Polynomial convergence rates for Markov kernels under nested modulated drift conditions

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Abstract

When a Markov kernel P satisfies a minorization condition and nested modulated drift conditions, Jarner and Roberts provided an asymptotic polynomial convergence rate in weighted total variation norm of $P^n(x,\cdot)$ to the P-invariant probability measure π. In connection with this polynomial asymptotics, we propose explicit and simple estimates on series of such weighted total variation norms, from which an estimate for the total variation norm of $P^n(x,\cdot)-\pi$ is deduced. The proofs are self-contained and based on the residual kernel and the Nummelin-type representation of π. No coupling technique is used.

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1 Introduction

Let $(\mathbb{X},\mathcal{X})$ be a measurable space, and let $\mathcal{M}^+$ (resp. $\mathcal{M}_*^+$) denote the set of finite nonnegative (resp. positive) measures on $(\mathbb{X},\mathcal{X})$. For any $\mu\in\mathcal{M}^+$ and any μ-integrable function $g:\mathbb{X}\to\mathbb{R}$, $\mu(g)$ denotes the integral $\int_{\mathbb{X}} g d\mu$. For any measurable function $V:\mathbb{X}\to[1,+\infty)$ and every measurable function $g:\mathbb{X}\to\mathbb{R}$, we set $\|g\|_V := \sup_{x\in\mathbb{X}}|g(x)|/V(x)\in[0,+\infty]$, and we define the space

$$\mathcal{B}_V := \{g:\mathbb{X}\to\mathbb{R}, \text{measurable such that } \|g\|_V < \infty\}.$$

If $(\mu_1,\mu_2)\in(\mathcal{M}^+)^2$ is such that $\mu_i(V)<\infty, i=1,2$, then the V-weighted total variation norm $\|\mu_1-\mu_2\|'_V$ is defined by

$$\|\mu_1-\mu_2\|'_V := \sup_{\|g\|_V\le 1}\big|\mu_1(g)-\mu_2(g)\big|. \tag{1}$$

If $V=1_{\mathbb{X}}$, then $\|\cdot\|'_{1_{\mathbb{X}}}=\|\cdot\|_{TV}$ is the standard total variation norm. Finally recall that a non-negative kernel $K(x,dy)\in\mathcal{M}^+$, $x\in\mathbb{X}$, is said to be a Markov (respectively submarkov) kernel if $K(x,\mathbb{X})=1$ (respectively $K(x,\mathbb{X})\le 1$) for any $x\in\mathbb{X}$. We denote by

$$\forall x\in\mathbb{X},\quad (Kg)(x) := \int_{\mathbb{X}} g(y)\,K(x,dy)$$

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the functional action of K, where $g : \mathbb{X} \to \mathbb{R}$ is any $K(x,\cdot)-$integrable function. For every $n \geq 1$ the $n-$th iterate kernel of $K(x, dy)$ is denoted by $K^n(x, dy)$, $x \in \mathbb{X}$, and K^n stands for its functional action. As usual K^0 is the identity map I by convention. If $\mu \in \mathcal{M}^+$ and K is a submarkov kernel, then the product μK is the finite non-negative measure defined by

$$\forall A \in \mathcal{X}, \quad (\mu K)(1_A) := \int_{\mathbb{X}} K(x, A)\, \mu(dx).$$

The measure μ is said to be $K-$invariant if $\mu K = \mu$.

Throughout this paper, P is a Markov kernel on $(\mathbb{X}, \mathcal{X})$ satisfying the following minorization condition $(\mathbf{S})$ (e.g. see [MT09])

$$\exists S \in \mathcal{X},\ \exists \nu \in \mathcal{M}_*^+, \quad \forall x \in \mathbb{X},\ \forall A \in \mathcal{X}, \quad P(x, A) \geq \nu(1_A)\, 1_S(x) \tag{$\mathbf{S}$}$$

(i.e. S is a small-set of 1-order for P), and we denote by R the associated submarkov residual kernel:

$$\forall x \in \mathbb{X},\ \forall A \in \mathcal{X}, \quad R(x, A) := P(x, A) - \nu(A) 1_S(x). \tag{2}$$

Moreover let us introduce the following well-known nested modulated drift conditions: There exists a collection $\{V_i\}_{i=0}^m$ of measurable functions from $\mathbb{X}$ to $[1, +\infty)$ with $m \geq 1$ such that

$$\forall i \in \{0, \ldots, m-1\}, \quad V_{i+1} \leq V_i \quad \text{and} \quad \exists b_i > 0,\ PV_i \leq V_i - V_{i+1} + b_i\, 1_S. \tag{$\boldsymbol{D}(V_0 : V_m)$}$$

The V_i's in $\boldsymbol{D}(V_0 : V_m)$ are called Lyapunov functions and are such that $V_m \leq \cdots \leq V_0$. The following statement was proved in [JR02, Th. 3.2], also see [FM03b, Th. 1]:

Theorem. *Assume that P is $\psi-$irreducible and aperiodic for some $\psi \in \mathcal{M}_*^+$ and that P satisfies Conditions $(\mathbf{S})$ and $\boldsymbol{D}(V_0 : V_m)$ with $m \geq 1$. Let π denote the $P-$invariant probability measure. Then*

$$\forall x \in \mathbb{X}, \quad \lim_{n \to +\infty} (n+1)^{m-1} \big\| P^n(x, \cdot) - \pi \big\|'_{V_m} = 0. \tag{3}$$

In [JR02, Th. 3.2] the condition $V_{i+1} \leq V_i$ is not assumed and the modulated drift inequalities write as $PV_i \leq V_i - V_{i+1} + b_i\, 1_{S_i}$ for some petite set S_i. The assumption $V_{i+1} \leq V_i$ is by no means restrictive since the Lyapunov functions V_i can be slightly modified in order to satisfy this condition, e.g. see Subsection 4.3.

Under Assumptions $(\mathbf{S})$ and $\boldsymbol{D}(V_0 : V_m)$ with $m \geq 2$, the purpose of this work is to provide quantitative estimates in connection with Property (3). Specifically we prove that there exists a positive function $U_m \in \mathcal{B}_{V_0}$ with a computable bound $\widehat{c}_m$ of $\|U_m\|_{V_0}$ such that

$$\begin{aligned}
\forall x \in \mathbb{X}, \quad \mathcal{S}_{m-2}(x) \;&:=\; \sum_{n=0}^{+\infty} (n+1)^{m-2} \|P^n(x, \cdot) - \pi\|'_{V_m} && \text{(4a)}\\
&\leq\; U_m(x) && \text{(4b)}\\
&\leq\; \widehat{c}_m\, V_0(x), && \text{(4c)}\\
\text{and} \quad \forall x \in \mathbb{X},\ \forall k \geq 0, \quad \big\|P^k(x, \cdot) - \pi\big\|_{TV} \;&\leq\; \frac{2^m}{k^{m-1}}\, U_m(x). && \text{(4d)}
\end{aligned}$$

In addition to Assumptions $(\mathbf{S})$-$\boldsymbol{D}(V_0 : V_m)$, the condition $\pi(1_S) > 1/2$ is required. This specific condition is discussed in Subsection 4.2. Estimates (4b)-(4c) are precisely stated in Section 2 (see Theorem 2.2) and proved in Section 5. Estimate (4d) is deduced from (4b) in Corollary 2.3. The key idea to prove (4b)-(4c) is that, for any $i = 1, \ldots, m$, the norm $\| \sum_{n=0}^{+\infty}(n+1)^{i-1} R^n V_i\|_{V_0}$ where R is the residual kernel in (2), can be simply bounded using Assumptions $\boldsymbol{D}(V_0 : V_m)$ (see Proposition 2.1). Then, the link between P^n and R^n (see (6)) and the Nummelin-type representation of π (see (5)) enable us to obtain (4b) with U_m expressed in terms of the functions $\Phi_i := \sum_{n=0}^{+\infty}(n+1)^i\, |P^n\phi_S|$ for $i = 0, \ldots, m-2$ with $\phi_S := 1_S - \pi(1_S)1_{\mathbb{X}}$. Moreover the norms $\|\Phi_i\|_{V_0}$ for $i = 0, \ldots, m-2$, can be explicitly bounded via recursive inequalities involving the data S, ν, m, V_0 and b_i of Conditions (**S**) and $\boldsymbol{D}(V_0 : V_m)$. This is illustrated for $m := 2$ and $m := 3$ in Subsection 3.1.

For a general overview on convergence rates of $P^n(x, \cdot)$ to π using drift conditions, we refer to the books [MT09, DMPS18, and the references therein]. Recall that the nested modulated drift conditions $\boldsymbol{D}(V_0 : V_m)$, first used in [TT94], were proved to hold in [JR02] under the single drift condition $PV \leq V - cV^\alpha + b1_S$ with some Lyapunov function V and some constants $\alpha \in [0, 1), b, c > 0$ (also see [FM00]), and in [FM03b, Prop. 4] under the more general single sub-geometric drift condition $PV \leq V - \phi \circ V + b1_S$ with suitable function ϕ. Also see [Del17] for an operator-type approach in sub-geometric case. Here, our basic assumption is directly $\boldsymbol{D}(V_0 : V_m)$, which must be implemented in practice anyway, regardless of the form of the starting single drift condition, see [FM03b, Rem. 3].

To the best of our knowledge there are very few works providing computable rates of convergence for series as defined in (4a). Using a coupling construction in the context of subgeometric Markov chains, such an issue is addressed in [AFV15, Th. 1] for series of the form $\sum_{n=0}^{+\infty} r(n)|(P^n g)(x) - (P^n g)(x')|$ where $(r(n))_{n\geq 0}$ is some sequence of positive real numbers related to a subgeometric drift condition. Then, under Jarner-Roberts's drift condition $PV \leq V - cV^\alpha + b1_S$, the case of series of the form $\sum_{n=0}^{+\infty}(n+1)^{\xi_\alpha}|(P^n g)(x) - \pi(g)|$ for some $\xi_\alpha > 0$ is covered by [AFV15, Cor. 1]. The results of [AFV15] are compared in more detail with ours in Subsection 4.3. For Markov kernels satisfying Conditions (**S**) and $\boldsymbol{D}(V_0 : V_m)$, Theorem 2.2 in Section 2 seems to be the first result providing a computable convergence rate for the series $\mathcal{S}_{m-2}(x)$ in (4a), even for $\mathcal{S}_0(x) = \sum_{n=0}^{+\infty} \|P^n(x, \cdot) - \pi\|'_{V_2}$. In fact, if P satisfies Conditions (**S**) and $\boldsymbol{D}(V_0 : V_1)$ (i.e. $m = 1$) and if P is $\psi-$irreducible, aperiodic and $\pi(V_0) < \infty$, we know from [MT09, Th. 14.0.1] that there exists a constant c such that

$$\forall x \in \mathbb{X}, \quad \sum_{n=0}^{+\infty} \big\|P^n(x, \cdot) - \pi\big\|'_{V_1} \leq c\, V_0(x).$$

But the constant c was previously unknown. Here, under Conditions $\boldsymbol{D}(V_0 : V_2)$ (i.e. $m = 2$) and without assuming $\pi(V_0) < \infty$, this inequality is derived from (4c), and the positive constant $\widehat{c}_2$ in (4c) is easily computed from the data S, ν, V_0, b_0 and b_1 of Conditions (**S**) and $\boldsymbol{D}(V_0 : V_2)$ (see Corollary 3.1). As detailed in Subsection 4.1 our assumptions in case $m := 2$ are in fact close to those in [MT09, Th. 14.0.1] due to the condition $\pi(V_0) < \infty$. Actually, for any $m \geq 2$ the constant $\widehat{c}_m$ in (4c) can be computed from the data S, ν, m, V_0, and b_i of Conditions (**S**) and $\boldsymbol{D}(V_0 : V_m)$. The use of both the residual kernel R in (2) and the Nummelin-type representation (5) of π is proved to be relevant for such a study, as already pointed out in [HL24a] for dealing with Poisson's equation under Assumptions (**S**) and $\boldsymbol{D}(V_0 : V_1)$.

Following on from the pioneering works [NT83, TT94], explicit bounds for $\|P^n(x, \cdot) - \pi\|_{TV}$ have been proposed in [FM03b, DMS07] thanks to coupling methods under the sub-geometric drift condition $PV \leq V - \phi \circ V + b1_S$ (recall that this encompasses Jarner-Roberts's drift condition). Also see [DFMS04] for various statements and examples on different rates of convergence, and [But14, DFM16] for rates of convergence in Wasserstein distance. Note that the polynomial asymptotics (3) ensures that $\|P^n(x,\cdot) - \pi\|_{TV} \le c(x)/n^{m-1}$ for every $x \in \mathbb{X}$, but with unknown constant $c(x)$ to our knowledge. In particular, although the sub-geometric drift condition induces nested modulated drift inequalities, the explicit bounds of $\|P^k(x,\cdot) - \pi\|_{TV}$ in [FM03b, Th. 2] and [DMS07, Th. 2.1] do not seem to provide any information on the quantitative polynomial rate of convergence in (3). Here Issue (4d) is directly linked to the polynomial asymptotics (3), and Estimate (20) in Corollary 2.3 of Section 2 seems to be the first one providing $c(x) = cV_0(x)$ with a computable constant c under Conditions $\boldsymbol{D}(V_0 : V_m)$.

Therefore, we propose a self-contained method for obtaining quantitative results on the asymptotic result [JR02, Th. 3.2]. Note that the coupling technique is not used. Although the bounds obtained in Theorem 2.2 and Corollary 2.3 have a much simpler formulation than in [AFV15] and [FM03b, DMS07], we do not claim that they are numerically better. Recall that the condition $\pi(1_S) > 1/2$ for the first-order small-set S is required here. This condition is discussed in Subsection 4.2 in link with several other works involving in fact this condition for studying the rates of convergence of iterates of Markov kernels.

2 The statements

Let us recall that if P satisfies Condition (**S**), then a necessary and sufficient condition for P to admit an invariant probability measure π on $(\mathbb{X}, \mathcal{X})$ such that $\pi(1_S) > 0$, is that $\sum_{k=0}^{+\infty} \nu(R^k 1_{\mathbb{X}}) < \infty$. Actually, under any of these two equivalent conditions,

$$\pi := \mu(1_{\mathbb{X}})^{-1}\, \mu \quad \text{with} \quad \mu := \sum_{n=0}^{+\infty} \nu R^n \in \mathcal{M}_*^+ \tag{5}$$

is an $P-$invariant probability measure, and we have $\mu(1_S) = 1$ and $\pi(1_S) = \mu(1_{\mathbb{X}})^{-1} > 0$. The Nummelin-type representation (5) of π is well-known under various assumptions on P, e.g. see [Num84, Th. 5.2, Cor. 5.2]), [MT09, Chap. 10]), and see [HL23] for a simple proof under the sole Condition (**S**). The assumptions in all the next statements ensure that the condition $\sum_{k=0}^{+\infty} \nu(R^k 1_{\mathbb{X}}) < \infty$ holds. Thus, throughout the paper, π is the $P-$invariant probability measure such that $\pi(1_S) > 0$ given in (5). Also recall that the key formula linking the kernels P^n, R^n and the finite non-negative measures νR^{k-1} is from [HL20, Prop. 2.1] (see also [Num84, Eq. (4.12)])

$$\forall n \ge 1, \quad P^n = R^n + \sum_{k=1}^{n} P^{n-k} 1_S \otimes \nu R^{k-1} \tag{6}$$

where, for any non-negative measurable function f and any $\eta \in \mathcal{M}^+$, we denote by $f \otimes \eta$ the following non-negative kernel: $\forall x \in \mathbb{X},\ \forall A \in \mathcal{X},\ (f \otimes \eta)(x, A) := f(x)\, \eta(1_A)$.

To prove that the series in (4a) converges, we first study the following functions defined on $\mathbb{X}$:

$$\forall i \in \{1, \ldots, m\}, \quad \sum_{n=0}^{+\infty} (n+1)^{i-1}\, R^n V_i.$$

To that effect, under Conditions $(\mathbf{S})$ and $\boldsymbol{D}(V_0 : V_m)$ we set

$$\forall i \in \{0,\ldots,m-1\}, \quad d_i := \max\left(0, \frac{b_i - \nu(V_i))}{\nu(1_{\mathbb{X}})}\right) \tag{7}$$

with constants b_i given in $\boldsymbol{D}(V_0 : V_m)$. Obviously, we have $d_i = 0$ when $b_i \le \nu(V_i)$. In particular, if S is an atom for P (i.e. $\forall x \in S,\ P(x,\cdot) = \nu$), then $d_i = 0$ for $0 \le i \le m-1$. Moreover define $(D_\ell)_{\ell=0}^{m-1}$ as follows:

$$D_0 := 1 + d_0 \quad \text{and} \quad \forall \ell \in \{1,\ldots,m-1\},\ D_\ell := (1+d_\ell)\sum_{j=0}^{\ell-1}\binom{\ell}{j} D_j \tag{8}$$

where $\binom{\ell}{j}$ is the standard binomial coefficient. The following Proposition 2.1 is proved in Subsection 5.1.

Proposition 2.1 *Assume that P satisfies Condition $(\mathbf{S})$ and $\boldsymbol{D}(V_0 : V_m)$ for some collection $\{V_i\}_{i=0}^m$ of Lyapunov functions with $m \ge 1$. Then we have for every $i \in \{1,\ldots,m\}$*

$$\sum_{n=0}^{+\infty} (n+1)^{i-1}\, R^n V_i \le D_{i-1}\, V_0 \tag{9a}$$

$$\sum_{n=0}^{+\infty} (n+1)^{i-1}\, \nu\big(R^n V_i\big) \le D_{i-1}\, \nu(V_0) < \infty. \tag{9b}$$

Note that, if P satisfies Conditions $(\mathbf{S})$ and $\boldsymbol{D}(V_0 : V_m)$, then we deduce from (9b) with $i = 1$ that

$$\pi(V_1) < \infty, \text{ so that } \pi(V_i) < \infty \text{ for } i = 1,\ldots,m. \tag{10}$$

Now, to obtain the positive function $U_m \in \mathcal{B}_{V_0}$ in Inequality (4b) under Conditions $(\mathbf{S})$ and $\boldsymbol{D}(V_0 : V_m)$ with $m \ge 2$, we need to study the following functions $\Phi_i : \mathbb{X} \to [0,+\infty]$ for $i \in \{0,\ldots,m-2\}$:

$$\Phi_i := \sum_{n=0}^{+\infty} (n+1)^i \left|P^n \phi_S\right| \quad \text{where} \quad \phi_S := 1_S - \pi(1_S) 1_{\mathbb{X}}. \tag{11}$$

Recall that, for every $m \ge 2$, there exists $\{a_{j,m}\}_{j=1}^{m-1} \in \mathbb{R}^{m-1}$ such that

$$\forall k \ge 1, \quad \Sigma_k^{m-2} := \sum_{n=1}^{k} n^{m-2} = \sum_{j=1}^{m-1} a_{j,m}\, k^j, \tag{12}$$

and that the real numbers $\{a_{j,m}\}_{j=1}^{m-1}$ can be computed by induction on m using binomial expansion (e.g. see Subsection 3.1 in cases $m := 2, 3$). Next, using D_j's in (8), define the following positive constants

$$\forall \ell \in \{1,\ldots,m-1\}, \quad E_\ell := \sum_{j=1}^{\ell} a_{j,\ell+1}\, D_j. \tag{13}$$

The next theorem is proved in Subsection 5.2.

Theorem 2.2 *Assume that P satisfies Conditions* $(\mathbf{S})$ *and* $\boldsymbol{D}(V_0 : V_m)$ *for some collection* $\{V_i\}_{i=0}^m$ *of Lyapunov functions with $m \geq 2$. Then the following inequalities hold in* $[0, +\infty]$*:*

$$\begin{aligned}
\forall g \in \mathcal{B}_{V_m},\ \forall x \in \mathbb{X}, \quad \mathcal{S}_{m-2}(g,x) &:= \sum_{n=0}^{+\infty} (n+1)^{m-2} \big|(P^n g)(x) - \pi(g)\big| \\
&\leq \|g - \pi(g)1_{\mathbb{X}}\|_{V_m} W_m(x) \qquad (14) \\
\text{and} \quad \forall x \in \mathbb{X}, \quad \mathcal{S}_{m-2}(x) &:= \sum_{n=0}^{+\infty} (n+1)^{m-2} \big\|P^n(x,\cdot) - \pi\big\|'_{V_m} \\
&\leq \theta_m W_m(x) \qquad (15)
\end{aligned}$$

where $\theta_m := 1 + \pi(V_m)\|1_{\mathbb{X}}\|_{V_m}$ and the function W_m is

$$W_m = D_{m-2} V_0 + \nu(V_0) \Big[\sum_{j=0}^{m-2} \binom{m-2}{j} D_j \, \Phi_{m-2-j} + \pi(1_S) \, E_{m-1} \, 1_{\mathbb{X}} \Big]. \qquad (16)$$

If $\pi(1_S) > 1/2$, then for every $i \in \{0, \ldots, m-2\}$ we have $\Phi_i \in \mathcal{B}_{V_0}$ and (with the convention $\sum_{j=1}^0 = 0$)

$$\Phi_i \leq \frac{1}{2\pi(1_S) - 1} \Big(D_i V_0 + \nu(V_0) \sum_{j=1}^{i} \binom{i}{j} D_j \Phi_{i-j} + \pi(1_S)\nu(V_0) E_{i+1} 1_{\mathbb{X}} \Big). \qquad (17)$$

Thus, if P satisfies all the assumptions of Theorem 2.2, then Estimates (4b)-(4c) in Section 1 are valid with $U_m(x) = \theta_m W_m(x)$. Indeed, Inequality (4b) is nothing else than (15). To derive Inequality (4c), first use (16) to get

$$\|W_m\|_{V_0} \leq D_{m-2} + \nu(V_0) \sum_{j=0}^{m-2} \binom{m-2}{j} D_j \, \|\Phi_{m-2-j}\|_{V_0} + \pi(1_S) \, \nu(V_0) E_{m-1} \|1_{\mathbb{X}}\|_{V_0}.$$

Next, if $\pi(1_S) > 1/2$ then the norms $(\|\Phi_i\|_{V_0})_{i=0}^{m-2}$ are recursively bounded from (17) by

$$\|\Phi_i\|_{V_0} \leq \frac{1}{2\pi(1_S) - 1} \Big(D_i + \nu(V_0) \big[\sum_{j=1}^{i} \binom{i}{j} D_j \, \|\Phi_{i-j}\|_{V_0} + \pi(1_S) \, E_{i+1} \|1_{\mathbb{X}}\|_{V_0} \big] \Big) \qquad (18)$$

from which the constant $\widehat{c}_m$ in (4c) is deduced. In the atomic case, recall that the d_i's (see (7)) are zero, so that the constants D_i defined in (8) and used in the previous estimates simply depend on the integer m.

Finally note that $\|1_{\mathbb{X}}\|_{V_m} \leq 1$ since $V_m \geq 1$ and that $\pi(V_m) \leq b_{m-1}$ from $PV_{m-1} \leq V_{m-1} - V_m + b_{m-1} \, 1_S$ and the $P-$invariance of π (recall that $\pi(V_{m-1}) < \infty$ from (10)). Thus the positive constant θ_m of Theorem 2.2 satisfies

$$\theta_m \leq 1 + b_{m-1}. \qquad (19)$$

Corollary 2.3 *Under all the assumptions of Theorem 2.2, we have*

$$\forall x \in \mathbb{X},\ \forall k \geq 0, \quad \big\|P^k(x,\cdot) - \pi\big\|_{TV} \leq \frac{2^m}{k^{m-1}} W_m(x) \qquad (20)$$

with W_m given in Theorem 2.2.

Proof. Note that V_m in $\boldsymbol{D}(V_0 : V_m)$ can be replaced with the function $1_{\mathbb{X}}$ since $V_m \geq 1_{\mathbb{X}}$, and that we have $\theta_m := 1 + \pi(1_{\mathbb{X}})\|1_{\mathbb{X}}\|_{1_{\mathbb{X}}} = 2$ in this case. Let $x \in \mathbb{X}$. Recall that the sequence $(\|P^n(x,\cdot) - \pi\|_{TV})_n$ is non-increasing. Let $j \geq 0$. Then we deduce from (15) that

$$(j+1)^{m-1} \left\|P^{2j}(x,\cdot) - \pi\right\|_{TV} \leq \sum_{n=j}^{2j} (n+1)^{m-2} \left\|P^n(x,\cdot) - \pi\right\|_{TV} \leq 2\, W_m(x)$$

thus

$$\left\|P^{2j}(x,\cdot) - \pi\right\|_{TV} \leq \frac{2^m}{(2j)^{m-1}}\, W_m(x).$$

Next, using $\sum_{n=j+1}^{2j+1}$, we obtain the same inequality for $\left\|P^{2j+1}(x,\cdot) - \pi\right\|_{TV}$ replacing $(2j)^{m-1}$ with $(2j+1)^{m-1}$. This proves (20). □

The material in Theorem 2.2 and Corollary 2.3 is fully detailed for $m := 2, 3$ in Subsection 3.1. In particular, the explicit constants are provided. Theorem 2.2 applies whenever P satisfies the minorization condition (**S**) and explicit modulated drift conditions are known: for such examples, e.g. see [FM00, FM03b, DFM16] in the context of Metropolis algorithm, [LH07, LH12] for queueing systems, [JT02] for Markov chains associated with the mean of Dirichlet processes.

3 Specific cases

Throughout this section, P is assumed to satisfy the minorization condition (**S**) with $\nu \in \mathcal{M}_*^+$ and $S \in \mathcal{X}$. Below the cases where P satisfies $\boldsymbol{D}(V_0 : V_m)$ with $m := 2$ and $m := 3$ are detailed. The case $m =: 2$ is compared with the classical statement [MT09, Th. 14.0.1] in Subsection 4.1. Finally, an application of the case $m := 3$ to geometric ergodicity is presented.

3.1 Cases $\boldsymbol{D}(V_0 : V_2)$ and $\boldsymbol{D}(V_0 : V_3)$

Recall that P satisfies Condition $\boldsymbol{D}(V_0 : V_2)$ (i.e. $m := 2$) if

$$\forall i \in \{0,1\}, \quad V_{i+1} \leq V_i \quad \text{and} \quad PV_i \leq V_i - V_{i+1} + b_i\, 1_S \tag{$\boldsymbol{D}(V_0 : V_2)$}$$

for some positive constants b_0, b_1 and Lyapunov functions V_0, V_1, V_2. Set (see (7))

$$\forall i \in \{0,1\}, \quad d_i := \max\left(0, \frac{b_i - \nu(V_i)}{\nu(1_{\mathbb{X}})}\right).$$

The main estimates of Theorem 2.2 and Corollary 2.3 are summarized as follows.

Corollary 3.1 *Let P satisfy Condition* (**S**) *with $\pi(1_S) > 1/2$ and $\boldsymbol{D}(V_0 : V_2)$. Then*

$$\forall g \in \mathcal{B}_{V_2},\ \forall x \in \mathbb{X},\ \mathcal{S}_0(g,x) = \sum_{n=0}^{+\infty} \left|(P^n g)(x) - \pi(g)\right| \leq \|g - \pi(g)1_{\mathbb{X}}\|_{V_2}\, \widehat{c}_2\, V_0(x), \tag{21a}$$

$$\forall x \in \mathbb{X},\ \mathcal{S}_0(x) := \sum_{n=0}^{+\infty} \left\|P^n(x,\cdot) - \pi\right\|'_{V_2} \leq (1+b_1)\, \widehat{c}_2\, V_0(x), \tag{21b}$$

$$\forall x \in \mathbb{X},\ \forall k \geq 0,\ \left\|P^k(x,\cdot) - \pi\right\|_{TV} \leq \frac{4}{k}\, \widehat{c}_2\, V_0(x), \tag{21c}$$

where $\widehat{c}_2 := c_0 + c_1\|1_{\mathbb{X}}\|_{V_0}$ *with*

$$c_0 := (1+d_0)\left(1+\frac{\nu(V_0)\,(1+d_0)}{2\pi(1_S)-1}\right) \quad c_1 := \nu(V_0)(1+d_0)(1+d_1)\left(\frac{\nu(V_0)\,(1+d_0)}{2\pi(1_S)-1}+1\right).$$

Proof. Note that $\Sigma_k^0 = k$, i.e. $a_{1,2} = 1$ in (12). Moreover we have $D_0 := 1 + d_0, D_1 := (1 + d_0)(1 + d_1)$ from (8) and $E_1 = D_1$ from (13). Then, the function W_2 is from (16) with $m := 2$

$$W_2 = (1+d_0)\,V_0 + \nu(V_0)\big[\,(1+d_0)\,\Phi_0 + \pi(1_S)\,(1+d_0)(1+d_1)\,1_{\mathbb{X}}\big]$$

and we have the following estimate from (17) with $i := 0$:

$$\Phi_0 \le \frac{(1+d_0)V_0 + \pi(1_S)\nu(V_0)(1+d_0)(1+d_1)1_{\mathbb{X}}}{2\pi(1_S)-1}.$$

It follows that $W_2 \le c_0V_0 + \pi(1_S)c_1 1_{\mathbb{X}} \le c_0V_0 + c_1 1_{\mathbb{X}} \le \widehat{c}_2 V_0$ with the constants $c_0, c_1, \widehat{c}_2$ defined in Corollary 3.1. Apply (14), (15), (19) and (20) with $m := 2$ to get (21a), (21b), (21c). □

Let P satisfy Condition $\boldsymbol{D}(V_0 : V_3)$ (i.e. $m := 3$) that is

$$\forall i \in \{0,1,2\}, \quad V_{i+1} \le V_i \quad \text{and} \quad PV_i \le V_i - V_{i+1} + b_i\,1_S \qquad (\boldsymbol{D}(V_0 : V_3))$$

for some positive constants b_0, b_1, b_2 and Lyapunov functions V_0, V_1, V_2, V_3. Set

$$\forall i \in \{0,1,2\}, \quad d_i := \max\left(0, \frac{b_i - \nu(V_i)}{\nu(1_{\mathbb{X}})}\right).$$

The main estimates of Theorem 2.2 and Corollary 2.3 in case $m := 3$ are summarized in the next corollary.

Corollary 3.2 *Let P satisfy Condition* (**S**) *with $\pi(1_S) > 1/2$ and Conditions $\boldsymbol{D}(V_0 : V_3)$. Then*

$$\forall g \in \mathcal{B}_{V_3},\, \forall x \in \mathbb{X}, \mathcal{S}_1(g,x) = \sum_{n=0}^{+\infty}(n+1)\big|(P^n g)(x) - \pi(g)\big| \le \|g - \pi(g)1_{\mathbb{X}}\|_{V_3}\,\widehat{c}_3\,V_0(x), \quad (22a)$$

$$\forall x \in \mathbb{X}, \mathcal{S}_1(x) := \sum_{n=0}^{+\infty}(n+1)\big\|P^n(x,\cdot) - \pi\big\|'_{V_3} \le (1+b_2)\,\widehat{c}_3\,V_0(x), \quad (22b)$$

$$\forall x \in \mathbb{X},\ \forall k \ge 0, \big\|P^k(x,\cdot) - \pi\big\|_{TV} \le \frac{8}{k^2}\,\widehat{c}_3\,V_0(x), \quad (22c)$$

where $\widehat{c}_3 := c_0 + c_1\|1_{\mathbb{X}}\|_{V_0}$ *with*

$$c_0 := D_1\left[1 + \frac{\nu(V_0)D_0}{2\pi(1_S)-1}\right]^2 c_1 := \nu(V_0)\left[E_2 + \frac{\nu(V_0){D_1}^2 + D_0E_2\nu(V_0)}{2\pi(1_S)-1} + \frac{\nu(V_0)^2D_0{D_1}^2}{(2\pi(1_S)-1)^2}\right],$$

$$\forall i \in \{0,1\},\ D_i := \prod_{j=0}^{i}(1+d_i),\ D_2 := (1+d_2)(D_0+2D_1),\ E_1 = D_1,\ E_2 := \frac{D_1+D_2}{2}. \quad (23)$$

Proof. Here we have $\Sigma_k^1 = k(k+1)/2$, i.e. $a_{1,3} = a_{2,3} = 1/2$ from (12). Thus we get (23) from (8) and (13). The claimed statements then follow as in Corollary 3.1, using (16) with $m := 3$

$$W_3 := D_1\,V_0 + \nu(V_0)\big[\,D_0\,\Phi_1 + D_1\,\Phi_0 + \pi(1_S)\,E_2\,1_{\mathbb{X}}\big]$$

and (17) with $i := 0, 1$

$$\Phi_0 \leq \frac{D_0 V_0 + \pi(1_S)\nu(V_0) E_1 1_{\mathbb{X}}}{2\pi(1_S) - 1}, \quad \Phi_1 \leq \frac{D_1 V_0 + \nu(V_0)\, D_1 \Phi_0 + \pi(1_S)\nu(V_0) E_2 1_{\mathbb{X}}}{2\pi(1_S) - 1}.$$

□

3.2 Conditions $\boldsymbol{D}(V_0 : V_m)$ under geometric drift condition

Assume that P satisfies Condition (**S**) for some $S \in \mathcal{X}$ and the following $V-$geometric drift condition

$$\exists \delta \in (0,1),\ \exists b \in (0, +\infty): \quad PV \leq \delta V + b\, 1_S \tag{24}$$

where $V : \mathbb{X} \to [1, +\infty)$ is a measurable function. Then, for every $m \geq 1$, P satisfies Condition $\boldsymbol{D}(V_0 : V_m)$ with

$$V_m := V \text{ and } \ \forall i \in \{0, \ldots, m-1\}, \quad V_i = \frac{V}{(1-\delta)^{m-i}} \ \text{ and } \ b_i := \frac{b}{(1-\delta)^{m-i}}. \tag{25}$$

Corollary 3.3 *Assume that P satisfies Conditions* (**S**)*–(24) with $\pi(1_S) > 1/2$. Then P is $V-$geometrically ergodic and for every $\tau \in (0,1)$, we have*

$$\forall g \in \mathcal{B}_V,\ \forall n \geq 0, \quad \|P^n g - \pi(g) 1_{\mathbb{X}}\|_V \leq \frac{\widehat{c}_3 (1 + \pi(V) \|1_{\mathbb{X}}\|_V)}{\tau\, (1-\delta)^3}\, \rho^n\, \|g\|_V \quad \text{with} \quad \rho := \tau^{1/n_0}$$

where $\widehat{c}_3$ is provided in Corollary 3.2 using $\boldsymbol{D}(V_0 : V_m)$ and V_i given in (25) with $m := 3$, and where n_0 is the smallest positive integer number such that $\widehat{c}_3(1 + \pi(V)\|1_{\mathbb{X}}\|_V)/(n_0+1) \leq \tau(1-\delta)^3$.

Proof. Using here Conditions $\boldsymbol{D}(V_0 : V_3)$ and V_0, V_1, V_2, V_3 given (25), it follows from (22a) that

$$\forall n \geq 1,\ \forall g \in \mathcal{B}_V,\ \forall x \in \mathbb{X}, \ \frac{|(P^n g)(x) - \pi(g)|}{V(x)} \leq \frac{d_3}{n+1} \|g\|_V \ \text{ with } \ d_3 := \frac{\widehat{c}_3 (1 + \pi(V)\|1_{\mathbb{X}}\|_V)}{(1-\delta)^3}$$

using $\|g - \pi(g) 1_{\mathbb{X}}\|_V \leq (1 + \pi(V)\|1_{\mathbb{X}}\|_V)\|g\|_V$. Let us still denote by $\|L\|_V$ the operator-norm of any bounded linear operator L on $(\mathcal{B}_V, \|\cdot\|_V)$, i.e. $\|L\|_V := \sup\{\|Lg\|_V : g \in \mathcal{B}_V, \|g\|_V \leq 1\}$. Then, we obtain from the above inequality that $\|P^n - \Pi\|_V \leq d_3/(n+1)$ with $\Pi := \pi(\cdot) 1_{\mathbb{X}}$. Let $\tau \in (0,1)$ and $n_0 \equiv n_0(\tau)$ be the smallest positive integer such that $d_3/(n_0+1) \leq \tau$. Then, writing $n = q n_0 + r$ with $r \in \{0, \ldots, n_0 - 1\}$, we deduce that

$$\forall n \geq 1, \quad \|P^n - \Pi\|_V \leq \|(P - \Pi)^r\|_V \times \big(\|(P-\Pi)^{n_0}\|_V\big)^q \leq \frac{d_3}{\tau}\, \rho^n \quad \text{with} \quad \rho := \tau^{1/n_0}$$

since $\|(P-\Pi)^r\|_V \leq d_3$ and $\tau^{-r/n_0} \leq \tau^{-1}$. □

4 Bibliographic comments

4.1 In Case $\boldsymbol{D}(V_0 : V_2)$, comparison with [MT09, Th. 14.0.1]

If P satisfies the assumptions of Corollary 3.1 (requiring $\boldsymbol{D}(V_0 : V_2)$) then we have from (21b)

$$\forall x \in \mathbb{X}, \quad \sum_{n=0}^{+\infty} \big\|P^n(x, \cdot) - \pi\big\|'_{V_2} \leq (1 + b_1)\, \widehat{c}_2\, V_0(x).$$

This statement may be surprising on first reading compared with the classical result [MT09, Th. 14.0.1]. Indeed, we know from [MT09, Th. 14.0.1] that, if P satisfies Condition (**S**) with some $S \in \mathcal{X}$ and the single modulated drift condition $PV \leq V - W + b\,1_S$ for some Lyapunov functions V and W such that $\pi(V) < \infty$, then there exist a $P-$absorbing set $A \in \mathcal{X}$ (i.e. $A \in \mathcal{X}$ is such that $P(x, A) = 1$ for every $x \in A$) and a constant $c > 0$ such that

$$\forall x \in A, \quad \sum_{n=0}^{+\infty} \left\|P^n(x, \cdot) - \pi\right\|_W' \leq c\, V(x) \tag{26}$$

provided that P is irreducible and aperiodic. Actually it can be proved that the additional assumption $\pi(V) < \infty$ in [MT09, Th. 14.0.1] generates a $V-$modulated drift condition on some $P-$absorbing set $A \in \mathcal{X}$ such that $\pi(1_A) = 1$, i.e. the drift inequality $PL \leq L - V + b' 1_S$ holds on A for some Lyapunov function $L \geq V$ (see [HL24b, Sec. 5]). Hence the assumptions of [MT09, Th. 14.0.1] involve in fact two nested modulated drift conditions.

To illustrate the previous discussion, consider the following so-called random walk $(X_n)_{n \geq 0}$ on the half line $\mathbb{X} = [0, +\infty)$

$$X_0 \in \mathbb{X} \quad \text{and} \quad \forall n \geq 1,\ X_n := \max\big(0, X_{n-1} + \vartheta_n\big) \tag{27}$$

where $\{\vartheta_n\}_{n \geq 1}$ is a sequence of i.i.d. $\mathbb{R}$-valued random variables assumed to be independent of X_0 and to satisfy $\mathbb{E}[\vartheta_1] < 0$ and $\mathbb{E}[\max(0, \vartheta_1)] < \infty$. Then it is well-known that the drift condition $PV \leq V - W + b\,1_S$ holds with $S = [0, s]$ for some $s > 0$ and with Lyapunov functions V, W defined on $\mathbb{X} = [0, +\infty)$ by $V(x) = 1 + x$ and $W = c_1 1_{\mathbb{X}}$ for some constant $c_1 > 0$. Moreover, it follows from [JT03, Prop. 3.5] that the condition $\int_{\mathbb{X}} x\, d\pi(x) < \infty$, i.e. $\pi(V) < \infty$, is equivalent to $\mathbb{E}[(\max(0, \vartheta_1))^2] < \infty$, so that the last moment condition is required to apply the statement [MT09, Th. 14.0.1]. However note that the condition $\mathbb{E}[(\max(0, \vartheta_1))^2] < \infty$ is precisely what ensures that Assumptions $\boldsymbol{D}(V_0 : V_2)$ hold with $V_0(x) = (1 + x)^2$ and $V_i(x) = c_i(1 + x)^{2-i}$ for $i = 1, 2$ with some $c_i > 0$ (e.g. see [JR02]). Accordingly the moment condition on ϑ_1 is indeed the same for applying (21b) or [MT09, Th. 14.0.1].

4.2 On the condition $\pi(1_S) > 1/2$

Let P satisfy Conditions (**S**) and $\boldsymbol{D}(V_0 : V_m)$. If $\pi(1_S) \leq 1/2$, then the explicit bound (17) for the Φ_i's in Theorem 2.2 cannot be applied. Accordingly, if $\pi(1_S) \leq 1/2$, then Corollary 2.3 does not apply. Observe that P obviously satisfies $\boldsymbol{D}(V_0 : V_m)$ for any set $S' \in \mathcal{X}$ containing S since $1_S \leq 1_{S'}$, and that $\pi(1_{S'}) > 1/2$ for S' large enough. However the same set S' must be used in the minorization condition, and unfortunately the existence of a minorizing positive measure ν' w.r.t. the set S' is not guaranteed when S' is too large. In other words, the naïve idea of enlarging the set S to obtain $\pi(1_S) > 1/2$ doesn't work in general. Surprisingly, whatever the method used, the condition $\pi(1_S) > 1/2$ required in Theorem 2.2 and Corollary 2.3 is often involved when dealing with explicit rates of convergence of the iterates of Markov kernels. For instance, using some refinements on the modulated drift condition, the authors in [FM03a, Prop. 13] present an explicit bound for [MT09, Th. 14.0.1], i.e. an explicit constant c in (26) (consider $\lambda = \delta_x$ and $= \pi$ in [FM03a, Prop. 13]). The assumption $\pi(1_D) > 1/2$ for some small-set D is actually also imposed in [FM03a, Prop. 13]. Indeed the Lyapunov function W in the modulated drift condition $PV \leq V - W + b1_S$ considered in [FM03a, Prop. 13] satisfies $W \geq b/(1-a)$ on $D^c := \mathbb{X} \setminus D$ for some $a \in (0,1)$ and some small-set $D \in \mathcal{X}$ containing the small-set S of the previous drift condition. Thus we have $\pi(1_{D^c}) \leq \pi(W)(1-a)/b$. Moreover, since the condition $\pi(V) < \infty$ is imposed in [FM03a, Prop. 13], we obtain that $\pi(W) \leq b\pi(1_S)$. Thus we have $\pi(1_{D^c}) \leq (1-a)\pi(1_S)$, from which we deduce that

$$\pi(1_D) \geq \pi(1_S) \geq \frac{\pi(1_{D^c})}{1-a} = \frac{1-\pi(1_D)}{1-a}.$$

Thus the condition $\pi(1_D) \geq 1/(2-a) > 1/2$ is indeed required. Similarly the assumption $\pi(1_D) > 1/2$ for some small-set D occurs in the nested modulated drift conditions in [FM03b, p. 78] introduced for the study of polynomial ergodicity (see [FM03b, Eq. (50)] and apply the previous arguments). Finally, as in the previous papers, a technical condition on the geometric drift inequality, again implying that $\pi(1_S) > 1/2$, is also assumed in [Ros95, Th. 12] to get a rate of convergence, see [Jer16] and [QH21, Prop. 17]. Accordingly the discussion in [QH21, QH22] concerning the trade-off that must be made in [Ros95, Th. 12] between, on the one hand, the condition $\pi(1_S) > 1/2$ requiring a sufficiently large small-set S and, on the other hand, the total mass $\nu(1_{\mathbb{X}})$ requiring S not to be too large, applies to the framework of Theorem 2.2 and Corollary 2.3.

Hence, in the papers cited above and in the present work, the condition $\pi(1_S) > 1/2$ is a strong assumption from a practical point of view. Mention that a way to overcome this condition in our work could be to introduce a small-set of higher order. This work is in progress.

4.3 Comparison with [AFV15] under Jarner-Roberts's drift condition

Throughout this subsection the Markov kernel P is assumed to satisfy the minorization condition (**S**). Recall that Jarner-Roberts's drift condition introduced in [JR02] is: There exists a Lyapunov function V such that

$$\exists \alpha \in [0,1),\ \exists b, c > 0, \quad PV \leq V - c\,V^{\alpha} + b\,1_S. \tag{28}$$

This is the most classical case leading to Markov kernels satisfying Conditions $\boldsymbol{D}(V_0 : V_m)$, also see [MT09, DMPS18, and the references therein]. Indeed P satisfies $\boldsymbol{D}(V_0 : V_m)$ with $m \equiv m(\alpha) := \lfloor (1-\alpha)^{-1} \rfloor \geq 1$, where $\lfloor \cdot \rfloor$ denotes the integer part function on $\mathbb{R}$, and with the Lyapunov functions

$$V_m := 1_{\mathbb{X}} \leq V_{m-1} := a_{m-1} V^{\alpha_{m-1}} \leq \cdots \leq V_1 := a_1 V^{\alpha_1} \leq V_0 := a_0 V \tag{29}$$

where $\alpha_1 := 1 - 1/m \in [0,1)$ and $\alpha_i = (\alpha_1 - 1)\,i + 1$ for $i = 2, \ldots, m-1$ when $m \geq 2$, and where a_i's are explicit constants strictly larger than one, see [JR02, Proof of Th. 3.6]. For the reader's convenience, the construction of V_i's is detailed in Appendix A. Hence, if $m \geq 2$ and $\pi(1_S) > 1/2$, then for any measurable and bounded $g : \mathbb{X} \to \mathbb{R}$, i.e. $g \in \mathcal{B}_{1_{\mathbb{X}}}$, and for any $x \in \mathbb{X}$, Theorem 2.2 provides an explicit bound for $\sum_{n=0}^{+\infty}(n+1)^{m-2}|(P^n g)(x) - \pi(g)|$. For instance the bounds (21a)-(21b)-(21c) in case $m := 2$, or the bounds (22a)-(22b)-(22c) in case $m := 3$, apply. Under the drift condition (28) (and some additional minor assumptions), it is proved in [AFV15, Cor. 1, homogeneous case with $\xi = 1$] that there exists a constant $C > 0$ such that for any $(x, x') \in \mathbb{X}^2$ and any $g \in \mathcal{B}_{1_{\mathbb{X}}}$

$$\sum_{n=0}^{+\infty}(n+1)^{m-1}|(P^n g)(x) - (P^n g)(x')| \leq C\,\|g\|_{1_{\mathbb{X}}}\big(V(x) + V(x') - 1\big).$$

Thus, if $\pi(V) < \infty$, then $\mathcal{S}_{m-1}(g,x) \leq C\,\|g\|_{1_{\mathbb{X}}}(V(x)+\pi(V)-1)$. The reason why $\mathcal{S}_{m-1}(g,x)$ can be estimated in [AFV15, Cor. 1], while Theorem 2.2 only provides an estimate for $\mathcal{S}_{m-2}(g,x)$, is the same as in Subsection 4.1, that is: The condition $\pi(V) < \infty$ is not guaranteed under Assumption (28) (we only know that $\pi(V^\alpha) < \infty$). Again note that the condition $\pi(V) < \infty$ is not required for using Theorem 2.2. Actually, assuming both (28) with $\alpha = 1-1/m$ and $\pi(V) < \infty$, is close to assuming Condition (28) with $\alpha = 1-1/(m+1)$. For instance, extending the arguments of Subsection 4.1, it follows from [JT03, Prop. 3.5] that the two last assumptions are identical for random walks on the half line. Note that (28) with $\alpha = 1 - 1/(m+1)$ implies $\boldsymbol{D}(V_0 : V_{m+1})$, so that for any $g \in \mathcal{B}_{1_{\mathbb{X}}}$ the series $\sum_{n=0}^{+\infty}(n+1)^{m-1}|P^n g - \pi(g)|$ can be estimated too using Theorem 2.2, as well as the sums studied in [AFV15] since

$$\forall (x,x') \in \mathbb{X}, \quad \sum_{n=0}^{+\infty}(n+1)^{m-1}|(P^n g)(x) - (P^n g)(x')| \leq \mathcal{S}_{m-1}(g,x) + \mathcal{S}_{m-1}(g,x')$$

from the triangular inequality. Series with the norms $\|P^n(x,\cdot)-\pi\|'_{V_m}$ (see (15)) and estimate of type (20) are not studied in [AFV15]. Finally mention that the comparison between the above constant C and that derived from Theorem 2.2 is not easy to address since the constant C in [AFV15, Cor. 1] is not completely computed. However note that this constant C involves the real number $\varepsilon_\nu^{-1} = \nu(1_{\mathbb{X}})^{-1}$ and the series $c_* := \sum_{j=0}^{+\infty}(1-\nu(1_{\mathbb{X}}))^j \prod_{k=0}^{j}(1+\delta_k M_1)$ for some $(\delta_k)_k \in \mathbb{R}^{\mathbb{N}}$ and some constant M_1. The bounds in Theorem 2.2 also involve the constant $\nu(1_{\mathbb{X}})^{-1}$ through d_i's in (7), but it only requires to compute finitely many constants of the form $\prod_{k=0}^{j}(1+d_\ell)$ (see (8)).

5 Proofs

5.1 Proof of Proposition 2.1

Under Assumption (**S**), recall that the residual kernel R defined in (2) is a submarkov kernel. The following simple result is from [HL24a, Lemma 2.2] and allows us to transform a modulated drift condition for P into a simpler drift condition for R which is in force for deriving Proposition 2.1.

Lemma 5.1 *Assume that P satisfies Condition* (**S**) *and $PV \leq V - W + b1_S$ for some $b > 0$ and some couple (V,W) of Lyapunov functions. Then we have*

$$RV_d \leq V_d - W \quad \textit{with} \quad V_d := V + d1_{\mathbb{X}} \geq V, \quad \textit{where} \quad d := \max\left(0, \frac{b-\nu(V))}{\nu(1_{\mathbb{X}})}\right).$$

Let us prove Inequalities (9a), that is with D_{i-1} defined in (8)

$$\forall i \in \{1,\dots,m\}, \quad \sum_{n=0}^{+\infty}(n+1)^{i-1}\,R^n V_i \leq D_{i-1}V_0.$$

We use an induction on m. Assume that $\boldsymbol{D}(V_0 : V_1)$ holds, that is $PV_0 \leq V_0 - V_1 + b_0\,1_S$. Then it follows from Lemma 5.1 applied to $(V,W) = (V_0,V_1)$ that $RV_{0,d_0} \leq V_{0,d_0} - V_1$ with $V_{0,d_0} := V_0 + d_0 1_{\mathbb{X}} \geq V_0$ where $d_0 = \max\{0,(b_0-\nu(V_0))/\nu(1_{\mathbb{X}})\}$. Equivalently we have

$V_1 \le V_{0,d_0} - RV_{0,d_0}$. Then for every $n \ge 0$ we obtain that $R^nV_1 \le R^nV_{0,d_0} - R^{n+1}V_{0,d_0}$. Hence we have for every $N \ge 1$

$$\sum_{n=0}^{N} R^n V_1 \le \sum_{n=0}^{N} \big[R^n V_{0,d_0} - R^{n+1}V_{0,d_0}\big] \le V_{0,d_0} \le (1+d_0)V_0.$$

This proves (9a) when $m = 1$. Now suppose that Inequalities (9a) are proved for some $m \ge 1$ and that Conditions $\boldsymbol{D}(V_0 : V_{m+1})$ hold for some collection $\{V_i\}_{i=0}^{m+1}$ of Lyapunov functions. Then it follows from Lemma 5.1 for $(V, W) = (V_m, V_{m+1})$ that $RV_{m,d_m} \le V_{m,d_m} - V_{m+1}$ with $V_{m,d_m} := V_m + d_m 1_{\mathbb{X}} \ge V_m$, where $d_m := \max\{0, (b_m - \nu(V_m)/\nu(1_{\mathbb{X}})\}$. Equivalently we have $V_{m+1} \le V_{m,d_m} - RV_{m,d_m}$, so that we obtain for every $N \ge 1$

$$\begin{aligned}
\sum_{n=0}^{N}(n+1)^m\, R^n V_{m+1} &\le \sum_{n=0}^{N}(n+1)^m\, R^n V_{m,d_m} - \sum_{n=0}^{N+1} n^m\, R^n V_{m,d_m} \\
&\le \sum_{n=0}^{N}\big[(n+1)^m - n^m\big]\, R^n V_{m,d_m} = \sum_{j=0}^{m-1}\binom{m}{j}\sum_{n=0}^{N} n^j\, R^n V_{m,d_m} \\
&\le \big(1+d_m\big)\sum_{j=0}^{m-1}\binom{m}{j}\sum_{n=0}^{N} n^j\, R^n V_{j+1} \\
&\le \big(1+d_m\big)\Big(\sum_{j=0}^{m-1}\binom{m}{j} D_j\Big)V_0 \;= D_m V_0
\end{aligned}$$

using the binomial expansion and $V_{m,d_m} \le (1+d_m)V_m \le (1+d_m)V_{j+1}$ for $j = 0,\dots,m-1$, the induction hypothesis, and using finally the definition of D_m. This gives Inequalities (9a) at order $m+1$. Finally (9b) follows from (9a). Indeed we have $\nu(V_0) < \infty$ since, for some $x \in S$, we have from Assumption (**S**): $\nu(V_0) \le (PV_0)(x) \le V_0(x) - V_1(x) + b_0 < \infty$.

5.2 Proof of Theorem 2.2

Let P satisfy Conditions (**S**) and $\boldsymbol{D}(V_0 : V_m)$ for some collection $\{V_i\}_{i=0}^{m}$ of Lyapunov functions with $m \ge 2$. Recall that $\phi_S := 1_S - \pi(1_S)1_{\mathbb{X}}$. For every $i \in \{0,\dots,m-2\}$ set:

$$\forall N \ge 1,\ \forall x \in \mathbb{X}, \quad \Phi_{i,N}(x) := \sum_{n=0}^{N}(n+1)^i\, \big|\big(P^n\phi_S\big)(x)\big|. \tag{30}$$

The following lemma plays a crucial role to prove Theorem 2.2.

Lemma 5.2 *Assume that P satisfies Conditions (**S**) and $\boldsymbol{D}(V_0 : V_\ell)$ for some collection $\{V_i\}_{i=0}^{\ell}$ of Lyapunov functions with $\ell \ge 2$. Let $(g_n)_{n\ge0} \in \mathcal{B}_{V_\ell}^{\mathbb{N}}$ and $\psi \in \mathcal{B}_{V_\ell}$ be such that $|g_n| \le \psi \le V_\ell$ and $\pi(g_n) = 0$ for every $n \ge 0$. Then we have for every $N \ge 1$ (with the standard convention $\sum_{j=1}^{0} = 0$)*

$$\begin{aligned}
\sum_{n=0}^{N}(n+1)^{\ell-2}\big|P^n g_n\big| \;\le\; & D_{\ell-2}\, V_0 + \Big(\sum_{k=1}^{+\infty}\nu(R^{k-1}\psi)\Big)\,\Phi_{\ell-2,N} \\
& + \nu(V_0)\Big[\sum_{j=1}^{\ell-2}\binom{\ell-2}{j} D_j\, \Phi_{\ell-2-j,N} + \pi(1_S)\, E_{\ell-1}\, 1_{\mathbb{X}}\Big].
\end{aligned} \tag{31}$$

Proof of Theorem 2.2. Note that $\Phi_{i,N} \le \Phi_i$ for every $N \ge 1$, with Φ_i given in (11). If $g \in \mathcal{B}_{V_m}$ is such that $\|g\|_{V_m} \le 1$ and $\pi(g) = 0$, then Inequality (14) in $[0, +\infty]$ with W_m given in (16) directly follows from Inequality (31) applied to $\ell := m$, $g_n = g, \psi = V_m$, and from

$$\sum_{k=1}^{+\infty} \nu(R^{k-1}V_m) \le \sum_{k=1}^{+\infty} \nu(R^{k-1}V_1) \le D_0\, \nu(V_0) \tag{32}$$

thanks to (9b) applied with $i = 1$. If $\pi(g) \neq 0$, replace g with $g - \pi(g)1_{\mathbb{X}}$.

Next, to prove Inequality (15), recall that $\theta_m = 1 + \pi(V_m)\|1_{\mathbb{X}}\|_{V_m}$, and first note that

$$\forall h \in \mathcal{B}_{V_m}, \quad \|h - \pi(h)1_{\mathbb{X}}\|_{V_m} \le \theta_m \|h\|_{V_m}.$$

Now let $(h_n)_{n\ge 0} \in \mathcal{B}_{V_m}^{\mathbb{N}}$ be such that $\|h_n\|_{V_m} \le 1$ and set $f_n := h_n - \pi(h_n)1_{\mathbb{X}}$. For any $n \ge 0$, we have $\|f_n\|_{V_m} \le \theta_m$, so that $g_n := f_n/\theta_m$ is such that $|g_n| \le V_m$ and $\pi(g_n) = \pi(f_n) = 0$. Then, applying Inequality (31) to $\ell := m$, $\psi = V_m$, we obtain that

$$\forall x \in \mathbb{X},\ \forall N \ge 1, \quad \sum_{n=0}^{N} (n+1)^{m-2} \big|(P^n h_n)(x) - \pi(h_n)\big| \le \theta_m\, W_m(x)$$

using again (32). Taking the supremum bound over the functions $h_0, \ldots, h_N$, we obtain that

$$\forall x \in \mathbb{X},\ \forall N \ge 1, \quad \sum_{n=0}^{N} (n+1)^{m-2} \|P^n(x, \cdot) - \pi\|'_{V_m} \le \theta_m\, W_m(x)$$

from which we deduce (15).

Now assume that $\pi(1_S) > 1/2$. Then we have:

$$\sum_{k=1}^{+\infty} \nu\big(R^{k-1}|\phi_S|\big) = 2\pi(1_{S^c}) < 1. \tag{33}$$

Indeed we have $\phi_S = (1 - \pi(1_S))1_S - \pi(1_S)1_{S^c}$, so that $|\phi_S| = (1 - \pi(1_S))1_S + \pi(1_S)1_{S^c}$. Recall that $\mu(1_S) = 1$ and $\pi = \pi(1_S)\mu$ (see (5)). Thus

$$\sum_{k=1}^{+\infty} \nu\big(R^{k-1}|\phi_S|\big) = (1 - \pi(1_S))\mu(1_S) + \pi(1_S)\mu(1_{S^c}) = 1 - \pi(1_S) + \pi(1_{S^c}) = 2\pi(1_{S^c}).$$

This proves (33).

Observe that Assumptions $\boldsymbol{D}(V_0 : V_m)$ obviously imply that, for every $i = 0, \ldots, m-2$, Assumptions $\boldsymbol{D}(V_0 : V_{i+2})$ hold too. Therefore, for any $i = 0, \ldots, m-2$, it follows from Inequality (31) with $\ell = i + 2$ applied to $g_n := \phi_S, \psi := |\phi_S|$, and from (33) that

$$\big(1 - 2\pi(1_{S^c})\big)\Phi_{i,N} \le D_i\, V_0 + \nu(V_0) \left[\sum_{j=1}^{i} \binom{i}{j} D_j\, \Phi_{i-j,N} + \pi(1_S)\, E_{i+1}\, 1_{\mathbb{X}} \right].$$

Recall that $\sum_{j=1}^{0} = 0$ by convention in (31). When $N \to +\infty$, the previous inequality for $i = 0$ shows that the series Φ_0 is convergent and satisfies (17) for $i = 0$. Next this inequality for $i \in \{1, \ldots, m-2\}$ ensures that the series Φ_i is convergent from the convergence of the $(\Phi_j)_{j=0}^{i-1}$, and that Φ_i satisfies Inequality (17). The proof of Theorem 2.2 is complete, provided that Lemma 5.2 is proved. □

Proof of Lemma 5.2. Let $(g_n)_{n\geq 0} \in \mathcal{B}_{V_\ell}^{\mathbb{N}}$ and $\psi \in \mathcal{B}_{V_\ell}$ be such that $|g_n| \leq \psi \leq V_\ell$ and $\pi(g_n) = 0$ for every $n \geq 0$. Note that $\mu(g_n) := \sum_{k=1}^{+\infty} \nu(R^{k-1}g_n) = 0$ since $\pi(g_n) = 0$ (see (5)). Then we get from Formula (6) and $\sum_{k=1}^{n} \nu(R^{k-1}g_n) = -\sum_{k=n+1}^{+\infty} \nu(R^{k-1}g_n)$ with the convention $\sum_{k=1}^{0} = 0$

$$\begin{aligned}\forall n \geq 0, \quad P^n g_n &= R^n g_n + \sum_{k=1}^{n} \nu(R^{k-1}g_n)\, P^{n-k} 1_S \\ &= R^n g_n + \sum_{k=1}^{n} \nu(R^{k-1}g_n)\, P^{n-k} \phi_S \,-\, \pi(1_S)\Big(\sum_{k=n+1}^{+\infty} \nu(R^{k-1}g_n)\Big) 1_{\mathbb{X}}. \end{aligned} \tag{34}$$

First, using the positivity of R and $|g_n| \leq V_\ell \leq V_{\ell-1}$, it follows from (9a) with $i = \ell - 1$ that

$$A_N := \sum_{n=0}^{N} (n+1)^{\ell-2}\, |R^n g_n| \leq \sum_{n=0}^{+\infty} (n+1)^{\ell-2}\, R^n |g_n| \leq \sum_{n=0}^{+\infty} (n+1)^{\ell-2}\, R^n V_{\ell-1} \leq D_{\ell-2}\, V_0. \tag{35}$$

Second, using again the convention $\sum_{k=1}^{0} = 0$ and the inequality $|g_n| \leq \psi$, we have

$$\begin{aligned} B_N := \sum_{n=0}^{N} (n+1)^{\ell-2} \Big| \sum_{k=1}^{n} \nu(R^{k-1}g_n)\, P^{n-k}\phi_S \Big| &\leq \sum_{n=0}^{N} (n+1)^{\ell-2} \sum_{k=1}^{n} \nu(R^{k-1}|g_n|)\, \big|P^{n-k}\phi_S\big| \\ &= \sum_{k=1}^{N} \nu(R^{k-1}|g_n|) \sum_{n=k}^{N} (n+1)^{\ell-2} \big|P^{n-k}\phi_S\big| \\ &\leq \sum_{k=1}^{N} \nu(R^{k-1}\psi) \sum_{n=0}^{N} (n+1+k)^{\ell-2} \big|P^{n}\phi_S\big| \\ &= \sum_{j=0}^{\ell-2} \binom{\ell-2}{j} \Big(\sum_{k=1}^{N} k^j\, \nu(R^{k-1}\psi) \Big) \Phi_{\ell-2-j,N} \\ &\leq \sum_{j=0}^{\ell-2} \binom{\ell-2}{j} \Big(\sum_{k=1}^{+\infty} k^j\, \nu(R^{k-1}\psi) \Big) \Phi_{\ell-2-j,N} \end{aligned}$$

where the $\Phi_{i,N}$'s are defined in (30). Then, separating the term for $j = 0$ in the last sum and using $\psi \leq V_\ell \leq V_{j+1}$ for $j = 1, \ldots, \ell - 2$, it follows from (9b) that

$$B_N \leq \Big(\sum_{k=1}^{+\infty} \nu(R^{k-1}\psi) \Big) \Phi_{\ell-2,N} + \nu(V_0) \sum_{j=1}^{\ell-2} \binom{\ell-2}{j} D_j\, \Phi_{\ell-2-j,N}. \tag{36}$$

Third, recall that, for any $k \geq 1$, $\Sigma_k^{\ell-2} := \sum_{n=1}^{k} n^{\ell-2} = \sum_{j=1}^{\ell-1} a_{j,\ell}\, k^j$ from (12). Then

$$
\begin{aligned}
C_N &:= \pi(1_S)\Big(\sum_{n=0}^{N}(n+1)^{\ell-2}\Big|\sum_{k=n+1}^{+\infty}\nu(R^{k-1}g_n)\Big|\Big)1_{\mathbb{X}} \\
&\leq \pi(1_S)\Big(\sum_{n=0}^{+\infty}(n+1)^{\ell-2}\sum_{k=n+1}^{+\infty}\nu(R^{k-1}|g_n|)\Big)1_{\mathbb{X}} \\
&\leq \pi(1_S)\Big(\sum_{n=0}^{+\infty}(n+1)^{\ell-2}\sum_{k=n+1}^{+\infty}\nu(R^{k-1}V_\ell)\Big)1_{\mathbb{X}} \ = \pi(1_S)\Big(\sum_{k=1}^{+\infty}\nu(R^{k-1}V_\ell)\sum_{n=1}^{k}n^{\ell-2}\Big)1_{\mathbb{X}} \\
&\leq \pi(1_S)\Big(\sum_{j=1}^{\ell-1}a_{j,\ell}\sum_{k=1}^{+\infty}k^j\,\nu(R^{k-1}V_\ell)\Big)1_{\mathbb{X}} \\
&\leq \pi(1_S)\nu(V_0)\Big(\sum_{j=1}^{\ell-1}a_{j,\ell}D_j\Big)1_{\mathbb{X}} = \pi(1_S)\,\nu(V_0)E_{\ell-1}1_{\mathbb{X}}
\end{aligned} \tag{37}
$$

using (9b) (note that $|g_n| \leq V_\ell \leq V_{j+1}$ for $j = 1, \ldots, \ell-1$) and the definition of $E_{\ell-1}$ in (13).

Finally, from the triangular inequality applied to (34), we obtain that

$$\sum_{n=0}^{N}(n+1)^{\ell-2}|P^n g_n| \leq A_N + B_N + C_N.$$

Therefore Inequality (31) follows from (35)-(37). The proof of Lemma 5.2 is complete. □

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A Construction of the Lyapunov functions V_i in Subsection 4.3

Assume that P satisfies Condition (**S**) and that there exists a Lyapunov function V such that

$$\exists\alpha \in [0,1),\ \exists b, c > 0, \quad PV \leq V - c\,V^{\alpha} + b\,1_S \tag{38}$$

with S given in (**S**). The construction of the Lyapunov functions V_i in $\boldsymbol{D}(V_0 : V_m)$ is based on the following fact. If W is a Lyapunov function and if $0 < \theta_2 < \theta_1 < 1$ are such that

$$\exists b, c > 0, \quad PW^{\theta_1} \leq W^{\theta_1} - c\,W^{\theta_2} + b\,1_S,$$

$$\text{then} \quad \exists b', c' > 0, \quad PW^{\theta_2} \leq W^{\theta_2} - c'\,W^{\theta_3} + b'\,1_S \qquad \text{with} \quad \theta_3 := 2\theta_2 - \theta_1. \tag{39}$$

Indeed we know from [JR02, Lem. 3.5] that

$$\forall \eta \in (0,1],\ \exists b_\eta, c_\eta > 0, \quad PW^{\eta\theta_1} \leq W^{\eta\theta_1} - c_\eta\,(W^{\theta_1})^{\theta_2/\theta_1 + \eta - 1} + b_\eta 1_S.$$

Then (39) is obtained with $\eta := \theta_2/\theta_1 < 1$. Next note that $\alpha_1 = 1 - 1/m \leq \alpha$, so that

$$PV \leq V - c\,V^{\alpha_1} + b\,1_S \tag{40}$$

from (38). Of course we can replace c with $c_1 < 1$. Recall that $m := \lfloor (1-\alpha)^{-1} \rfloor$. Then:

- If $\alpha_1 = 0$, i.e. $m = 1$ or $\alpha \in [0, 1/2)$, then $\boldsymbol{D}(V_0 : V_1)$ holds with $V_0 := c_1^{-1} V \geq V_1 := 1_{\mathbb{X}}$.
- If $\alpha_1 = 1/2$, i.e. $m = 2$ or $\alpha \in [1/2, 2/3)$, then we deduce from (40) and Property (39) applied to $W := V, \theta_1 = 1, \theta_2 = \alpha_1$ that

$$\exists b_1, c_2 > 0, \quad PV^{\alpha_1} \leq V^{\alpha_1} - c_2\,V^{\alpha_2} + b_1 1_S \tag{41}$$

  with $\alpha_2 := 2\alpha_1 - 1 = 0$. Again note that we can choose $c_2 < 1$. Then the procedure stops, and Conditions $\boldsymbol{D}(V_0 : V_2)$ hold with $V_0 := c_1^{-1} c_2^{-1} V \geq V_1 := c_2^{-1} V^{\alpha_1} \geq V_2 := 1_{\mathbb{X}}$.
- If $\alpha_1 > 1/2$, then Property (39) can be used recursively to provide inequalities of the form $PV^{\alpha_{i-1}} \leq V^{\alpha_{i-1}} - c_i\,V^{\alpha_i} + b_{i-1} 1_S$ with $c_i < 1$ and $\alpha_i = 2\alpha_{i-1} - \alpha_{i-2} = (\alpha_1 - 1)\,i + 1$. Actually (39) can only be used until the value $i = m$ since $\alpha_m = 0$ and $\alpha_i < 0$ for $i > m$. Then Conditions $\boldsymbol{D}(V_0 : V_m)$ hold with V_i given in (29), where $a_i = [\prod_{k=i+1}^{m} c_k]^{-1}$.