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COMPUTABLE BOUNDS OF ℓ^2-SPECTRAL GAP FOR DISCRETE MARKOV CHAINS WITH BAND TRANSITION MATRICES

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Abstract

We analyse the $\ell^2(\pi)$-convergence rate of irreducible and aperiodic Markov chains with N-band transition probability matrix P and with invariant distribution π. This analysis is heavily based on: first the study of the essential spectral radius $r_{ess}(P_{|\ell^2(\pi)})$ of $P_{|\ell^2(\pi)}$ derived from Hennion's quasi-compactness criteria; second the connection between the Spectral Gap property (SG$_2$) of P on $\ell^2(\pi)$ and the V-geometric ergodicity of P. Specifically, (SG$_2$) is shown to hold under the condition

$$\alpha_0 := \sum_{m=-N}^{N} \limsup_{i\to+\infty} \sqrt{P(i,i+m)\,P^*(i+m,i)}\ <\ 1.$$

Moreover $r_{ess}(P_{|\ell^2(\pi)}) \le \alpha_0$. Effective bounds on the convergence rate can be provided from a truncation procedure.

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1. Introduction

Let $P := (P(i,j))_{(i,j)\in\mathbb{N}^2}$ be a Markov kernel on the countable state space $\mathbb{N}$. Throughout the paper we assume that P is irreducible and aperiodic, that P has a unique invariant probability measure denoted by $\pi := (\pi(i))_{i\in\mathbb{N}}$, and finally that

$$\exists i_0 \in \mathbb{N},\ \exists N \in \mathbb{N}^*,\ \forall i \ge i_0\ :\quad |i-j| > N \implies P(i,j) = 0. \tag{\textbf{AS1}}$$

We denote by $(\ell^2(\pi), \|\cdot\|_2)$ the Hilbert space of sequences $(f(i))_{i\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ such that $\|f\|_2 := \big[\sum_{i\ge0} |f(i)|^2\,\pi(i)\big]^{1/2} < \infty$. Then P defines a linear contraction on $\ell^2(\pi)$, and its adjoint operator P^* on $\ell^2(\pi)$ is defined by $P^*(i,j) := \pi(j)\,P(j,i)/\pi(i)$. If $\pi(f) := \sum_{i\ge0} f(i)\,\pi(i)$, then the kernel P is said to have the spectral gap property on $\ell^2(\pi)$ if there exists $\rho \in (0,1)$ and $C \in (0,+\infty)$ such that

$$\forall n \ge 1, \forall f \in \ell^2(\pi), \quad \|P^n f - \Pi f\|_2 \le C\,\rho^n\,\|f\|_2 \quad \text{with} \quad \Pi f := \pi(f)1_{\mathbb{N}}. \tag{$\mathbf{SG}_2$}$$

A standard issue is to compute the value (or to find an upper bound) of

$$\varrho_2 := \inf\{\rho \in (0,1) : (\mathbf{SG}_2) \text{ holds true}\}. \tag{1}$$

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In this work the quasi-compactness criteria of [3] is used to study ($\mathbf{SG}_2$) and to estimate ϱ_2. In Section 2 it is proved that ($\mathbf{SG}_2$) holds when

$$\alpha_0 := \sum_{m=-N}^{N} \limsup_{i\to+\infty} \sqrt{P(i,i+m)\,P^*(i+m,i)} \; < \; 1. \tag{\textbf{AS2}}$$

Moreover $r_{ess}(P_{|\ell^2(\pi)}) \leq \alpha_0$. We refer to [3] for the definition of the essential spectral radius $r_{ess}(T)$ and for quasi-compactness of a bounded linear operator T on a Banach space. Under the assumptions

$$\forall m = -N,\dots,N, \quad P(i,i+m) \xrightarrow[i\to+\infty]{} a_m \in [0,1] \tag{\textbf{AS3}}$$

$$\frac{\pi(i+1)}{\pi(i)} \xrightarrow[i\to+\infty]{} \tau \in [0,1) \tag{\textbf{AS4}}$$

$$\sum_{k=-N}^{N} k\,a_k \; < 0, \tag{\textbf{NERI}}$$

Property (**AS2**) holds (hence ($\mathbf{SG}_2$)) and α_0 can be explicitly computed in function of τ and the a_m's. Moreover, using the inequality $r_{ess}(P_{|\ell^2(\pi)}) \leq \alpha_0$, Property ($\mathbf{SG}_2$) is proved to be connected to the V-geometric ergodicity of P for $V := (\pi(n)^{-1/2})_{n\in\mathbb{N}}$. In particular, denoting the minimal V-geometrical ergodic rate by ϱ_V, it is proved that, either ϱ_2 and ϱ_V are both less than α_0, or $\varrho_2 = \varrho_V$. As a result, an accurate bound of ϱ_2 can be obtained for random walks (RW) with i.d. bounded increments using the results of [5]. Actually, any estimation of ϱ_V, for instance that derived in Section 3 from the truncation procedure of [4], provides an estimation of ϱ_2. We point out that all the previous results hold without any reversibility properties.

The spectral gap property for Markov processes has been widely investigated in the discrete and continuous-time cases (e.g. see [10, 2]). There exist different definitions of the spectral gap property according that we are concerned with discrete or continuous-time case (e.g. see [14, 8]). The focus of our paper is on the discrete time case. In the reversible case, the equivalence between the geometrical ergodicity and ($\mathbf{SG}_2$) is proved in [9] and Inequality $\varrho_2 \leq \varrho_V$ is obtained in [1, Th.6.1.]. This equivalence fails in the non-reversible case (see [7]). The link between ϱ_2 and ϱ_V stated in our Proposition 1 is obtained with no reversibility condition. Formulae for ϱ_2 are provided in [11, 13] in terms of isoperimetric constants which are related to P in reversible case and to P and P^* in non-reversible case. However, to the best of our knowledge, no explicit value (or upper bounds) of ϱ_2 can be derived from these formulae for discrete Markov chains with band transition matrices. Our explicit bound $r_{ess}(P_{|\ell^2(\pi)}) \leq \alpha_0$ in Theorem 1 is the preliminary key results in this work. Recall that $r_{ess}(P_{|\ell^2(\pi)})$ is a natural lower bound of ϱ_2 (apply [5, Prop. 2.1] with the Banach space $\ell^2(\pi)$). The essential spectral radius of Markov operators on a $\mathbb{L}^2$-type space is investigated for Markov chains with general state space in [12], but no explicit bound for $r_{ess}(P_{|\ell^2(\pi)})$ can be derived a priori from these theoretical results for Markov chains with band transition matrices, except in the reversible case [12, Th. 5.5.].

2. Property ($\mathbf{SG_2}$) and V-geometrical ergodicity

Theorem 1. *If Condition* (**AS2**) *holds, then P satisfies* ($\mathbf{SG}_2$). *Moreover we have* $r_{ess}(P_{|\ell^2(\pi)}) \leq \alpha_0$.

Proof. Let us introduce $\ell^1(\pi) := \{(f(i))_{i\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}\colon\ \|f\|_1 := \sum_{i\geq 0} |f(i)|\, \pi(i) < \infty\}$.

Lemma 2.1. *For any $\alpha > \alpha_0$, there exists a positive constant $L \equiv L(\alpha)$ such that*

$$\forall f \in \ell^2(\pi), \quad \|Pf\|_2 \leq \alpha\, \|f\|_2 + L\|f\|_1.$$

Since the identity map is compact from $\ell^2(\pi)$ into $\ell^1(\pi)$ (from the Cantor diagonal procedure), it follows from Lemma 2.1 and from [3] that P is quasi-compact on $\ell^2(\pi)$ with $r_{ess}(P_{|\ell^2(\pi)}) \leq \alpha$. Since α can be chosen arbitrarily close to α_0, this gives $r_{ess}(P_{|\ell^2(\pi)}) \leq \alpha_0$. Then ($\mathbf{SG}_2$) is deduced from aperiodicity and irreducibility assumptions. The proof of Theorem 1 is complete.

Proof of Lemma 2.1. Under Assumption (**AS1**), define

$$\forall i \geq i_0,\ \forall m = -N, \ldots, N, \quad \beta_m(i) := \sqrt{P(i, i+m)\, P^*(i+m, i)}. \tag{2}$$

Let $\alpha > \alpha_0$, with α_0 given in (**AS2**). Fix $\ell \equiv \ell(\alpha) \geq i_0$ such that $\sum_{m=-N}^{N} \sup_{i\geq\ell} \beta_m(i) \leq \alpha$. For $f \in \ell^2(\pi)$ we have from Minkowski's inequality and the band structure of P for $i \geq \ell$

$$\begin{aligned}
\|Pf\|_2 \ &\leq\ \Big[\sum_{i<\ell} \big|(Pf)(i)\big|^2 \pi(i)\Big]^{1/2} + \Big[\sum_{i\geq\ell} \Big|\sum_{m=-N}^{N} P(i, i+m)\, f(i+m)\Big|^2 \pi(i)\Big]^{1/2} \\
&\leq\ L \sum_{i<\ell} |(Pf)(i)|\, \pi(i) + \Big[\sum_{i\geq\ell} \Big|\sum_{m=-N}^{N} P(i, i+m)\, f(i+m)\Big|^2 \pi(i)\Big]^{1/2}
\end{aligned} \tag{3}$$

where $L \equiv L_\ell > 0$ comes from equivalence of norms on $\mathbb{C}^\ell$. Moreover we have $\sum_{i<\ell} |(Pf)(i)|\, \pi(i) \leq \|Pf\|_1 \leq \|f\|_1$. To control the second term in (3), define $F_m = (F_m(i))_{i\in\mathbb{N}} \in \ell^2(\pi)$ by $F_m(i) := P(i, i+m)\, f(i+m)(1 - 1_{\{0,\ldots,\ell-1\}}(i))$ for $-N \leq m \leq N$. Then

$$\Big[\sum_{i\geq\ell} \Big|\sum_{m=-N}^{N} P(i, i+m)\, f(i+m)\Big|^2 \pi(i)\Big]^{1/2} = \Big\|\sum_{m=-N}^{N} F_m\Big\|_2 \leq \sum_{m=-N}^{N} \|F_m\|_2.$$

$$\begin{aligned}
\text{and} \quad \|F_m\|_2^2 \ &=\ \sum_{i\geq\ell} P(i, i+m)^2\, |f(i+m)|^2 \pi(i) \\
&=\ \sum_{i\geq\ell} P(i, i+m) \frac{\pi(i)\, P(i, i+m)}{\pi(i+m)}\, |f(i+m)|^2 \pi(i+m) \\
&\leq\ \sup_{i\geq\ell} \beta_m(i)^2\, \|f\|_2^2 \qquad \text{(from the definition of } P^* \text{ and from (2)).}
\end{aligned}$$

The statement in Lemma 2.1 can be deduced from the previous inequality and from (3).

The core of our approach to estimate ϱ_2 is the relationship between Property ($\mathbf{SG}_2$) and the $V-$geometric ergodicity. Indeed, specify Theorem 1 in terms of the $V-$geometric ergodicity with $V := (\pi(n)^{-1/2})_{n\in\mathbb{N}}$. Let $(\mathcal{B}_V, \|\cdot\|_V)$ denote the space of sequences $(g(n))_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ such that $\|g\|_V := \sup_{n\in\mathbb{N}} V(n)^{-1}\,|g(n)| < \infty$. Recall that P is said to be V-geometrically ergodic if P satisfies the spectral gap property on $\mathcal{B}_V$, namely: there exist $C \in (0,+\infty)$ and $\rho \in (0,1)$ such that

$$\forall n \geq 1, \forall f \in \mathcal{B}_V, \quad \|P^n f - \Pi f\|_V \leq C\,\rho^n\,\|f\|_V. \qquad (\mathbf{SG}_V)$$

When this property holds, we define

$$\varrho_V := \inf\{\rho \in (0,1) : (\mathbf{SG}_V) \text{ holds true}\}. \qquad (4)$$

Remark 2.1. Under Assumptions *(**AS3**)* and *(**AS4**)*, we have

$$\alpha_0 := \sum_{m=-N}^{N} \limsup_{i\to+\infty} \sqrt{P(i,i+m)\,P^*(i+m,i)} = \begin{cases} \displaystyle\sum_{m=-N}^{N} a_m\,\tau^{-m/2} & \text{if } \tau \in (0,1) \\ a_0 & \text{if } \tau = 0. \end{cases} \qquad (5)$$

Indeed, if *(**AS4**)* holds with $\tau \in (0,1)$, then the claimed formula follows from the definition of P^*. If $\tau = 0$ in *(**AS4**)*, then $a_m = 0$ for every $m = 1,\ldots,N$ from $\sum_{m=-N}^{N} P(i+m,i)\,\pi(i+m)/\pi(i) = 1$. Thus $a_{-m} = 0$ when $m < 0$. Hence $\alpha_0 = a_0$.

Proposition 1. *If P and π satisfy Assumptions* (**AS3**)*,* (**AS4**) *and* (**NERI**)*, then P satisfies* (**AS2**) *(with $\alpha_0 < 1$ given in (5)). Moreover P satisfies both* ($\mathbf{SG}_2$) *and* ($\mathbf{SG}_V$) *with $V := (\pi(n)^{-1/2})_{n\in\mathbb{N}}$, we have $\max(r_{ess}(P_{|\mathcal{B}_V}), r_{ess}(P_{|\ell^2(\pi)})) \leq \alpha_0$, and the next assertions hold:*

1. *if $\varrho_V \leq \alpha_0$, then $\varrho_2 \leq \alpha_0$;*
2. *if $\varrho_V > \alpha_0$, then $\varrho_2 = \varrho_V$.*

Proof. If $\tau = 0$ in (**AS4**), then $\alpha_0 = a_0 < 1$ from (5) and (**NERI**). Now assume that (**AS4**) holds with $\tau \in (0,1)$. Then $\alpha_0 = \sum_{m=-N}^{N} a_m\,\tau^{-m/2} = \psi(\sqrt{\tau})$, where: $\forall t > 0,\ \psi(t) := \sum_{k=-N}^{N} a_k\,t^{-k}$. Moreover it easily follows from the invariance of π that $\psi(\tau) = 1$. Inequality $\alpha_0 = \psi(\sqrt{\tau}) < 1$ is deduced from the following assertions: $\forall t \in (\tau,1),\ \psi(t) < 1$ and $\forall t \in (0,\tau)\cup(1,+\infty),\ \psi(t) > 1$. To prove these properties, note that $\psi(\tau) = \psi(1) = 1$ and that ψ is convex on $(0,+\infty)$. Moreover we have $\lim_{t\to+\infty}\psi(t) = +\infty$ since $a_k > 0$ for some $k < 0$ (use $\psi(\tau) = \psi(1) = 1$ and $\tau \in (0,1)$). Similarly, $\lim_{t\to 0^+}\psi(t) = +\infty$ since $a_k > 0$ for some $k > 0$. This gives the desired properties on ψ since $\psi'(1) > 0$ from (**NERI**).

($\mathbf{SG}_2$) and $r_{ess}(P_{|\ell^2(\pi)}) \leq \alpha_0$ follow from Theorem 1. Next ($\mathbf{SG}_V$) is deduced from the well-known link between geometric ergodicity and the following drift inequality:

$$\forall \alpha \in (\alpha_0,1),\ \exists L \equiv L_\alpha > 0, \quad PV \leq \alpha V + L\,1_{\mathbb{N}}. \qquad (6)$$

This inequality holds from $\lim_i (PV)(i)/V(i) = \alpha_0$.

Then ($\mathbf{SG}_V$) is derived from (6) using aperiodicity and irreducibility. It also follows from (6) that $r_{ess}(P_{|\mathcal{B}_V}) \leq \alpha$ (see [5, Prop. 3.1]). Thus $r_{ess}(P_{|\mathcal{B}_V}) \leq \alpha_0$.

Now we prove the statements 1. and 2. using the spectral properties of [5, Prop. 2.1] of both $P_{|\ell^2(\pi)}$ and $P_{|\mathcal{B}_V}$ (due to quasi-compactness, see [3]). We will also use the following obvious inclusion: $\ell^2(\pi) \subset \mathcal{B}_V$. In particular every eigenvalue of $P_{|\ell^2(\pi)}$ is also an eigenvalue for $P_{|\mathcal{B}_V}$. First assume that $\varrho_V \leq \alpha_0$. Then there is no eigenvalue for $P_{|\mathcal{B}_V}$ in the annulus $\Gamma := \{\lambda \in \mathbb{C} : \alpha_0 < |\lambda| < 1\}$ since $r_{ess}(P_{|\mathcal{B}_V}) \leq \alpha_0$. From $\ell^2(\pi) \subset \mathcal{B}_V$ it follows that there is also no eigenvalue for $P_{|\ell^2(\pi)}$ in this annulus. Hence $\varrho_2 \leq \alpha_0$ since $r_{ess}(P_{|\ell^2(\pi)}) \leq \alpha_0$. Second assume that $\varrho_V > \alpha_0$. Then $P_{|\mathcal{B}_V}$ admits an eigenvalue $\lambda \in \mathbb{C}$ such that $|\lambda| = \varrho_V$. Let $f \in \mathcal{B}_V$, $f \neq 0$, such that $Pf = \lambda f$. We know from [5, Prop. 2.2] that there exists some $\beta \equiv \beta_\lambda \in (0,1)$ such that $|f(n)| = \mathrm{O}(V(n)^\beta) = \mathrm{O}(\pi(n)^{-\beta/2})$, so that $|f(n)|^2\pi(n) = \mathrm{O}(\pi(n)^{(1-\beta)})$, thus $f \in \ell^2(\pi)$ from (**AS4**). We have proved that $\varrho_2 \geq \varrho_V$. Finally the converse inequality is true since every eigenvalue of $P_{|\ell^2(\pi)}$ is an eigenvalue for $P_{|\mathcal{B}_V}$. Thus $\varrho_2 = \varrho_V$.

From Proposition 1, any estimation of ϱ_V provides an estimation of ϱ_2. This is illustrated in Example 2.1 and Corollary 3.1. Markov chains in Example 2.1 have been studied in details in [5, Section 3]. Also mention that further technical details are reported in [6].

Example 2.1. (*RWs with i.d. bounded increments.*) Let P be defined as follows. There exist some positive integers $c, g, d \in \mathbb{N}^*$ such that

$$\forall i \in \{0, \ldots, g-1\}, \quad \sum_{j=0}^{c} P(i,j) = 1;$$

$$\forall i \geq g, \forall j \in \mathbb{N}, \quad P(i,j) = \begin{cases} a_{j-i} & \text{if } \ i-g \leq j \leq i+d \\ 0 & \text{otherwise.} \end{cases}$$

$$(a_{-g}, \ldots, a_d) \in [0,1]^{g+d+1} : a_{-g} > 0, \ a_d > 0, \ \sum_{k=-g}^{d} a_k = 1.$$

Assume that P is aperiodic and irreducible, and satisfies ***(NERI)***. Then P has a unique invariant distribution π. It can be derived from standard results of linear difference equation that $\pi(n) \sim c\,\tau^n$ when $n \to +\infty$, with $\tau \in (0,1)$ defined by $\psi(\tau) = 1$, where $\psi(t) := \sum_{k=-N}^{N} a_k\, t^{-k}$. Thus, if $\gamma := \tau^{-1/2}$, then $\mathcal{B}_V = \{(g(n))_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}},\ \sup_{n\in\mathbb{N}} \gamma^{-n}\, |g(n)| < \infty\}$. Then we know from [5, Prop. 3.2] that $r_{ess}(P_{|\mathcal{B}_V}) = \alpha_0$ with α_0 given in (5), and that ϱ_V can be computed from an algebraic polynomial elimination. From this computation, Proposition 1 provides an accurate estimation of ϱ_2. Property ***(SG$_2$)*** was proved in [13, Th. 2] under an extra weak reversibility assumption (with no explicit bound on ϱ_2). However, except in case $g = d = 1$ where reversibility is automatic, an RW with i.d. bounded increments is not reversible or even weak reversible in general. No reversibility condition is required here.

3. Bound for ϱ_2 via truncation

Let P be any Markov kernel on $\mathbb{N}$, and let us consider the k-th truncated (and augmented on the last column) matrix P_k associated with P as in [4]. If $\sigma(P_k)$ denotes the set of eigenvalues of P_k, define $\rho_k := \max\big\{|\lambda|,\ \lambda \in \sigma(P_k),\ |\lambda| < 1\big\}$. The weak perturbation method in [4] provides the following general result where Condition (**AS1**) is not required and V is any unbounded increasing sequence.

Proposition 2. *Let P be an irreducible and aperiodic Markov kernel on $\mathbb{N}$ satisfying the following drift inequality for some unbounded increasing sequence $(V(n))_{n\in\mathbb{N}}$:*

$$\exists\delta \in [0,1[,\ \exists L > 0, \quad PV \leq \delta V + L\,1_{\mathbb{N}}. \tag{8}$$

Let ϱ_V be defined in (4). Then, either $\varrho_V \leq \delta$ and $\limsup_k \rho_k \leq \delta$, or $\varrho_V > \delta$ and $\varrho_V = \lim_k \rho_k$.

Proof. Condition (8) ensures that the assumptions of [4, Lem. 6.1] are satisfied, so that $r_{ess}(P_{|\mathcal{B}_V}) \leq \delta$. Then, using standard duality arguments, the spectral rank-stability property [4, Lem. 7.2] applies to $P_{|\mathcal{B}_V}$ and P_k. If $\varrho_V \leq \delta$, then, for each r such that $\delta < r < 1$, $\lambda = 1$ is the unique eigenvalue of $P_{|\mathcal{B}_V}$ in $C_r := \{\lambda \in \mathbb{C} : r < |\lambda| \leq 1\}$ (see [3]). From [4, Lem. 7.2] this property holds for P_k when k is large enough, so that $\limsup_k \rho_k \leq r$. Thus $\limsup_k \rho_k \leq \delta$ since r is arbitrarily close to δ. Now assume that $\varrho_V > \delta$, and let r be such that $\delta < r < \varrho_V$. Then $P_{|\mathcal{B}_V}$ has a finite number of eigenvalues in C_r, say $\lambda_0, \lambda_1, \ldots, \lambda_N$, with $\lambda_0 = 1$, $|\lambda_1| = \varrho_V$ and $|\lambda_k| \leq \varrho_V$ for $k = 2, \ldots, N$ (see [3]). For $a \in \mathbb{C}$ and $\varepsilon > 0$ we define $D(a,\varepsilon) := \{z \in \mathbb{C} : |z-a| < \varepsilon\}$. Now consider any $\varepsilon > 0$ such that the disks $D(\lambda_k,\varepsilon)$ for $k = 0,\ldots,N$ are disjoint and are contained in C_r pour $k \geq 1$. From [4, Lem. 7.2], for k large enough, 1 is the only eigenvalue of P_k in $D(1,\varepsilon)$, the others eigenvalues of P_k in C_r are contained in $\cup_{k=1}^N D(\lambda_k,\varepsilon)$, and finally each $D(\lambda_k,\varepsilon)$ contains at least one eigenvalue of P_k. Thus each eigenvalue $\lambda \neq 1$ of P_k in C_r has modulus less than $\varrho_V + \varepsilon$, so that $\rho_k \leq \varrho_V + \varepsilon$. Moreover the disk $D(\lambda_1,\varepsilon)$ contains at least an eigenvalue λ of P_k, so that $\rho_k \geq |\lambda| \geq \varrho_V - \varepsilon$. Thus, for k large enough, we have $\varrho_V - \varepsilon \leq \rho_k \leq \varrho_V + \varepsilon$.

Under the assumptions of Proposition 1 we deduce the following result from Proposition 2.

Corollary 3.1. *If P satisfies the assumptions of Proposition 1, then the following properties hold true with α_0 given in (5):*

1. $\varrho_2 \leq \alpha_0 \iff \varrho_V \leq \alpha_0$, *and in this case we have* $\limsup_k \rho_k \leq \alpha_0$;
2. $\varrho_2 > \alpha_0 \iff \varrho_V > \alpha_0$, *and in this case we have* $\varrho_2 = \varrho_V = \lim_k \rho_k$.

As usual the reversible case is simpler. In particular we can take $C = 1$ and $\rho = \varrho_2$ in $(\mathbf{SG}_2)$. Details and numerical illustrations for Metropolis-Hastings kernels are reported in [6].

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