Polynomial convergence rates for Markov kernels under nested modulated drift conditions

Loïc Hervé, James Ledoux

Polynomial convergence rates for Markov kernels under nested modulated drift conditions

Loïc HERVÉ, and James LEDOUX *

version: Monday 18th December, 2023 – 17:52

Abstract

When a Markov kernel P satisfies a minorization condition and nested modulated drift conditions, Jarner and Roberts provided in [JR02, Th. 3.2] an asymptotic polynomial convergence rate in weighted total variation norm of $P^n(x,\cdot)$ to the P-invariant probability measure π. In connection with this polynomial asymptotics, we propose explicit and simple estimates on series of such weighted total variation norms, from which an estimate for the total variation norm of $P^n(x,\cdot) - \pi$ is deduced. The proofs are self-contained and based on the residual kernel and the Nummelin-type representation of π. No coupling technique is used.

AMS subject classification : 60J05

Keywords : Drift conditions; Invariant probability measure; Minorization condition; Residual kernel

1 Introduction

Let $(\mathbb{X}, \mathcal{X})$ be a measurable space, and let $\mathcal{M}^+$ (resp. $\mathcal{M}^+_*$) denote the set of finite nonnegative (resp. positive) measures on $(\mathbb{X}, \mathcal{X})$. For any $\mu \in \mathcal{M}^+$ and any μ-integrable function $g : \mathbb{X} \to \mathbb{R}$, $\mu(g)$ denotes the integral $\int_{\mathbb{X}} g d\mu$. Any measurable function $V : \mathbb{X} \to [1, +\infty)$ is called a Lyapunov function. For every measurable function $g : \mathbb{X} \to \mathbb{R}$, we set $\|g\|_V := \sup_{x\in\mathbb{X}} |g(x)|/V(x) \in [0, +\infty]$, and we define the space

$$\mathcal{B}_V := \{g : \mathbb{X} \to \mathbb{R}, \text{measurable such that } \|g\|_V < \infty\}.$$

If $(\mu_1, \mu_2) \in (\mathcal{M}^+)^2$ is such that $\mu_i(V) < \infty, i = 1, 2$, then the V-weighted total variation norm $\|\mu_1 - \mu_2\|'_V$ is defined by

$$\|\mu_1 - \mu_2\|'_V := \sup_{\|g\|_V \le 1} \big|\mu_1(g) - \mu_2(g)\big|. \tag{1}$$

If $V = 1_{\mathbb{X}}$, then $\|\cdot\|'_{1_{\mathbb{X}}} = \|\cdot\|_{TV}$ is the standard total variation norm. Finally recall that a nonnegative kernel $K(x, dy) \in \mathcal{M}^+$, $x \in \mathbb{X}$, is said to be a Markov (respectively submarkov)

*Univ Rennes, INSA Rennes, CNRS, IRMAR-UMR 6625, F-35000, France. Loic.Herve@insa-rennes.fr, James.Ledoux@insa-rennes.fr

kernel if $K(x,\mathbb{X}) = 1$ (respectively $K(x,\mathbb{X}) \le 1$) for any $x \in \mathbb{X}$. We denote by

$$\forall x \in \mathbb{X}, \quad (Kg)(x) := \int_{\mathbb{X}} g(y)\, K(x,dy)$$

the functional action of K, where $g : \mathbb{X} \to \mathbb{R}$ is any $K(x,\cdot)-$integrable function. For every $n \ge 1$ the $n-$th iterate kernel of $K(x,dy)$ is denoted by $K^n(x,dy)$, $x \in \mathbb{X}$, and K^n stands for its functional action. As usual K^0 is the identity map I by convention.

Throughout this paper, P is a Markov kernel on $(\mathbb{X},\mathcal{X})$ satisfying the following minorization condition (**S**) (e.g. see [MT09])

$$\exists S \in \mathcal{X},\ \exists \nu \in \mathcal{M}_*^+, \quad \forall x \in \mathbb{X},\ \forall A \in \mathcal{X}, \quad P(x,A) \ge \nu(1_A)\, 1_S(x), \tag{$\mathbf{S}$}$$

(i.e. S is a small-set of 1-order for P), and we denote by R the associated submarkov residual kernel:

$$\forall x \in \mathbb{X},\ \forall A \in \mathcal{X}, \quad R(x,A) := P(x,A) - \nu(A) 1_S(x). \tag{2}$$

Moreover let us introduce the following well-known nested modulated drift conditions: There exists a collection $\{V_i\}_{i=0}^m$ of Lyapunov functions with $m \ge 1$ such that

$$\forall i \in \{0,\dots,m-1\}, \quad V_{i+1} \le V_i \quad \text{and} \quad \exists b_i > 0,\ PV_i \le V_i - V_{i+1} + b_i\, 1_S. \tag{$\boldsymbol{D}(V_0:V_m)$}$$

The following statement was proved in [JR02, Th. 3.2], also see [FM03, Th. 1]:

Theorem 1 *Assume that P is $\psi-$irreducible and aperiodic for some $\psi \in \mathcal{M}_*^+$ and that P satisfies Conditions* (**S**) *and $\boldsymbol{D}(V_0 : V_m)$ with $m \ge 1$. Let π denote the $P-$invariant probability measure. Then*

$$\forall x \in \mathbb{X}, \quad \lim_{n \to +\infty} (n+1)^{m-1} \big\| P^n(x,\cdot) - \pi \big\|'_{V_m} = 0. \tag{3}$$

In [JR02, Th. 3.2] the condition $V_{i+1} \le V_i$ is not assumed and the modulated drift inequalities write as $PV_i \le V_i - V_{i+1} + b_i\, 1_{S_i}$ for some petite set S_i. However the assumption $V_{i+1} \le V_i$ is by no means restrictive since the Lyapunov functions V_i can be slightly modified in order to satisfy this condition, e.g. see Comment 2.6.

Under Assumptions (**S**) and $\boldsymbol{D}(V_0 : V_m)$ with $m \ge 2$, the purpose of this work is to provide quantitative estimates in connection with Property (3). Specifically we prove that there exists a positive function $U_m \in \mathcal{B}_{V_0}$ with a computable bound $\widehat{c}_m$ of $\|U_m\|_{V_0}$ such that

$$\begin{aligned}
\forall x \in \mathbb{X}, \quad \mathcal{S}_{m-2}(x) \ &:=\ \sum_{n=0}^{+\infty} (n+1)^{m-2} \|P^n(x,\cdot) - \pi\|'_{V_m} && \text{(4a)}\\
&\le\ U_m(x) && \text{(4b)}\\
&\le\ \widehat{c}_m\, V_0(x), && \text{(4c)}\\
\text{and} \quad \forall x \in \mathbb{X},\ \forall k \ge 0, \quad \big\|P^k(x,\cdot) - \pi\big\|_{TV} \ &\le\ \frac{2^{m-1}}{k^{m-1}}\, U_m(x). && \text{(4d)}
\end{aligned}$$

Estimates (4b)-(4c) are precisely stated in Section 2 (see Theorem 2.2) and proved in Section 3. Estimate (4d) is deduced from (4b) in Corollary 2.3. The key idea to prove (4b)-(4c) is that, for any $i = 1,\dots,m$, the norm $\| \sum_{n=0}^{+\infty} (n+1)^{i-1}\, R^n V_i \|_{V_0}$ where R is the residual kernel in (2), can be simply bounded using Assumptions $\boldsymbol{D}(V_0 : V_m)$ (see Proposition 2.1). Then, the link between P^n and R^n (see (6)) and the Nummelin-type representation of π (see (5)) enable us to obtain (4b) with U_m expressed in terms of the functions $\Phi_i := \sum_{n=0}^{+\infty}(n+1)^i\,|P^n\phi_S|$ for $i = 1,\dots,m-2$ with $\phi_S := 1_S - \pi(1_S)1_{\mathbb{X}}$. Moreover the norms $\|\Phi_i\|_{V_0}$ for $i = 1,\dots,m-2$, can be explicitly bounded via recursive inequalities involving the data S, ν, m, V_0 and b_i of Conditions $(\mathbf{S})$ and $\boldsymbol{D}(V_0 : V_m)$. This is illustrated for $m = 2$ and $m = 3$.

For a general overview on convergence rates of $P^n(x,\cdot)$ to π using drift conditions, we refer to the books [MT09, DMPS18, and the references therein]. Recall that the nested modulated drift conditions $\boldsymbol{D}(V_0 : V_m)$, first used in [TT94], were proved to hold in [JR02] under the single drift condition $PV \le V - cV^\alpha + b1_S$ with some Lyapunov function V and some constants $\alpha \in [0,1), b, c > 0$ (also see [FM00]), and in [FM03, Prop. 4] under the more general single sub-geometric drift condition $PV \le V - \phi\circ V + b1_S$ with suitable function ϕ. Here, our basic assumption is directly $\boldsymbol{D}(V_0 : V_m)$, which must be implemented in practice anyway, regardless of the form of the starting single drift condition, see [FM03, Rem. 3].

To the best of our knowledge there are very few works providing computable rates of convergence for series as defined in (4a). Using a coupling construction in the context of subgeometric Markov chains, such an issue is addressed in [AFV15, Th. 1] for series of the form $\sum_{n=0}^{+\infty} r(n)|(P^ng)(x) - (P^ng)(x')|$ where $(r(n))_{n\ge0}$ is some sequence of positive real numbers related to a subgeometric drift condition. Then, under Jarner-Roberts's drift condition $PV \le V - cV^\alpha + b1_S$, the case of series of the form $\sum_{n=0}^{+\infty}(n+1)^{\xi_\alpha}|(P^ng)(x) - \pi(g)|$ for some $\xi_\alpha > 0$ is covered by [AFV15, Cor. 1] (see Comment 2.6 for details). For Markov kernels satisfying Conditions $(\mathbf{S})$ and $\boldsymbol{D}(V_0 : V_m)$, Theorem 2.2 in Section 2 seems to be the first result providing a computable convergence rate for the series $\mathcal{S}_{m-2}(x)$ in (4a), even for $\mathcal{S}_0(x) = \sum_{n=0}^{+\infty}\|P^n(x,\cdot) - \pi\|'_{V_2}$. In fact, if P satisfies Conditions $(\mathbf{S})$ and $\boldsymbol{D}(V_0 : V_1)$ (i.e. $m = 1$) and if P is ψ−irreducible, aperiodic and $\pi(V_0) < \infty$, we know from [MT09, Th. 14.0.1] that there exists a constant c such that, for every $x \in \mathbb{X}$, we have $\mathcal{S}_0(x) \le c\,(1 + V_0(x))$. But the constant c was previously unknown. Here, under Conditions $\boldsymbol{D}(V_0 : V_2)$ (i.e. $m = 2$) and without assuming $\pi(V_0) < \infty$, this inequality is derived from (4c), and the positive constant $\widehat{c}_2$ in (4c) is easily computed from the data S, ν, V_0, b_0 and b_1 of Conditions $(\mathbf{S})$ and $\boldsymbol{D}(V_0 : V_2)$ (see (22) and Comment 2.5 for further comparisons with [MT09, Th. 14.0.1]). Actually we prove that, for any $m \ge 2$, the constant $\widehat{c}_m$ in (4c) can be computed from the data S, ν, m, V_0, and b_i of Conditions $(\mathbf{S})$ and $\boldsymbol{D}(V_0 : V_m)$. The use of both the residual kernel R in (2) and the Nummelin-type representation (5) of π is proved to be relevant for such a study, as already pointed out in [HL23a] for dealing with Poisson's equation under Assumptions $(\mathbf{S})$ and $\boldsymbol{D}(V_0 : V_1)$.

Following on from the pioneering works [NT83, TT94], explicit bounds for $\|P^n(x,\cdot)-\pi\|_{TV}$ have been proposed in [FM03, DMS07] thanks to coupling methods under the sub-geometric drift condition $PV \le V - \phi\circ V + b1_S$ (recall that this encompasses Jarner-Roberts's drift condition). Also see [DFMS04] for various statements and examples on different rates of convergence, and [But14, DFM16] for rates of convergence in Wasserstein distance. Note that the polynomial asymptotics (3) ensures that $\|P^n(x,\cdot) - \pi\|_{TV} \le c(x)/n^{m-1}$ for every $x \in \mathbb{X}$, but with unknown constant $c(x)$ to our knowledge. In particular, although the sub-geometric drift condition induces nested modulated drift inequalities, the explicit bounds of $\|P^k(x,\cdot) - \pi\|_{TV}$ in [FM03, Th. 2] and [DMS07, Th. 2.1] do not seem to provide any information on the quantitative polynomial rate of convergence in (3). Here Issue (4d) is directly linked to the polynomial asymptotics (3), and Estimate (19) in Corollary 2.3 of

Section 2 seems to be the first one providing $c(x) = cV_0(x)$ with a computable constant c under Conditions $\boldsymbol{D}(V_0 : V_m)$.

Therefore, we propose a self-contained method for obtaining quantitative results on the asymptotic result [JR02, Th. 3.2]. Note that the coupling technique is not used. Although the bounds obtained in Theorem 2.2 and Corollary 2.3 have a much simpler formulation than in [AFV15] and [FM03, DMS07], we do not claim that they are numerically better.

2 The statements

Let us recall that if P satisfies Condition (**S**), then a necessary and sufficient condition for P to admit an invariant probability measure π on $(\mathbb{X}, \mathcal{X})$ such that $\pi(1_S) > 0$, is that $\sum_{k=0}^{+\infty} \nu(R^k 1_{\mathbb{X}}) < \infty$. Actually, under any of these two equivalent conditions,

$$\pi := \mu(1_{\mathbb{X}})^{-1}\, \mu \quad \text{with} \quad \mu := \sum_{n=0}^{+\infty} \nu R^n \in \mathcal{M}_*^+ \tag{5}$$

is an $P-$invariant probability measure, and we have $\mu(1_S) = 1$ and $\pi(1_S) = \mu(1_{\mathbb{X}})^{-1} > 0$. The Nummelin-type representation (5) of π is well-known under various assumptions on P, e.g. see [Num84, Th. 5.2, Cor. 5.2]), [MT09, Chap. 10]), and see [HL23b] for a simple proof under the sole Condition (**S**). The assumptions in all the next statements ensure that the condition $\sum_{k=0}^{+\infty} \nu(R^k 1_{\mathbb{X}}) < \infty$ holds. Thus, throughout the paper, π is the $P-$invariant probability measure such that $\pi(1_S) > 0$ given in (5). Also recall that the key formula linking the kernels P^n, R^n and the non-negative measures $\nu(R^{k-1}\cdot)$ is from [HL20, Prop. 2.1]

$$\forall n \geq 1, \quad P^n = R^n + \sum_{k=1}^{n} \nu(R^{k-1}\cdot)\, P^{n-k} 1_S. \tag{6}$$

To prove that the series in (4a) converges, we first study the following functions:

$$\forall i \in \{1, \ldots, m\},\ \forall x \in \mathbb{X}, \quad R_i(x) := \sum_{n=0}^{+\infty} (n+1)^{i-1}\, (R^n V_i)(x).$$

To that effect, under Conditions (**S**) and $\boldsymbol{D}(V_0 : V_m)$ we set

$$\forall i \in \{0, \ldots, m-1\}, \quad d_i := \max\left(0, \frac{b_i - \nu(V_i))}{\nu(1_{\mathbb{X}})}\right) \tag{7}$$

with constants b_i given in $\boldsymbol{D}(V_0 : V_m)$. Obviously, we have $d_i = 0$ when $b_i \leq \nu(V_i)$. In particular, if S is an atom for P (i.e. $\forall x \in S,\ P(x, \cdot) = \nu$), then $d_i = 0$ for $0 \leq i \leq m-1$. Moreover define $(D_\ell)_{\ell=0}^{m-1}$ as follows:

$$D_0 := 1 + d_0 \quad \text{and} \quad \forall \ell \in \{1, \ldots, m-1\},\ D_\ell := (1 + d_\ell) \sum_{j=0}^{\ell-1} C_\ell^j D_j. \tag{8}$$

The following Proposition 2.1 is proved in Subsection 3.1.

Proposition 2.1 *Assume that P satisfies Condition* $(\mathbf{S})$ *and* $\boldsymbol{D}(V_0 : V_m)$ *for some collection* $\{V_i\}_{i=0}^m$ *of Lyapunov functions with* $m \geq 1$. *Then we have for every* $i \in \{1, \ldots, m\}$

$$\sum_{n=0}^{+\infty} (n+1)^{i-1}\, R^n V_i \leq D_{i-1}\, V_0 \tag{9a}$$

$$\sum_{n=0}^{+\infty} (n+1)^{i-1}\, \nu\big(R^n V_i\big) \leq D_{i-1}\, \nu(V_0) < \infty. \tag{9b}$$

Note that, if P satisfies Conditions $(\mathbf{S})$ and $\boldsymbol{D}(V_0 : V_m)$, then we deduce from (9b) applied with $i = 1$ that

$$\pi(V_1) < \infty, \text{ so that } \pi(V_i) < \infty \text{ for } i = 1, \ldots, m. \tag{10}$$

Now, to obtain the positive function $U_m \in \mathcal{B}_{V_0}$ in Inequality (4b) under Conditions $(\mathbf{S})$ and $\boldsymbol{D}(V_0 : V_m)$ with $m \geq 2$, we need to study the following functions for $i \in \{0, \ldots, m-2\}$:

$$\forall x \in \mathbb{X}, \quad \Phi_i(x) := \sum_{n=0}^{+\infty} (n+1)^i \left|\big(P^n \phi_S\big)(x)\right| \quad \text{where} \quad \phi_S := 1_S - \pi(1_S) 1_{\mathbb{X}}. \tag{11}$$

Recall that, for every $m \geq 2$, there exists $\{a_{j,m}\}_{j=1}^{m-1} \in \mathbb{R}^{m-1}$ such that

$$\forall k \geq 1, \quad \Sigma_k^{m-2} := \sum_{n=1}^{k} n^{m-2} = \sum_{j=1}^{m-1} a_{j,m}\, k^j, \tag{12}$$

and that the real numbers $\{a_{j,m}\}_{j=1}^{m-1}$ can be computed by induction on m using binomial expansion. Next, using D_j's in (8), define the following positive constants

$$\forall \ell \in \{1, \ldots, m-1\}, \quad E_\ell := \sum_{j=1}^{\ell} a_{j,\ell+1}\, D_j. \tag{13}$$

The next theorem is proved in Subsection 3.2.

Theorem 2.2 *Assume that P satisfies Conditions* $(\mathbf{S})$ *and* $\boldsymbol{D}(V_0 : V_m)$ *for some collection* $\{V_i\}_{i=0}^m$ *of Lyapunov functions with* $m \geq 2$. *Then the following inequalities hold in* $[0, +\infty]$:

$$\begin{aligned} \forall g \in \mathcal{B}_{V_m},\ \forall x \in \mathbb{X}, \quad \mathcal{S}_{m-2}(g, x) &:= \sum_{n=0}^{+\infty} (n+1)^{m-2} \big|(P^n g)(x) - \pi(g)\big| \\ &\leq \|g - \pi(g) 1_{\mathbb{X}}\|_{V_m}\, W_m(x) \end{aligned} \tag{14}$$

$$\begin{aligned} \textit{and} \quad \forall x \in \mathbb{X}, \quad \mathcal{S}_{m-2}(x) &:= \sum_{n=0}^{+\infty} (n+1)^{m-2} \big\|P^n(x, \cdot) - \pi\big\|'_{V_m} \\ &\leq \theta_m\, W_m(x) \end{aligned} \tag{15}$$

where $\theta_m := 1 + \pi(V_m) \|1_{\mathbb{X}}\|_{V_m}$ *and the function* W_m *is*

$$W_m = D_{m-2}\, V_0 + \nu(V_0) \Big[\sum_{j=0}^{m-2} C_{m-2}^{j} D_j\, \Phi_{m-2-j} + \pi(1_S)\, E_{m-1}\, 1_{\mathbb{X}} \Big]. \tag{16}$$

If $\pi(1_S) > 1/2$, then for every $i \in \{0, \dots, m-2\}$ we have $\Phi_i \in \mathcal{B}_{V_0}$ and (with the convention $\sum_{j=1}^{0} = 0$)

$$\Phi_i \leq \frac{1}{2\pi(1_S) - 1}\Big(D_i V_0 + \nu(V_0)\sum_{j=1}^{i} C_i^j \, D_j \Phi_{i-j} + \pi(1_S)\nu(V_0)E_{i+1}1_{\mathbb{X}}\Big). \tag{17}$$

Thus, if P satisfies all the assumptions of Theorem 2.2, then Estimates (4b)-(4c) in Section 1 are valid with $U_m(x) = \theta_m W_m(x)$. Indeed, Inequality (4b) is nothing else than (15). To derive Inequality (4c), first use (16) to get

$$\|W_m\|_{V_0} \leq D_{m-2} + \nu(V_0)\sum_{j=0}^{m-2} C_{m-2}^j D_j \, \|\Phi_{m-2-j}\|_{V_0} + \pi(1_S)\,\nu(V_0)E_{m-1}\|1_{\mathbb{X}}\|_{V_0}.$$

Next, if $\pi(1_S) > 1/2$ then the norms $(\|\Phi_i\|_{V_0})_{i=0}^{m-2}$ are recursively bounded from (17) by

$$\|\Phi_i\|_{V_0} \leq \frac{1}{2\pi(1_S) - 1}\Big(D_i + \nu(V_0)\big[\sum_{j=1}^{i} C_i^j \, D_j \, \|\Phi_{i-j}\|_{V_0} + \pi(1_S)\, E_{i+1}\|1_{\mathbb{X}}\|_{V_0}\big]\Big) \tag{18}$$

from which the constant $\widehat{c}_m$ in (4c) is deduced. In the atomic case, recall that the d_i's (see (7)) are zero, so that the constants D_i defined in (8) and used in the previous estimates simply depend on the integer m. We refer to Comment 2.4 for a discussion of the technical condition $\pi(1_S) > 1/2$. Finally note that $\|1_{\mathbb{X}}\|_{V_m} \leq 1$ since $V_m \geq 1$ and that $\pi(V_m) \leq b_{m-1}$ from $PV_{m-1} \leq V_{m-1} - V_m + b_{m-1}\,1_S$ and the $P-$invariance of π (recall that $\pi(V_{m-1}) < \infty$ from (10)). Thus the positive constant θ_m of Theorem 2.2 satisfies

$$\theta_m \leq 1 + b_{m-1}.$$

Corollary 2.3 *Under all the assumptions of Theorem 2.2, we have*

$$\forall k \geq 0, \quad \big\|P^k(x,\cdot) - \pi\big\|_{TV} \leq \frac{2^{m-1}\,\theta_m}{k^{m-1}}\, W_m(x) \tag{19}$$

with θ_m and W_m given in Theorem 2.2.

Proof. Note that V_m in $\boldsymbol{D}(V_0 : V_m)$ can be replaced with the function $1_{\mathbb{X}}$ since $V_m \geq 1_{\mathbb{X}}$. Let $x \in \mathbb{X}$. Recall that the sequence $(\|P^n(x,\cdot) - \pi\|_{TV})_n$ is non-increasing. Let $j \geq 0$. Then we deduce from (15) that

$$(j+1)^{m-1}\,\big\|P^{2j}(x,\cdot) - \pi\big\|_{TV} \leq \sum_{n=j}^{2j}(n+1)^{m-2}\big\|P^n(x,\cdot) - \pi\big\|_{TV} \leq \theta_m\, W_m(x)$$

thus

$$\big\|P^{2j}(x,\cdot) - \pi\big\|_{TV} \leq \frac{2^{m-1}\,\theta_m}{(2j)^{m-1}}\, W_m(x).$$

Next, using $\sum_{n=j+1}^{2j+1}$, we obtain the same inequality for $\big\|P^{2j+1}(x,\cdot) - \pi\big\|_{TV}$ replacing $(2j)^{m-1}$ with $(2j+1)^{m-1}$. This proves (19). □

Comment 2.4 *The condition $\pi(1_S) > 1/2$ is a technical assumption which is only used to get the estimate (17) of Φ_i (see (30)). Let us explain why this condition is not restrictive in general and how the multiplicative constant $(2\pi(1_S)-1)^{-1}$ can be explicitly controlled. Assume that P satisfies the drift condition $PV \leq V - W + b1_{S_0}$ for some $S_0 \in \mathcal{X}$, $b > 0$ and Lyapunov functions $V \geq W$. First note that this condition can be reduced to*

$$b_0 := \sup_{x\in S_0} \big[(PV)(x) - V(x) + W(x)\big] < \infty \quad \textit{and} \quad PV - V + W \leq 0 \quad \textit{on } S_0^c.$$

Consequently, for every $S \supset S_0$, we have $PV \leq V - W + b_0 1_S$ (note that b_0 only depends on S_0). Second observe that if $\pi(V) < \infty$ then

$$\pi(1_S) \geq \frac{m_S}{m_S + b_0 - 1} \quad \textit{with} \quad m_S := \inf_{x\in S^c} W(x) \tag{20}$$

provided that $m_S > 1 - b_0$. This easily follows from $b_0\,\pi(1_S) \geq \pi(W) \geq \pi(1_S) + m_S\,\pi(1_{S^c})$ and $\pi(1_{S^c}) = 1 - \pi(1_S)$. Thus, if $m_S \to +\infty$ when S growths, then $\pi(1_S) > 1/2$ for S large enough. Moreover it follows from (20) that

$$\frac{1}{2\pi(1_S) - 1} \leq \frac{m_S + b_0 - 1}{m_S - b_0 + 1}.$$

In general the Lyapunov functions V_i in Conditions $\boldsymbol{D}(V_0 : V_m)$ are unbounded on $\mathbb{X}$, excepted possibly V_m which may be $1_{\mathbb{X}}$, so that the above condition $m_S \to +\infty$ is realistic when applied to the couple $(V, W) = (V_{m-2}, V_{m-1})$ for instance. Finally note that, for the choice of S in practice, a trade-off must be found between the value of $\pi(1_S)$ and the mass $\nu(1_{\mathbb{X}})$ of the positive measure $\nu \equiv \nu_S$ in Condition (**S**)*. Indeed, the larger the set S is, then the closer $\pi(1_S)$ is to one, but the smaller the mass $\nu_S(1_{\mathbb{X}})$ is, and so the bigger the values d_i in (7) are.*

Theorem 2.2 applies whenever explicit modulated drift conditions are known: for such examples, e.g. see [FM00, FM03, DFM16] in the context of Metropolis algorithm, [LH07, LH12] for queueing systems, [JT02] for Markov chains associated with the mean of Dirichlet processes. Now let us detail the cases $m = 2$ and $m = 3$.

Case $m = 2$

Assume that P satisfies Condition (**S**) with $\pi(1_S) > 1/2$ and $\boldsymbol{D}(V_0 : V_2)$ for some Lyapunov functions V_0, V_1, V_2. Note that $\Sigma_k^0 := k$, i.e. $a_{1,2} = 1$ in (12). We have $E_1 = D_1$ from (13). Consequently it follows from (14) and (16) applied with $m = 2$ and from (17) with $i = 0$ that

$$\begin{aligned}\forall g \in \mathcal{B}_{V_2},\ \forall x \in \mathbb{X}, \quad \mathcal{S}_0(g,x) = \sum_{n=0}^{+\infty} \big|(P^n g)(x) - \pi(g)\big| &\leq \|g - \pi(g)1_{\mathbb{X}}\|_{V_m}\, W_2(x) \\ &\leq \|g - \pi(g)1_{\mathbb{X}}\|_{V_m}\, \widehat{c}_2\, V_0(x)\end{aligned} \tag{21}$$

where

$$\begin{aligned}W_2 = D_0\, V_0 + \nu(V_0)\, D_0\, \Phi_0 + \pi(1_S)\nu(V_0) E_1\, 1_{\mathbb{X}} \ &\leq\ c_0 V_0 + c_1 1_{\mathbb{X}} \\ \text{with } c_0 = (1 + d_0)\left(1 + \frac{\nu(V_0)\,(1+d_0)}{2\pi(1_S) - 1}\right) \quad & c_1 = \nu(V_0) D_1 \left(\frac{\nu(V_0)\,(1+d_0)}{2\pi(1_S)-1} + 1\right) \\ \text{and } \widehat{c}_2 = c_0 + c_1 \|1_{\mathbb{X}}\|_{V_0} &\end{aligned} \tag{22}$$

where $D_1 = (1+d_0)(1+d_1)$ and $d_i = \max(0, (b_i - \nu(V_i))/\nu(1_{\mathbb{X}}))$ for $i = 0, 1$. Similarly Inequalities (15) and (19) hold with W_2 defined in (22).

Comment 2.5 *If P is $\psi-$irreducible, aperiodic, and satisfies Conditions* $(\mathbf{S})$ *and* $\boldsymbol{D}(V_0 : V_1)$ *with $\pi(V_0) < \infty$, then we know from [MT09, Th. 14.0.1] that there exists a (non explicit) constant c such that, for any $x \in \mathbb{X}$, we have $\mathcal{S}_0(x) \le c\,(1 + V_0(x))$. Recall that the condition $\pi(V_0) < \infty$ is not guaranteed under Assumptions $\boldsymbol{D}(V_0 : V_1)$ (we only know that $\pi(V_1) < \infty$ from (10)). We do not need to assume $\pi(V_0) < \infty$ for (21), but Assumptions $\boldsymbol{D}(V_0 : V_2)$ are required. Actually, Conditions $\boldsymbol{D}(V_0 : V_1)$ with the additional condition $\pi(V_0) < \infty$ may be close, or even identical, to Conditions $\boldsymbol{D}(V_0 : V_2)$ For instance, let $(X_n)_{n\ge0}$ be the following so-called random walk on the half line $\mathbb{X} = [0, +\infty)$*

$$X_0 \in \mathbb{X} \quad \textit{and} \quad \forall n \ge 1,\ X_n := \max\big(0, X_{n-1} + W_n\big) \tag{23}$$

where $\{W_n\}_{n\ge1}$ is a sequence of $\mathbb{R}$-valued i.i.d. random variables assumed to be independent of X_0 and to satisfy $\mathbb{E}[W_1] < 0$ and $\mathbb{E}[\max(0, W_1)] < \infty$. Then it is well-known that Conditions $\boldsymbol{D}(V_0 : V_1)$ hold with $S = [0, s]$ for some $s > 0$ and with Lyapunov functions V_0, V_1 defined on $\mathbb{X} = [0, +\infty)$ by $V_0(x) = 1 + x$ and $V_1(x) = c_1 1_{\mathbb{X}}$ for some constant $c_1 > 0$. Moreover, it follows from [JT03, Prop. 3.5] that the condition $\int_{\mathbb{X}} x\, d\pi(x) < \infty$, i.e. $\pi(V_0) < \infty$, is equivalent to $\mathbb{E}[(\max(0, W_1))^2] < \infty$, so that the last moment condition is required to apply the statement [MT09, Th. 14.0.1]. However note that the condition $\mathbb{E}[(\max(0, W_1))^2] < \infty$ is precisely what ensures that Assumptions $\boldsymbol{D}(V_0 : V_2)$ hold with $V_0(x) = (1 + x)^2$ and $V_i(x) = c_i(1 + x)^{2-i}$ for $i = 1, 2$ with some $c_i > 0$ (e.g. see [JR02]). Accordingly, in this example, the moment condition on W_1 is the same for applying (21) *or [MT09, Th. 14.0.1].*

Case $m = 3$

Assume that P satisfies Condition $(\mathbf{S})$ with $\pi(1_S) > 1/2$ and Conditions $\boldsymbol{D}(V_0 : V_3)$ for some Lyapunov functions V_0, V_1, V_2, V_3. Here we have $\Sigma^1_k = k(k + 1)/2$, i.e. $a_{1,3} = a_{2,3} = 1/2$ from (12). Thus we get from (8) and (13)

$$i = 0, 1,\ D_i = \prod_{j=0}^{i}(1 + d_i),\ D_2 = (1 + d_2)(D_0 + 2D_1),\ E_1 = D_1,\ E_2 = \frac{D_1 + D_2}{2}$$

with $d_i = \max(0, (b_i - \nu(V_i))/\nu(1_{\mathbb{X}}))$ for $i = 0, 1, 2$. Consequently it follows from (14) and (16) applied with $m = 3$ and from (17) with $i = 0, 1$ that

$$\begin{aligned}\forall g \in \mathcal{B}_{V_3},\ \forall x \in \mathbb{X},\quad \sum_{n=0}^{+\infty}(n + 1)\big|(P^n g)(x) - \pi(g)\big| &\le \|g - \pi(g)1_{\mathbb{X}}\|_{V_3}\, W_3(x) \\ &\le \|g - \pi(g)1_{\mathbb{X}}\|_{V_3}\, \widehat{c}_3\, V_0(x)\end{aligned}$$

where

$$W_3 = D_1\, V_0 + \nu(V_0)\, D_0\, \Phi_1 + \nu(V_0) D_1\, \Phi_0 + \pi(1_S)\, \nu(V_0) E_2\, 1_{\mathbb{X}} \ \le\ c_0 V_0 + c_1 1_{\mathbb{X}} \tag{24}$$

$$\text{with } c_0 = D_1\left[1 + \frac{\nu(V_0)D_0}{2\pi(1_S) - 1}\right]^2 c_1 = \nu(V_0)\left[E_2 + \frac{\nu(V_0){D_1}^2 + D_0 E_2 \nu(V_0)}{2\pi(1_S) - 1} + \frac{\nu(V_0)^2 D_0 {D_1}^2}{(2\pi(1_S) - 1)^2}\right]$$

$$\widehat{c}_3 = c_0 + c_1\|1_{\mathbb{X}}\|_{V_0}.$$

Similarly Inequalities (15) and (19) hold with W_3 defined in (24).

Comment 2.6 (Comparison with [AFV15] under Jarner-Roberts's drift condition) *Thoughout this comment, the Markov kernel P is assumed to satisfies the minoration condition* (**S**)*. Recall that Jarner-Roberts's drift condition introduced in [JR02] is: There exists a Lyapunov function V such that*

$$\exists\alpha \in [0,1),\ \exists b_0, c > 0, \quad PV \le V - c\,V^{\alpha} + b_0\,1_S. \tag{25}$$

This is the most classical case leading to Markov kernels satisfying Conditions $\boldsymbol{D}(V_0 : V_m)$, also see [MT09, DMPS18, and the references therein]. Indeed P satisfies $\boldsymbol{D}(V_0 : V_m)$ with $m \equiv m(\alpha) := \lfloor (1-\alpha)^{-1} \rfloor \ge 1$, where $\lfloor\cdot\rfloor$ denotes the integer part function on $\mathbb{R}$, and with the Lyapunov functions

$$V_m := 1_{\mathbb{X}} \le V_{m-1} := a_{m-1}V^{\alpha_{m-1}} \le \cdots \le V_1 := a_1 V^{\alpha_1} \le V_0 := a_0 V \tag{26}$$

where $\alpha_1 := 1 - 1/m \in [0,1)$ and $\alpha_i = (\alpha_1 - 1)\,i + 1$ for $i = 2,\ldots,m-1$ when $m \ge 2$, and where a_i's are explicit constants strictly larger than one, see [JR02, Proof of Th. 3.6]. For the reader's convenience, the construction of V_i's is detailed in Appendix A. Hence, if $m \ge 2$ and $\pi(1_S) > 1/2$, then for any measurable and bounded $g : \mathbb{X} \to \mathbb{R}$, i.e. $g \in \mathcal{B}_{1_{\mathbb{X}}}$, and for any $x \in \mathbb{X}$, Theorem 2.2 provides an explicit bound for $\sum_{n=0}^{+\infty}(n+1)^{m-2}|(P^n g)(x) - \pi(g)|$. For instance the bounds (22) *in case $m = 2$, or the bounds* (24) *in case $m = 3$, apply. Under the drift condition* (25) *(and some additional minor assumptions), it is proved in [AFV15, Cor. 1, homogeneous case with $\xi = 1$] that there exists a constant $C > 0$ such that for any $(x, x') \in \mathbb{X}^2$ and any $g \in \mathcal{B}_{1_{\mathbb{X}}}$*

$$\sum_{n=0}^{+\infty}(n+1)^{m-1}|(P^n g)(x) - (P^n g)(x')| \le C\,\|g\|_{1_{\mathbb{X}}}\big(V(x) + V(x') - 1\big).$$

Thus, if $\pi(V) < \infty$, then $\mathcal{S}_{m-1}(g,x) \le C\,\|g\|_{1_{\mathbb{X}}}(V(x) + \pi(V) - 1)$. The reason why $\mathcal{S}_{m-1}(g,x)$ can be estimated in [AFV15, Cor. 1], while Theorem 2.2 only provides an estimate for $\mathcal{S}_{m-2}(g,x)$, is the same as in Comment 2.5, that is: The condition $\pi(V) < \infty$ is not guaranteed under Assumption (25) (we only know that $\pi(V^{\alpha}) < \infty$). Again note that the condition $\pi(V) < \infty$ is not required for using Theorem 2.2. Actually, assuming both (25) with $\alpha = 1 - 1/m$ and $\pi(V) < \infty$, is close to assuming Condition (25) *with $\alpha = 1 - 1/(m+1)$. For instance, extending the arguments of Comment 2.5, it follows from [JT03, Prop. 3.5] that the two last assumptions are identical for random walks on the half line. Note that (25) with $\alpha = 1 - 1/(m+1)$ implies $\boldsymbol{D}(V_0 : V_{m+1})$, so that for any $g \in \mathcal{B}_{1_{\mathbb{X}}}$ the series $\sum_{n=0}^{+\infty}(n+1)^{m-1}|P^n g - \pi(g)|$ can be estimated too using Theorem 2.2, as well as the sums studied in [AFV15] since*

$$\forall (x,x') \in \mathbb{X}, \quad \sum_{n=0}^{+\infty}(n+1)^{m-1}|(P^n g)(x) - (P^n g)(x')| \le \mathcal{S}_{m-1}(g,x) + \mathcal{S}_{m-1}(g,x')$$

from the triangular inequality. Series with the norms $\|P^n(x,\cdot) - \pi\|'_{V_m}$ (see (15)*) and estimate of type* (19) *are not studied in [AFV15]. Finally mention that the comparison between the above constant C and that derived from Theorem 2.2 is not easy to address since the constant C in [AFV15, Cor. 1] is not completely computed. However note that this constant C involves the real number $\varepsilon_\nu^{-1} = \nu(1_{\mathbb{X}})^{-1}$ and the series $c_* := \sum_{j=0}^{+\infty}(1 - \nu(1_{\mathbb{X}}))^j \prod_{k=0}^{j}(1 + \delta_k M_1)$ for some $(\delta_k)_k \in \mathbb{R}^{\mathbb{N}}$ and some constant M_1. The bounds in Theorem 2.2 also involve the constant $\nu(1_{\mathbb{X}})^{-1}$ through d_i's in* (7)*, but it only requires to compute finitely many constants of the form $\prod_{k=0}^{j}(1 + d_\ell)$ (see* (8)*).*

3 Proofs

3.1 Proof of Proposition 2.1

Recall that the residual kernel R defined in (2) under Assumption ($\mathbf{S}$) is a submarkov kernel. The following simple result is from [HL23a, Lemma 2.2] and allows us to transform a modulated drift condition for P into a simpler drift condition for R which is in force for deriving Proposition 2.1.

Lemma 3.1 *Assume that P satisfies Condition ($\mathbf{S}$) and $PV \leq V - W + b1_S$ for some $b > 0$ and some couple (V, W) of Lyapunov functions. Then we have*

$$RV_d \leq V_d - W \quad \textit{with} \quad V_d := V + d1_{\mathbb{X}} \geq V, \quad \textit{where} \quad d := \max\left(0, \frac{b - \nu(V))}{\nu(1_{\mathbb{X}})}\right).$$

Let us prove Inequalities (9a), that is with D_{i-1} defined in (8)

$$\forall i \in \{1, \dots, m\}, \quad \sum_{n=0}^{+\infty} (n+1)^{i-1}\, R^n V_i \leq D_{i-1} V_0.$$

We use an induction on m. Assume that $\boldsymbol{D}(V_0 : V_1)$ holds, that is $PV_0 \leq V_0 - V_1 + b_0\, 1_S$. Then it follows from Lemma 3.1 applied to $(V, W) = (V_0, V_1)$ that $RV_{0,d_0} \leq V_{0,d_0} - V_1$ with $V_{0,d_0} := V_0 + d_0 1_{\mathbb{X}} \geq V_0$ where $d_0 = \max\{0, (b_0 - \nu(V_0))/\nu(1_{\mathbb{X}})\}$. Equivalently we have $V_1 \leq V_{0,d_0} - RV_{0,d_0}$. Then for every $n \geq 0$ we obtain that $R^n V_1 \leq R^n V_{0,d_0} - R^{n+1} V_{0,d_0}$. Hence we have for every $N \geq 1$

$$\sum_{n=0}^{N} R^n V_1 \leq \sum_{n=0}^{N} \left[R^n V_{0,d_0} - R^{n+1} V_{0,d_0}\right] \leq V_{0,d_0} \leq (1 + d_0) V_0.$$

This proves (9a) when $m = 1$. Now suppose that Inequalities (9a) are proved for some $m \geq 1$ and that Conditions $\boldsymbol{D}(V_0 : V_{m+1})$ hold for some collection $\{V_i\}_{i=0}^{m+1}$ of Lyapunov functions. Then it follows from Lemma 3.1 for $(V, W) = (V_m, V_{m+1})$ that $RV_{m,d_m} \leq V_{m,d_m} - V_{m+1}$ with $V_{m,d_m} := V_m + d_m 1_{\mathbb{X}} \geq V_m$, where $d_m := \max\{0, (b_m - \nu(V_m)/\nu(1_{\mathbb{X}})\}$. Equivalently we have $V_{m+1} \leq V_{m,d_m} - RV_{m,d_m}$, so that we obtain for every $N \geq 1$

$$\begin{aligned}
\sum_{n=0}^{N} (n+1)^m\, R^n V_{m+1} \quad &\leq \quad \sum_{n=0}^{N} (n+1)^m\, R^n V_{m,d_m} - \sum_{n=0}^{N+1} n^m\, R^n V_{m,d_m} \\
&\leq \quad \sum_{n=0}^{N} \left[(n+1)^m - n^m\right] R^n V_{m,d_m} = \sum_{j=0}^{m-1} C_m^j \sum_{n=0}^{N} n^j\, R^n V_{m,d_m} \\
&\leq \quad \left(1 + d_m\right) \sum_{j=0}^{m-1} C_m^j \sum_{n=0}^{N} n^j\, R^n V_{j+1} \\
&\leq \quad \left(1 + d_m\right) \left(\sum_{j=0}^{m-1} C_m^j D_j\right) V_0 \ = D_m V_0
\end{aligned}$$

using the binomial expansion and $V_{m,d_m} \leq (1 + d_m) V_m \leq (1 + d_m) V_{j+1}$ for $j = 0, \dots, m-1$, and using finally the definition of D_m. This gives Inequalities (9a) at order $m+1$. Finally (9b) follows from (9a) since, for some $x \in S$, we have $\nu(V_0) \leq (PV_0)(x) < \infty$ from Assumption ($\mathbf{S}$) and $(PV_0)(x) \leq V_0(x) - V_1(x) + b_0$.

3.2 Proof of Theorem 2.2

Assume that P satisfies Conditions $(\mathbf{S})$ and $\boldsymbol{D}(V_0 : V_m)$ for some collection $\{V_i\}_{i=0}^m$ of Lyapunov functions with $m \geq 2$. Recall that $\phi_S := 1_S - \pi(1_S)1_{\mathbb{X}}$. For every $i \in \{0, \ldots, m-2\}$ set:

$$\forall N \geq 1,\ \forall x \in \mathbb{X}, \quad \Phi_{i,N}(x) := \sum_{n=0}^{N} (n+1)^i \left|\big(P^n \phi_S\big)(x)\right|. \tag{27}$$

The following lemma plays a crucial role to prove Theorem 2.2.

Lemma 3.2 *Assume that P satisfies Conditions $(\mathbf{S})$ and $\boldsymbol{D}(V_0 : V_\ell)$ for some collection $\{V_i\}_{i=0}^\ell$ of Lyapunov functions with $\ell \geq 2$. Let $(g_n)_{n\geq 0} \in \mathcal{B}_{V_\ell}^{\mathbb{N}}$ and $\psi \in \mathcal{B}_{V_\ell}$ such that $|g_n| \leq \psi \leq V_\ell$ and $\pi(g_n) = 0$ for every $n \geq 0$. Then we have for every $N \geq 1$ (with the usual convention $\sum_{j=1}^{0} = 0$)*

$$\begin{aligned} \sum_{n=0}^{N} (n+1)^{\ell-2} \big|P^n g_n\big| \quad &\leq \quad D_{\ell-2}\, V_0 + \left(\sum_{k=1}^{+\infty} \nu(R^{k-1}\psi)\right) \Phi_{\ell-2,N} \\ &\qquad + \nu(V_0) \Bigg[\sum_{j=1}^{\ell-2} C_{\ell-2}^{j} D_j\, \Phi_{\ell-2-j,N} + \pi(1_S)\, E_{\ell-1}\, 1_{\mathbb{X}}\Bigg]. \end{aligned} \tag{28}$$

Proof of Theorem 2.2. Note that $\Phi_{i,N} \leq \Phi_i$ for every $N \geq 1$, with Φ_i given in (11). If $g \in \mathcal{B}_{V_m}$ is such that $\|g\|_{V_m} \leq 1$ and $\pi(g) = 0$, then Inequality (14) in $[0, +\infty]$ with W_m given in (16) directly follows from Inequality (28) applied to $\ell := m$, $g_n = g, \psi = V_m$, and from

$$\sum_{k=1}^{+\infty} \nu(R^{k-1} V_m) \leq \sum_{k=1}^{+\infty} \nu(R^{k-1} V_1) \leq D_0\, \nu(V_0) \tag{29}$$

thanks to (9b) applied with $i = 1$. If $\pi(g) \neq 0$, replace g with $g - \pi(g)1_{\mathbb{X}}$.

Next, to prove Inequality (15), recall that $\theta_m = 1 + \pi(V_m)\|1_{\mathbb{X}}\|_{V_m}$, and first note that

$$\forall h \in \mathcal{B}_{V_m}, \quad \|h - \pi(h)1_{\mathbb{X}}\|_{V_m} \leq \theta_m \|h\|_{V_m}.$$

Now let $(h_n)_{n\geq 0} \in \mathcal{B}_{V_m}^{\mathbb{N}}$ be such that $\|h_n\|_{V_m} \leq 1$ and set $f_n := h_n - \pi(h_n)1_{\mathbb{X}}$. For any $n \geq 0$, we have $\|f_n\|_{V_m} \leq \theta_m$, so that $g_n := f_n/\theta_m$ is such that $|g_n| \leq V_m$ and $\pi(g_n) = \pi(f_n) = 0$. Then, applying Inequality (28) to $\ell := m$, $\psi = V_m$, we obtain that

$$\forall N \geq 1, \quad \sum_{n=0}^{N} (n+1)^{m-2} \big|(P^n h_n)(x) - \pi(h_n)\big| \leq \theta_m\, W_m(x)$$

using again (29). Taking the suppremum bound over the functions $h_0, \ldots, h_N$, we obtain that

$$\forall N \geq 1, \quad \sum_{n=0}^{N} (n+1)^{m-2} \|P^n(x, \cdot) - \pi\|_{V_m}' \leq \theta_m\, W_m(x)$$

from which we deduce (15).

Now assume that $\pi(1_S) > 1/2$. Then we have:

$$\sum_{k=1}^{+\infty} \nu\big(R^{k-1}|\phi_S|\big) \leq 2\pi(1_{S^c}) < 1. \tag{30}$$

Indeed we have $\phi_S = (1-\pi(1_S))1_S - \pi(1_S)1_{S^c}$, so that $|\phi_S| = (1-\pi(1_S))1_S + \pi(1_S)1_{S^c}$. Recall that $\mu(1_S) = 1$ and $\pi = \pi(1_S)\mu$ (see (5)). Thus

$$\sum_{k=1}^{+\infty} \nu\big(R^{k-1}|\phi_S|\big) \leq (1-\pi(1_S))\mu(1_S) + \pi(1_S)\mu(1_{S^c}) = 1 - \pi(1_S) + \pi(1_{S^c}) = 2\pi(1_{S^c}).$$

This proves (30).

Observe that Assumptions $\boldsymbol{D}(V_0 : V_m)$ obviously imply that, for every $i = 0,\ldots,m-2$, Assumptions $\boldsymbol{D}(V_0 : V_{i+2})$ hold too. Therefore, for any $i = 0,\ldots,m-2$, it follows from Inequality (28) with $\ell = i+2$ applied to $g_n := \phi_S, \psi := |\phi_S|$, and from (30) that

$$\big(1 - 2\pi(1_{S^c})\big)\Phi_{i,N} \leq D_i\, V_0 + \nu(V_0)\left[\sum_{j=1}^{i} C_i^j D_j\, \Phi_{i-j,N} + \pi(1_S)\, E_{i+1}\, 1_{\mathbb{X}}\right].$$

Recall that $\sum_{j=1}^{0} = 0$ by convention in (28). When $N \to +\infty$, the previous inequality for $i = 0$ shows that the series Φ_0 is convergent and satisfies (17) for $i = 0$. Next this inequality for $i \in \{1,\ldots,m-2\}$ ensures that the series Φ_i is convergent from the convergence of the $(\Phi_j)_{j=0}^{i-1}$, and that Φ_i satisfies Inequality (17). The proof of Theorem 2.2 is complete, provided that Lemma 3.2 is proved. □

Proof of Lemma 3.2. Let $(g_n)_{n\geq 0} \in \mathcal{B}_{V_\ell}^{\mathbb{N}}$ and $\psi \in \mathcal{B}_{V_\ell}$ such that $|g_n| \leq \psi \leq V_\ell$ and $\pi(g_n) = 0$ for every $n \geq 0$. Note that $\mu(g_n) := \sum_{k=1}^{+\infty} \nu(R^{k-1}g_n) = 0$ since $\pi(g_n) = 0$ (see (5)). Then we get from Formula (6) and $\sum_{k=1}^{n} \nu(R^{k-1}g_n) = -\sum_{k=n+1}^{+\infty} \nu(R^{k-1}g_n)$ with the convention $\sum_{k=1}^{0} = 0$

$$\begin{aligned}
\forall n \geq 0, \quad P^n g_n &= R^n g_n + \sum_{k=1}^{n} \nu(R^{k-1}g_n)\, P^{n-k}1_S \\
&= R^n g_n + \sum_{k=1}^{n} \nu(R^{k-1}g_n)\, P^{n-k}\phi_S \;-\; \pi(1_S)\bigg(\sum_{k=n+1}^{+\infty} \nu(R^{k-1}g_n)\bigg)1_{\mathbb{X}}. \quad (31)
\end{aligned}$$

First, using the positivity of R and $|g_n| \leq V_\ell \leq V_{\ell-1}$, it follows from (9a) with $i = \ell - 1$ that

$$A_N := \sum_{n=0}^{N} (n+1)^{\ell-2}\, |R^n g_n| \leq \sum_{n=0}^{+\infty} (n+1)^{\ell-2}\, R^n|g_n| \leq \sum_{n=0}^{+\infty} (n+1)^{\ell-2}\, R^n V_{\ell-1} \leq D_{\ell-2}\, V_0. \quad (32)$$

Second, using again the convention $\sum_{k=1}^{0} = 0$ and the inequality $|g_n| \leq \psi$, we have

$$\begin{aligned}
B_N := \sum_{n=0}^{N} (n+1)^{\ell-2} \bigg|\sum_{k=1}^{n} \nu(R^{k-1}g_n)\, P^{n-k}\phi_S\bigg| \;&\leq\; \sum_{n=0}^{N} (n+1)^{\ell-2} \sum_{k=1}^{n} \nu(R^{k-1}|g_n|)\, \big|P^{n-k}\phi_S\big| \\
&= \sum_{k=1}^{N} \nu(R^{k-1}|g_n|) \sum_{n=k}^{N} (n+1)^{\ell-2} \big|P^{n-k}\phi_S\big| \\
&\leq \sum_{k=1}^{N} \nu(R^{k-1}\psi) \sum_{n=0}^{N} (n+1+k)^{\ell-2} \big|P^{n}\phi_S\big| \\
&= \sum_{j=0}^{\ell-2} C_{\ell-2}^j \bigg(\sum_{k=1}^{N} k^j\, \nu(R^{k-1}\psi)\bigg) \Phi_{\ell-2-j,N} \\
&\leq \sum_{j=0}^{\ell-2} C_{\ell-2}^j \bigg(\sum_{k=1}^{+\infty} k^j\, \nu(R^{k-1}\psi)\bigg) \Phi_{\ell-2-j,N}
\end{aligned}$$

where the $\Phi_{i,N}$'s are defined in (27). Then, separating the term for $j=0$ in the last sum and using $\psi \le V_\ell \le V_{j+1}$ for $j=1,\dots,\ell-2$, it follows from (9b) that

$$B_N \le \Big(\sum_{k=1}^{+\infty} \nu(R^{k-1}\psi)\Big)\Phi_{\ell-2,N} + \nu(V_0)\sum_{j=1}^{\ell-2} C_{\ell-2}^j \, D_j \, \Phi_{\ell-2-j,N}. \tag{33}$$

Third, recall that, for any $k \ge 1$, $\Sigma_k^{\ell-2} := \sum_{n=1}^k n^{\ell-2} = \sum_{j=1}^{\ell-1} a_{j,\ell}\, k^j$ from (12). Then

$$\begin{aligned}
C_N &:= \pi(1_S)\Big(\sum_{n=0}^{N}(n+1)^{\ell-2}\Big|\sum_{k=n+1}^{+\infty}\nu(R^{k-1}g_n)\Big|\Big)1_{\mathbb{X}}\\
&\le \pi(1_S)\Big(\sum_{n=0}^{+\infty}(n+1)^{\ell-2}\sum_{k=n+1}^{+\infty}\nu(R^{k-1}|g_n|)\Big)1_{\mathbb{X}}\\
&\le \pi(1_S)\Big(\sum_{n=0}^{+\infty}(n+1)^{\ell-2}\sum_{k=n+1}^{+\infty}\nu(R^{k-1}V_\ell)\Big)1_{\mathbb{X}} \;= \pi(1_S)\Big(\sum_{k=1}^{+\infty}\nu(R^{k-1}V_\ell)\sum_{n=1}^{k}n^{\ell-2}\Big)1_{\mathbb{X}}\\
&\le \pi(1_S)\Big(\sum_{j=1}^{\ell-1}a_{j,\ell}\sum_{k=1}^{+\infty}k^j\,\nu(R^{k-1}V_\ell)\Big)1_{\mathbb{X}}\\
&\le \pi(1_S)\nu(V_0)\Big(\sum_{j=1}^{\ell-1}a_{j,\ell}D_j\Big)1_{\mathbb{X}} = \pi(1_S)\,\nu(V_0)E_{\ell-1}1_{\mathbb{X}}
\end{aligned} \tag{34}$$

using (9b) (note that $|g_n| \le V_\ell \le V_{j+1}$ for $j=1,\dots,\ell-1$) and the definition of $E_{\ell-1}$ in (13).

From the triangular inequality applied to (31), we obtain that

$$\sum_{n=0}^{N}(n+1)^{\ell-2}|P^n g_n| \le A_N + B_N + C_N.$$

Therefore Inequality (28) follows from (32)-(34). The proof of Lemma 3.2 is complete. □

References

[AFV15] C. Andrieu, G. Fort, and M. Vihola. Quantitative convergence rates for subgeometric Markov chains. *J. Appl. Probab.*, 52(2):391–404, 2015.

[But14] O. Butkovsky. Subgeometric rates of convergence of Markov processes in the Wasserstein metric. *Ann. Appl. Probab.*, 24(2):526–552, 2014.

[DFM16] A. Durmus, G. Fort, and E. Moulines. Subgeometric rates of convergence in Wasserstein distance for Markov chains. *Ann. Inst. Henri Poincaré Probab. Stat.*, 52(4):1799–1822, 2016.

[DFMS04] R. Douc, G. Fort, E. Moulines, and P. Soulier. Practical drift conditions for subgeometric rates of convergence. *Ann. Appl. Probab.*, 14(3):1353–1377, 2004.

[DMPS18] R. Douc, E. Moulines, P. Priouret, and P. Soulier. *Markov chains.* Springer, 2018.

[DMS07] R. Douc, E. Moulines, and P. Soulier. Computable convergence rates for subgeometric ergodic Markov chains. *Bernoulli*, 13(3):831–848, 2007.

[FM00] G. Fort and E. Moulines. V-subgeometric ergodicity for a Hastings-Metropolis algorithm. *Statist. Probab. Lett.*, 49(4):401–410, 2000.

[FM03] G. Fort and E. Moulines. Polynomial ergodicity of Markov transition kernels. *Stochastic Process. Appl.*, 103(1):57–99, 2003.

[HL20] L. Hervé and J. Ledoux. V-geometrical ergodicity of Markov kernels via finite-rank approximations. *Electron. Commun. Probab.*, 25(23):1–12, March 2020.

[HL23a] L. Hervé and J. Ledoux. Computable bounds for solutions to Poisson's equation and perturbation of Markov kernels. hal-04259531, 2023.

[HL23b] L. Hervé and J. Ledoux. Rate of convergence of Nummelin-type representation of the invariant distribution of a Markov chain via the residual kernel. *Electron. Commun. Probab.*, 28:1–13, 2023. hal-04283397.

[JR02] S. F. Jarner and G. O. Roberts. Polynomial convergence rates of Markov chains. *Ann. Appl. Probab.*, 12(1):224 – 247, 2002.

[JT02] S. F. Jarner and R. L. Tweedie. Convergence rates and moments of Markov chains associated with the mean of Dirichlet processes. *Stochastic Process. Appl.*, 101(2):257–271, 2002.

[JT03] S. F. Jarner and R. L. Tweedie. Necessary conditions for geometric and polynomial ergodicity of random-walk-type. *Bernoulli*, 9(4):559 – 578, 2003.

[LH07] Y. Y. Liu and Z. T. Hou. Explicit convergence rates of the embedded $M/G/1$ queue. *Acta Math. Sin. (Engl. Ser.)*, 23(7):1289–1296, 2007.

[LH12] X. Li and Z. Hou. Subgeometric rates of convergence of the $GI/G/1$ queueing system. *Acta Math. Sci. Ser. B (Engl. Ed.)*, 32(5):1983–1996, 2012.

[MT09] S. P. Meyn and R. L. Tweedie. *Markov chains and stochastic stability.* Cambridge University Press, 2th edition, 2009.

[NT83] E. Nummelin and P. Tuominen. The rate of convergence in Orey's theorem for Harris recurrent Markov chains with applications to renewal theory. *Stochastic Process. Appl.*, 15(3):295–311, 1983.

[Num84] E. Nummelin. *General irreducible Markov chains and nonnegative operators.* Cambridge University Press, Cambridge, 1984.

[TT94] P. Tuominen and R. L. Tweedie. Subgeometric rates of convergence of f-ergodic Markov chains. *Adv. in Appl. Probab.*, 26(3):775–798, 1994.

A Construction of the Lyapunov functions V_i in Comment 2.6

Assume that P satisfies Condition (**S**) and that there exists a Lyapunov function V such that

$$\exists \alpha \in [0,1),\ \exists b_0, c > 0, \quad PV \le V - c\,V^{\alpha} + b_0\,1_S \tag{35}$$

with S given in (**S**). The construction of the Lyapunov functions V_i in $\boldsymbol{D}(V_0 : V_m)$ is based on the following fact. If W is a Lyapunov function and if $0 < \theta_2 < \theta_1 < 1$ are such that

$$\exists b, c > 0, \quad PW^{\theta_1} \le W^{\theta_1} - c\,W^{\theta_2} + b\,1_S,$$

$$\text{then} \quad \exists b', c' > 0, \quad PW^{\theta_2} \le W^{\theta_2} - c'\,W^{\theta_3} + b'\,1_S \qquad \text{with} \quad \theta_3 := 2\theta_2 - \theta_1. \tag{36}$$

Indeed we know from [JR02, Lem. 3.5] that

$$\forall \eta \in (0,1],\ \exists b_\eta, c_\eta > 0, \quad PW^{\eta\theta_1} \le W^{\eta\theta_1} - c_\eta\,(W^{\theta_1})^{\theta_2/\theta_1 + \eta - 1} + b_\eta 1_S.$$

Then (36) is obtained with $\eta := \theta_2/\theta_1 < 1$. Next note that $\alpha_1 = 1 - 1/m \le \alpha$, so that

$$PV \le V - c\,V^{\alpha_1} + b_0\,1_S \tag{37}$$

from (35). Of course we can replace c with $c_1 < 1$. Recall that $m := \lfloor (1-\alpha)^{-1} \rfloor$. Then:

- If $\alpha_1 = 0$, i.e. $m = 1$ or $\alpha \in [0, 1/2)$, then $\boldsymbol{D}(V_0 : V_1)$ holds with $V_0 := c_1^{-1} V \ge V_1 := 1_{\mathbb{X}}$.
- If $\alpha_1 = 1/2$, i.e. $m = 2$ or $\alpha \in [1/2, 2/3)$, then we deduce from (37) and Property (36) applied to $W := V, \theta_1 = 1, \theta_2 = \alpha_1$ that

$$\exists b_1, c_2 > 0, \quad PV^{\alpha_1} \le V^{\alpha_1} - c_2\,V^{\alpha_2} + b_1 1_S \tag{38}$$

  with $\alpha_2 := 2\alpha_1 - 1 = 0$. Again note that we can choose $c_2 < 1$. Then the procedure stops, and Conditions $\boldsymbol{D}(V_0 : V_2)$ hold with $V_0 := c_1^{-1} c_2^{-1} V \ge V_1 := c_2^{-1} V^{\alpha_1} \ge V_2 := 1_{\mathbb{X}}$.
- If $\alpha_1 > 1/2$, then Property (36) can be used recursively to provide inequalities of the form $PV^{\alpha_{i-1}} \le V^{\alpha_{i-1}} - c_i\,V^{\alpha_i} + b_{i-1} 1_S$ with $c_i < 1$ and $\alpha_i = 2\alpha_{i-1} - \alpha_{i-2} = (\alpha_1 - 1)\,i + 1$. Actually (36) can only be used until the value $i = m$ since $\alpha_m = 0$ and $\alpha_i < 0$ for $i > m$. Then Conditions $\boldsymbol{D}(V_0 : V_m)$ hold with V_i given in (26), where $a_i = [\prod_{k=i+1}^{m} c_k]^{-1}$.