this prior study of Poisson's equation plays a crucial role at the beginning of the proof of Theorem 6.2 and that the convergent series in (64a)-(64b) are repeatedly used in the proof of Lemmas 6.3-6.4. A standard use of Poisson's equation is to prove a central limit theorem (C.L.T.). Let P be a Markov kernel satisfying Conditions ($\boldsymbol{M}_{\nu,\psi}$) and the $V-$geometric drift condition $\boldsymbol{G}_\psi(\delta, V)$. Then P satisfies Condition $\boldsymbol{D}_\psi(V_0, V_1)$ with V_0, V_1, b_0 given in (63). Consequently, if $\pi(V^2) < \infty$, then the conclusions of Glynn-Meyn's C.L.T., recalled page 44, hold true (note that $\mathcal{B}_{V_1} = \mathcal{B}_V$ here). Mention that the residual kernel R and its iterates have been considered in [KM03] to investigate the eigenvectors belonging to the dominated eigenvalue of the Laplace kernels associated with $V-$geometrically ergodic Markov kernel P. This issue called "multiplicative Poisson equation" in [KM03] is used to prove limit theorems for the underlying Markov chain (also see [KM05]). This question is not addressed in our work.

The key idea in this section is thus to apply Theorem 5.4 under the modulated drift Condition $\boldsymbol{D}_\psi(V_0, V_1)$ provided by the geometric drift condition $\boldsymbol{G}_\psi(\delta, V)$. Recall that the main argument for Theorem 5.4 is the residual-type drift inequality introduced in Subsection 5.2. An alternative residual-type drift inequality is proposed under Conditions ($\boldsymbol{M}_{\nu,1_S}$)-$\boldsymbol{G}_\psi(\delta, V)$ in [HL24b], showing that there exists $\alpha_0 \in (0, 1]$ such that $PV^{\alpha_0} \le \delta^{\alpha_0} V^{\alpha_0} + \nu(V^{\alpha_0})1_S$. Hence the residual kernel R satisfies the drift inequality $RV^{\alpha_0} \le \delta^{\alpha_0} V^{\alpha_0}$, from which bounds for the $V^{\alpha_0}-$weighted norm of solutions to Poisson's equation are provided, as well as bounds in the $V^{\alpha_0}-$geometric ergodicity. The bounds in [HL24b] involve the constant $(1 - \delta^{\alpha_0})^{-1}$, which is large when α_0 is close to zero. In such a case, the bounds (66) and (67) for the $V-$weighted norm of solutions to Poisson's equation may be more relevant.

A Probabilistic terminology

The split chain (e.g. see [Num84, Num02]). Let $(X_n)_{n\geq 0}$ be a Markov chain on the space $(\mathbb{X}, \mathcal{X})$ with kernel transition P satisfying condition $(\boldsymbol{M}_{\nu,\psi})$ with $\nu \in \mathcal{M}_{+,b}^*, \psi \in \mathcal{B}_+^*$, that is

$$R := P - \psi \otimes \nu \geq 0.$$

Let us introduce the bivariate Markov chain $((X_n, Y_n))_{n\geq 0}$ with the state space $\mathbb{X} \times \{0,1\}$ and the following transition kernel $\widehat{P}$: for every bounded measurable function f on $\mathbb{X} \times \{0,1\}$

$$\mathbb{E}[f(X_{n+1}, Y_{n+1}) \mid \sigma(X_k, Y_k, k \leq n)] = \mathbb{E}[f(X_{n+1}, Y_{n+1}) \mid \sigma(X_n)] = (\widehat{P}f)(X_n)$$

with

$$\forall A \in \mathcal{X}, \quad \widehat{P}(x, A \times \{0\}) = R(x, A) \quad \widehat{P}(x, A \times \{1\}) = \psi(x)\, \nu(1_A).$$

$((X_n, Y_n))_{n\geq 0}$ is called *the split chain* associated with $(X_n)_{n\geq 0}$. Note that, for any $A \in \mathcal{X}$, $\widehat{P}(x, A \times \{0,1\}) = \widehat{P}(x, A \times \{0\}) + \widehat{P}(x, A \times \{1\}) = P(x, A)$ so that the marginal process $(X_n)_{n\geq 0}$ is indeed the original Markov with transition kernel P. Next, for any $f \in \mathcal{B}$ and $x \in \mathbb{X}$, $\mathbb{E}[f(X_{n+1}) \mid X_n = x, Y_{n+1} = 1] = \nu(1_{\mathbb{X}})^{-1}\nu(f)$ for every $n \geq 1$. It follows that the set $\mathbb{X} \times \{1\}$ is an atom for the split chain. Let $\sigma_{\{1\}} := \inf\{n \geq 1, Y_n = 1\}$ be the return time to the atom $\mathbb{X} \times \{1\}$ of the split chain or the return time of $(Y_n)_{n\geq 0}$ to state 1. It is a regeneration times of the split chain. Such a material leads to using the so-called regeneration method to analyze the split chain and to deduce, when possible, the properties of the original Markov chain.

Probabilistic counterparts of various quantities in this document.

Let us introduce the probability measure $\widehat{\nu} = \nu(1_{\mathbb{X}})^{-1}\nu$ on $\mathbb{X}$. The probability $\mathbb{P}$ when $\mathbb{X}_0$ has probability distribution η, is denoted by $\mathbb{P}_\eta$ and $\mathbb{E}_\eta$ is the expectation under $\mathbb{P}_\eta$.

$\forall A \in \mathcal{X}$ and $\forall x \in \mathcal{X}$:

- $(R^n 1_A)(x) = R^n(x, A) = \mathbb{P}_x\{X_n \in A, \sigma_{\{1\}} > n\}$;
  $(R^n 1_{\mathbb{X}})(x) = R^n(x, \mathbb{X}) = \mathbb{P}_x\{\sigma_{\{1\}} > n\}$;
  $\sum_{n=0}^{+\infty} (R^n 1_{\mathbb{X}})(x) = \mathbb{E}_x[\sigma_{\{1\}}]$;
- $h_R^\infty(x) := \lim_n (R^n 1_{\mathbb{X}})(x) = \mathbb{P}_x\{\sigma_{\{1\}} = +\infty\}$;
- $(R^{n-1}\psi)(x) = \mathbb{P}_x\{\sigma_{\{1\}} = n\}/\nu(1_{\mathbb{X}})$, $\sum_{k=1}^n (R^{k-1}\psi)(x) = \mathbb{P}_x\{\sigma_{\{1\}} \leq n\}/\nu(1_{\mathbb{X}})$;
  $\sum_{n=1}^{+\infty} (R^{n-1}\psi)(x) = \mathbb{P}_x\{\sigma_{\{1\}} < \infty\}/\nu(1_{\mathbb{X}})$;
- $\mu_R(1_A) = \nu(1_{\mathbb{X}}) \sum_{n=0}^{+\infty} \mathbb{P}_{\widehat{\nu}}\{X_n \in A, \sigma_{\{1\}} > n\}$, $\mu_R(1_{\mathbb{X}}) = \nu(1_{\mathbb{X}})\mathbb{E}_{\widehat{\nu}}[\sigma_{\{1\}}]$
  $\mu_R(\psi) = \mathbb{P}_{\widehat{\nu}}\{\sigma_{\{1\}} < \infty\}$.
- Formula (16). For any $n \geq 1$, let $L_n := \min\{k = 0, \ldots, n-1 : Y_{n-k} = 1\}$, be the time elapsed since the last visit of $(Y_n)_{n\geq 0}$ to 1 before time n. Then $\{\sigma_{\{1\}} \leq n\} = \sqcup_{k=0}^{n-1}\{L_n = k\}$ and Formula (16) has the following probabilistic meaning
  $\mathbb{P}_x\{X_n \in A\} \;=\; \mathbb{P}_x\{X_n \in A, \sigma_{\{1\}} > n\} + \sum_{k=0}^{n-1} \mathbb{P}_x\{X_n \in A, L_n = k\}$.

B Proof of Theorem 4.12

From the definition of d in (41), there exists an integer $\ell_0 \geq 1$ such that the power series $\rho(z) := \sum_{n=1}^{+\infty} \nu(R^{n-1}\psi)\, z^n$ writes as

$$\forall z \in \overline{D}, \quad \rho(z) := \sum_{k=\ell_0}^{+\infty} \nu(R^{kd-1}\psi)\, z^{kd}. \tag{70}$$

The proof of Theorem 4.12 is based on the two following lemmas.

Lemma B.1 *Let P satisfy Condition $(\boldsymbol{M}_{\nu,\psi})$ with $\mu_R(1_{\mathbb{X}}) < \infty$. Then*

$$\lim_{n \to +\infty} P^{dn}\psi = \zeta_\psi := \frac{1}{m_d} \sum_{k=0}^{+\infty} R^{kd}\psi \quad \textit{with} \quad m_d := \sum_{k=\ell_0}^{+\infty} k\, \nu(R^{kd-1}\psi) < \infty. \tag{71}$$

Proof. Using the definition of the integer d, the arguments here are close to those used in the proof of the direct implication in Lemma 4.9. Note that $\sum_{k=0}^{+\infty} R^{dk}\psi$ is a bounded function on $\mathbb{X}$ from Proposition 3.4, and that $m_d < \infty$ from Remark 4.10. Now define

$$\forall z \in \overline{D}, \quad \mathcal{P}_d(z) := \sum_{n=0}^{+\infty} z^n P^{dn}\psi, \quad \mathcal{R}_d(z) := \sum_{n=0}^{+\infty} z^n R^{dn}\psi, \quad \rho_d(z) := \sum_{k=\ell_0}^{+\infty} \nu(R^{kd-1}\psi)\, z^k.$$

Note that the power series ρ in (70) satisfies $\rho(z) = \rho_d(z^d)$. Thus $\rho_d(z)$ is not a power series in z^q for any integer $q \geq 2$: Indeed, otherwise we would have $\rho_d(z) := \sum_{\ell=\ell_0'}^{+\infty} \nu(R^{q\ell d-1}\psi)\, z^{q\ell}$ for some integers $\ell_0' \geq 1$ and $q \geq 2$, thus

$$\rho(z) = \sum_{\ell=\ell_0'}^{+\infty} \nu(R^{q\ell d-1}\psi)\, z^{q\ell d},$$

which contradicts the definition (41) of d. Moreover observe that $|\rho_d(z)| < 1$ for every $z \in D = \{z \in \mathbb{C} : |z| < 1\}$ since $\mu_R(\psi) = \sum_{k=\ell_0}^{+\infty} \nu(R^{kd-1}\psi) = 1$ from Theorem 3.6. Now using (16) applied to ψ and the definition of d (see (70)) it follows that $P^{dn}\psi = R^{dn}\psi$ for every $n \in \{0, \ldots, \ell_0 - 1\}$ and that

$$\forall n \geq \ell_0, \quad P^{dn}\psi = R^{dn}\psi + \sum_{k=\ell_0}^{n} \nu(R^{dk-1}\psi) P^{d(n-k)}\psi.$$

Considering the associated power series and interchanging sums for the last term, we easily obtain that

$$\forall z \in \overline{D}, \quad \mathcal{P}_d(z) = \mathcal{R}_d(z)\, U_d(z) \quad \text{with} \quad U_d(z) := \frac{1}{1 - \rho_d(z)}. \tag{72}$$

Next, we deduce from the Erdös-Feller-Pollard renewal theorem [EFP49] that the coefficients $u_{d,k}$ of the power series $U_d(z) = \sum_{k=0}^{+\infty} u_{d,k} z^k$ in (72) satisfy: $\lim_k u_{d,k} = 1/m_d$. Then, identifying the coefficients in Equation (72) (Cauchy product), we obtain that $P^{dn}\psi = \sum_{k=0}^{n} u_{d,n-k} R^{dk}\psi$ for every $n \geq 0$. Since $\sum_{k=0}^{+\infty} R^{dk}\psi < \infty$ from Proposition 3.4, Property (71) follows from Lebesgue theorem w.r.t. discrete measure. □

Lemma B.2 *Let P satisfy Condition $(\boldsymbol{M}_{\nu,\psi})$ with $\mu_R(1_{\mathbb{X}}) < \infty$ and $h_R^\infty = 0$. Then there exists a sequence $(\varepsilon_n)_n \in \mathcal{B}^{\mathbb{N}}$ such that $\lim_n \varepsilon_n = 0$ (point-wise convergence) and*

$$\forall h \in \mathcal{B},\ \|h\|_{1_{\mathbb{X}}} \leq 1,\ \exists \xi_h \in \mathcal{B}, \quad |P^{dn}h - \xi_h| \leq \varepsilon_n.$$

Proof. Here, using the definition of the integer d, the arguments are close to those used in the proof of Lemma 4.11. For $r = 0, \ldots, d-1$ set $\zeta_{r,\psi} := P^r\zeta_\psi$ with ζ_ψ given in (71). Note that $\zeta_{r,\psi} \in \mathcal{B}$, and that $\lim_n P^{dn+r}\psi = \zeta_{r,\psi}$ (point-wise convergence) from Lebesgue's theorem w.r.t. $P^r(x, dy)$ for each $x \in \mathbb{X}$. Now for every $h \in \mathcal{B}$ define $\xi_h \in \mathcal{B}$ by

$$\xi_h := \sum_{r=0}^{d-1} \bigg(\sum_{j=1}^{+\infty} \nu(R^{dj-r-1}h) \bigg) \zeta_{r,\psi}. \tag{73}$$

Then using again (16) and observing that every integer $k = 1, \ldots, dn$ writes as $k = dj - r$ for $r = 0, \ldots, d-1$ and $j = 1, \ldots, n$, we obtain that for every $n \geq 1$

$$P^{dn}h - \xi_h = R^{dn}h + \sum_{r=0}^{d-1}\sum_{j=1}^{n} \nu(R^{dj-r-1}h)\big(P^{d(n-j)+r}\psi - \zeta_{r,\psi}\big) - \sum_{r=0}^{d-1}\bigg(\sum_{j=n+1}^{+\infty} \nu(R^{dj-r-1}h)\bigg)\zeta_{r,\psi}.$$

Thus, if $\|h\|_{1_{\mathbb{X}}} \leq 1$ (i.e. $|h| \leq 1_{\mathbb{X}}$), then we have $|P^{dn}h - \xi_h| \leq \varepsilon_n$ with $\varepsilon_n \in \mathcal{B}$ defined by

$$\varepsilon_n := R^{dn}1_{\mathbb{X}} + \sum_{r=0}^{d-1}\sum_{j=1}^{n} \nu(R^{dj-r-1}1_{\mathbb{X}})\big|P^{d(n-j)+r}\psi - \zeta_{r,\psi}\big| + \sum_{r=0}^{d-1} \|\zeta_{r,\psi}\|_{1_{\mathbb{X}}} \sum_{j=n+1}^{+\infty} \nu(R^{dj-r-1}1_{\mathbb{X}}).$$

We have $\lim_n \varepsilon_n = 0$ (point-wise convergence). Indeed, the last term converges to zero when $n \to +\infty$ since $\sum_{k=0}^{+\infty} \nu(R^k 1_{\mathbb{X}}) = \mu_R(1_{\mathbb{X}}) < \infty$; The second sum above converges to zero when $n \to +\infty$ from Lebesgue's theorem w.r.t. discrete measure recalling that $\lim_n P^{dn+r}\psi = \zeta_{r,\psi}$; Finally $\lim_n R^{dn}1_{\mathbb{X}} = 0$ from $h_R^\infty = 0$.

□

Proof of Theorem 4.12. Let $g \in \mathcal{B}$ be such that $|g| \leq 1_{\mathbb{X}}$. Note that for $r = 0, \ldots, d-1$ we have $|P^r g| \leq P^r|g| \leq P^r 1_{\mathbb{X}} = 1_{\mathbb{X}}$. Thus for $r = 0, \ldots, d-1$ we can consider $\xi_{r,g} := \xi_{P^r g}$, where $\xi_{P^r g}$ is the function of Lemma B.2 associated to $h = P^r g$. Let $\gamma_g = \frac{1}{d}\sum_{r=0}^{d-1} \xi_{r,g}$. Then

$$\bigg|\gamma_g - \frac{1}{d}\sum_{r=0}^{d-1} P^{nd+r}g\bigg| = \frac{1}{d}\sum_{r=0}^{d-1} \big|\xi_{r,g} - P^{nd}(P^r g)\big| \leq \varepsilon_n \tag{74}$$

from Lemma B.2. Thus we have $\gamma_g = \lim_n \frac{1}{d}\sum_{r=0}^{d-1} P^{nd+r}g$ (point-wise convergence). From Lebesgue's theorem w.r.t. $P(x, dy)$ for each $x \in \mathbb{X}$, we then obtain that

$$P\gamma_g = \lim_{n\to+\infty} \frac{1}{d}\sum_{r=1}^{d} P^{nd+r}g = \gamma_g \tag{75}$$

the last equality being obviously deduced from $\lim_{n\to+\infty} P^{nd+d}g = \lim_{n\to+\infty} P^{nd}g$. Thus γ_g is a $P-$harmonic function, so that $\gamma_g = c(g)1_{\mathbb{X}}$ for some constant c_g from Theorem 4.1. Moreover, using the second equality of (75) and applying Lebesgue's theorem w.r.t. the $P-$invariant probability measure π_R, we obtain that $\pi_R(g) = \pi_R(\gamma_g)$, so $\gamma_g = \pi_R(g)1_{\mathbb{X}}$. Finally, applying the function inequality (74) to any fixed $x \in \mathbb{X}$ and taking the supremum on all the functions $g \in \mathcal{B}$ such that $|g| \leq 1_{\mathbb{X}}$, we obtain the desired total variation convergence of Theorem 4.12 since $\lim_n \varepsilon_n(x) = 0$ from Lemma B.2. □

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