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Multi-Resolution Analysis of Multiplicity d: Applications to Dyadic Interpolation

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This paper studies the Multi-Resolution Analyses of multiplicity d ($d \in \mathbb{N}^*$), that is, the families $(V_n)_{n\in\mathbb{Z}}$ of closed subspaces in $\mathbb{L}^2(\mathbb{R})$ such that $V_n \subset V_{n+1}, V_{n+1} = DV_n$, where $Df(x) = f(2x)$, and such that there exists a Riesz basis for V_0 of the form $\{\phi_i(\cdot - k), i = 1,\ldots,d, k \in \mathbb{Z}\}$, with $\phi_1,\ldots,\phi_d \in V_0$. Using the Fourier transform, we prove that $\hat{\Phi}(\lambda) = {}^t[\hat{\phi}_1(\lambda),\ldots,\hat{\phi}_d(\lambda)] = H(\lambda/2)\hat{\Phi}(\lambda/2)$, where H is in the set $\mathcal{M}_d$ of continuous 1-periodic functions taking values in $\mathcal{M}(d,\mathbb{C})$. If $d = 1$, the definition corresponds to the standard Multi-Resolution Analyses, and one can characterize the regular 1-periodic complex-valued functions H (called, then, scaling filters) which yield a Multi-Resolution Analysis. In this paper, we generalize this study to $d \geqslant 2$ by giving conditions on $H \in \mathcal{M}_d$ so that there exists $\hat{\Phi} = {}^t[\hat{\phi}_1,\ldots,\hat{\phi}_d]$ in $\mathbb{L}^2(\mathbb{R},\mathbb{C}^d)$ solution of $\hat{\Phi}(\lambda) = H(\lambda/2)\hat{\Phi}(\lambda/2)$, and so that the integer translates of $\phi_1,\ldots,\phi_d$ form a Riesz family. Then, the latter span the space V_0 of a Multi-Resolution Analysis of multiplicity d. We show that the conditions on H focus on the zeros of $\det H(\cdot)$ and on simple spectral hypotheses for the operator P_H defined on $\mathcal{M}_d$ by $P_HF(\lambda) = H(\lambda/2)F(\lambda/2)H(\lambda/2)^* + H(\lambda/2 + 1/2)F(\lambda/2 + 1/2)H(\lambda/2 + 1/2)^*$. Finally, we explore connections with the order r dyadic interpolation schemes, where $r \in \mathbb{N}^*$.

Contents.

1. INTRODUCTION

A Multi-Resolution Analysis is an increasing family, $\ldots \subset V_{n-1} \subset V_n \subset V_{n+1} \ldots, n \in \mathbb{Z}$, of closed subspaces of $\mathbb{L}^2(\mathbb{R})$ with the following properties: (a) $\cup_{n\in\mathbb{Z}} V_n$ is dense in $\mathbb{L}^2(\mathbb{R})$ and $\cap_{n\in\mathbb{Z}} V_n = \{0\}$; (b) $V_n = D^n V_0$, where $Df(x) = f(2x)$; (c) there is a function $g \in V_0$ (called a scaling function) such that $\{g(\cdot - k), k \in \mathbb{Z}\}$ forms a Riesz basis for V_0. Moreover, there exists, in the orthogonal complement of V_0 in V_1, a function ψ (called the wavelet) such that $\{2^{j/2}\psi(2^j \cdot - k), j,k \in \mathbb{Z}\}$ constitutes an orthonormal basis for $\mathbb{L}^2(\mathbb{R})$. The concepts of Multi-Resolution Analysis and wavelet basis have been introduced by Mallat [20] and Meyer [22]. The scaling function g satisfies $\hat{g}(\lambda) = m_0(\lambda/2)\hat{g}(\lambda/2)$ and $\hat{g}(\lambda) = \prod_{k=1}^{+\infty} m_0(\lambda/2^k)$, where m_0 is a 1-periodic complex-valued function, such that $m_0(0) = 1$. Here we use the following convention for the Fourier transform:

$$\hat{f}(\lambda) = \int_{\mathbb{R}} f(x)e^{-2i\pi\lambda x}dx.$$

The converse problem is to characterize the 1-periodic complex-valued functions H such that $\hat{\phi}(\lambda) = \prod_{k=1}^{+\infty} H(\lambda/2^k) \in \mathbb{L}^2(\mathbb{R})$, and such that the inverse Fourier transform of $\hat{\phi}$, ϕ, generates by integer translates an orthonormal family [5, 7], and more generally a Riesz family [15, 24]. Then H is called a scaling filter: under a mild additional hypothesis, the space spanned by the $\phi(\cdot - k)$ constitutes the set V_0 of a Multi-Resolution Analysis (with $g = \phi$ and $m_0 = H$). Recall that, in order to characterize the scaling filters, the above cited papers use the operator P defined by

$$Pf(\lambda) = \left|H\left(\frac{\lambda}{2}\right)\right|^2 f\left(\frac{\lambda}{2}\right) + \left|H\left(\frac{\lambda}{2} + \frac{1}{2}\right)\right|^2 f\left(\frac{\lambda}{2} + \frac{1}{2}\right),$$

where f is a complex-valued continuous function defined

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on [0, 1]. The operator P has been introduced in the theory of Wavelets in [6, 7].

A *Multi-Resolution Analysis* $(V_n)_{n\in\mathbb{Z}}$ of multiplicity $d \geqslant 2$ is an increasing family of closed subspaces of $\mathbb{L}^2(\mathbb{R})$ satisfying the above statements (a) (b), and the following condition: there are d functions $g_1, \dots, g_d$ (called scaling functions) such that $\{g_i(\cdot - k), k \in \mathbb{Z}, i = 1, \dots, d\}$ is a Riesz basis for V_0. We can construct the corresponding wavelets, functions $\psi_1, \dots, \psi_d$, such that the family $\{2^{j/2}\psi_i(2^j \cdot -k), j, k \in \mathbb{Z}, i = 1, \dots, d\}$ forms an orthonormal basis for $\mathbb{L}^2(\mathbb{R})$ (see [12]). The standard polynomial interpolation spaces provide examples of Multi-Resolution Analyses of multiplicity $d \geqslant 2$ [12, 1]. These examples generalize the ones obtained for $d = 1$ with splines [22]. The above notions are developed in Section 2.

The main purpose of this paper is to construct Multi-Resolution Analyses of multiplicity $d \geqslant 2$ by extending the method of scaling filters. First let $g_1, \dots, g_d$ be d scaling functions of a Multi-Resolution Analysis of multiplicity $d \geqslant 2$. We prove in Section 3 that $g_1, \dots, g_d$ satisfy the scaling matrix equation

$$G(x) = \sum_{k\in\mathbb{Z}} M_k G(2x + k),$$

where $G(x) = {}^t[g_1(x), \dots, g_d(x)]$, and the M_k are $d \times d$ matrices. Using the Fourier transform, we obtain $\hat{G}(\lambda) = M(\lambda/2)\hat{G}(\lambda/2)$, and by induction

$$\hat{G}(\lambda) = M\left(\frac{\lambda}{2}\right) M\left(\frac{\lambda}{4}\right) \cdots M\left(\frac{\lambda}{2^n}\right) \hat{G}\left(\frac{\lambda}{2^n}\right), \forall n \in \mathbb{N}*,$$

where $\hat{G}(\lambda) = {}^t[\hat{g}_1(\lambda), \dots, \hat{g}_d(\lambda)]$ and $M(\lambda) = (1/2)\sum_{k\in\mathbb{Z}} e^{2i\pi\lambda k} M_k$.

Conversely, let H be a 1-periodic regular function, taking values in $\mathcal{M}(d, \mathbb{C})$. In this work, we give conditions on H so that it satisfies the following properties:

(A) For all $\lambda \in \mathbb{R}$, the matrix sequence $\{H(\lambda/2)H(\lambda/4) \cdots H(\lambda/2^n), n \geqslant 1\}$ converges.

(B) There exists a vector $\vec{x} \in \mathbb{R}^d$ such that

$$\begin{aligned}\hat{\Phi}(\lambda) &= {}^t\left[\hat{\phi}_1(\lambda), \dots, \hat{\phi}_d(\lambda)\right] \\ &= \lim_{n\to+\infty} H\left(\frac{\lambda}{2}\right) H\left(\frac{\lambda}{4}\right) \cdots H\left(\frac{\lambda}{2^n}\right) \vec{x} \quad (*)\end{aligned}$$

is in $\mathbb{L}^2(\mathbb{R}, \mathbb{C}^d)$, and such that the integer translates of the inverse Fourier transforms $\phi_1, \dots, \phi_d$ of $\hat{\phi}_1, \dots, \hat{\phi}_d$ form a Riesz family.

If (A) and (B) hold, H is called a scaling $d \times d$ matrix filter. In that case, under mild additional assumptions, if V_0 denotes the space spanned by the integer translates of $\phi_1, \dots, \phi_d$, then the family $\{V_n = D^n V_0, n \in \mathbb{Z}\}$ constitutes a Multi-Resolution Analysis of multiplicity d (with $g_i = \phi_i$ and $M = H$).

Suppose that H is Holderian—the d^2 complex-valued functions given by the coefficients of H are uniformly Holderian. Then, we prove in Section 5.1 that *a sufficient condition for H to verify (A) is that $H(0) = \mathrm{diag}(1, \mu_2, \dots, \mu_d)$ with $|\mu_i| < 1$ for $i = 2, \dots, d$.*

The conditions for (B) are more difficult, and depend on spectral properties of the operator P_H defined by

$$P_H F(\lambda) = H\left(\frac{\lambda}{2}\right) F\left(\frac{\lambda}{2}\right) H\left(\frac{\lambda}{2}\right)^* + H\left(\frac{\lambda}{2} + \frac{1}{2}\right) F\left(\frac{\lambda}{2} + \frac{1}{2}\right) H\left(\frac{\lambda}{2} + \frac{1}{2}\right)^*.$$

Here F is a continuous function defined on [0, 1] and taking values in $\mathcal{M}(d, \mathbb{C})$. Of course, if $d = 1$, we have $P_H = P$. The use of P_H for (A) (B) has been developed in [14]. However, a recent result of Hennion [13] allowed us to clarify the spectral study of P_H (Section 4), and consequently to simplify the conditions given in [14] (Section 5). In particular, if H is Holderian, then P_H acts on the space of Holderian matrix functions, and the spectral radius ρ of P_H (on this space) is an eigenvalue of finite index, $\nu(\rho)$. On the same way, if H is of finite length—$H(\lambda) = \sum_{k=p}^{q} e^{2i\pi k\lambda} H_k$ where H_k are $d\times d$ matrices—then P_H acts on a finite dimensional space. In that case, we denote by P_N the matrix obtained from the restriction of P_H to this space, by ρ_N the greatest positive eigenvalue of P_N, and by ν_N the index of ρ_N.

Suppose that H is Holderian and satisfies $H(0) = \mathrm{diag}(1, \mu_2, \dots, \mu_d)$ with $|\mu_i| < 1$ for $i = 2, \dots, d$, and $H(1/2)^* \vec{e}_1 = \vec{0}$. Let us consider the functions $\hat{\phi}_1, \dots, \hat{\phi}_d$ defined by $(*)$ with $\vec{x} = \vec{e}_1$. The following properties are proved in Section 5:

A necessary condition for H to be a scaling matrix filter is that $\rho = 1$ and $\nu(\rho) = 1$. Conversely, if $\rho = 1$ and $\nu(\rho) = 1$, then $\hat{\phi}_1, \dots, \hat{\phi}_d \in \mathbb{L}^2(\mathbb{R})$. When H is of finite length, the same statements hold with ρ_N, ν_N instead of $\rho, \nu(\rho)$, and in addition, $\phi_1, \dots, \phi_d$ are compactly supported. The Riesz family property in (B) is satisfied under additional hypotheses which are similar to those obtained for $d = 1$ in [7, 15]. If det $H(\cdot)$ has a finite number of zeros, these hypotheses are very simple.

The Multi-Resolution Analyses of multiplicity $d \geqslant 2$ provide a theoretical framework for order r dyadic interpolation schemes, where $r \in \mathbb{N}^*$. The latter involves constructing, from any family of scalars $\{a_j(k), k \in \mathbb{Z}, j = 0, \dots, r\}$, a function f defined on all the dyadic points of $\mathbb{R}$, and which admits an extension $\tilde{f}$ of class $\mathscr{C}^r$ on $\mathbb{R}$, such that $\tilde{f}^{(j)}(k) = a_j(k)$, for all $k \in \mathbb{Z}$, and all $j \in \{0, \dots, r\}$ (see [21]). Section 6 studies the connection between order r dyadic interpolation schemes and Multi-Resolution Analyses of multiplicity $r + 1$.

For $d = 1$, the operator P is also used to estimate the regularity of the scaling function ϕ associated to a given scaling filter H (see, for instance, [8, 15, 24]). In section 7, we investigate this problem for $d \geqslant 2$ by calculating, for any scaling $d \times d$ matrix filter of finite length, the Sobolev integer coefficients of the scaling functions $\phi_1, \ldots, \phi_d$–that is $s \in \mathbb{N}^*$ such that

$$\int_{-\infty}^{+\infty} \left(1 + |\lambda|^{2s}\right) \times \left[\left|\hat{\phi}_1(\lambda)\right|^2 + \cdots + \left|\hat{\phi}_d(\lambda)\right|^2\right] d\lambda < +\infty.$$

2. DEFINITION AND EXAMPLES OF MULTI-RESOLUTION ANALYSES OF MULTIPLICITY d

Let $d \in \mathbb{N}^*$. We denote by $(\vec{e}_1, \ldots, \vec{e}_d)$ the canonical basis for $\mathbb{C}^d$, by $\langle , \rangle$ the usual Hermitian product on $\mathbb{C}^d$, and by $|| \cdot ||_2$ the associated Hermitian norm. Let $\mathcal{M}(d, \mathbb{C})$ be the space of $d \times d$ complex matrices, and let I_d be the idendity matrix. For any matrix A, we write A^* for the adjoint matrix of A. If A, B are two $d \times d$ Hermitian matrices such that $\langle A\vec{x}, \vec{x}\rangle \leqslant \langle B\vec{x}, \vec{x}\rangle$ for all $\vec{x} \in \mathbb{C}^d$, we use the standard notation $A \leqslant B$.

If $(E, || \cdot ||_E)$ is a Hilbert space, recall that a countable family $\{f_i, i \in I\}$ of vectors in E is a Riesz family if, for all $(c_i)_{i \in I} \in l^2(I)$,

$$\frac{1}{C} \sum_{i \in I} |c_i|^2 \leqslant \|\sum_{i \in I} c_i f_i\|_E^2 \leqslant C \sum_{i \in I} |c_i|^2,$$

where $C > 0$ is a constant independant of the c_i. If the vectors f_i span E, we say that $\{f_i, i \in I\}$ is a Riesz basis for E.

DEFINITION. A family $(V_n)_{n \in Z}$ of closed subspaces of $\mathbb{L}^2(\mathbb{R})$ is called a *Multi-Resolution Analysis of multiplicity d* if it satisfies the following properties:

1. $\bigcap_{n \in \mathbb{Z}} V_n = \{\vec{0}\}$ and $\overline{\bigcup_{n \in \mathbb{Z}} V_n} = \mathbb{L}^2(\mathbb{R})$.
2. $V_n \subset V_{n+1}$.
3. $V_{n+1} = DV_n$ where $Df(x) = f(2x)$.
4. There exist in V_0 d functions $g_1, \ldots, g_d$ (called scaling functions) such that the family $\{g_i(\cdot - k), k \in \mathbb{Z}, i = 1, \ldots, d\}$ forms a Riesz basis for V_0.

EXAMPLE 1. Let $(V_n)_{n \in \mathbb{Z}}$ be a Multi-Resolution Analysis of multiplicity 1, and let ϕ and ψ be, respectively, the scaling function and the wavelet. Define $\mathcal{V}_0 = V_1$, and $\mathcal{V}_n = D^n \mathcal{V}_0$ for all $n \in \mathbb{Z}$. We know that $\{\phi(\cdot - k), \psi(\cdot - k), k \in \mathbb{Z}\}$ forms a Riesz basis for $\mathcal{V}_0$. Therefore the family $(\mathcal{V}_n)_{n \in \mathbb{Z}}$ is a Multi-Resolution Analysis of multiplicity 2.

Of course we may choose another basis for $\mathcal{V}_0$. For instance if $(V_n)_{n \in \mathbb{Z}}$ is the Multi-Resolution Analysis of multiplicity 1 with respect to the quadratic splines [22], then $\mathcal{V}_0$ also admits the Riesz basis $\{g_i(\cdot - k), k \in \mathbb{Z}, i = 0, 1\}$ where

$$g_i(x) = p_i(x) 1_{[0,1]}(x) + (-1)^i p_i(-x) 1_{[-1,0]}(x), \quad i = 0, 1,$$

with

$$p_0(x) = (1 - 2x^2)\, 1_{[0,1/2]}(x) + 2(x-1)^2\, 1_{]1/2,1]}(x),$$
$$p_1(x) = x\left(1 - \frac{3}{2}x\right) 1_{[0,1/2]}(x) + \frac{1}{2}(x-1)^2\, 1_{]1/2,1]}(x).$$

Note that $g_i^{(j)}(k) = \delta_{i,j}\delta_{0,k}$, for all $i, j \in \{0, 1\}$ and $k \in \mathbb{Z}$, where δ denotes the usual Kronecker's symbol. Thus every function f in $\mathcal{V}_0$ can be expressed as the sum

$$f(x) = \sum_{k \in \mathbb{Z}} f(k) g_0(x-k) + \sum_{k \in \mathbb{Z}} f'(k) g_1(x-k).$$

EXAMPLE 2. HERMITE INTERPOLATION. Given an integer $r \geqslant 1, E_r$ denotes the space of functions of class $\mathscr{C}^{r-1}$ on $\mathbb{R}$, whose restriction to every interval $[k, k+1], k \in \mathbb{Z}$, coincides with a polynomial function of degree $\leqslant 2r - 1$.

If $f \in E_r$, then $f(\cdot/2) \in E_r$. Otherwise, for all $i \in \{0, \ldots, r-1\}$, there exists an unique function g_i in E_r such that $g_i^{(j)}(k) = \delta_{i,j}\delta_{0,k}$ for all $j \in \{0, \ldots, r-1\}$ and all $k \in \mathbb{Z}$. Every function f in E_r can be written as

$$f(x) = \sum_{j=0}^{r-1} \sum_{k \in \mathbb{Z}} f^{(j)}(k) g_j(x-k).$$

Using an idea of Auscher [3], let us prove that the integer translates of $g_0, \ldots, g_{r-1}$ constitute a Riesz basis for $V_0(r) = E_r \cap \mathbb{L}^2(\mathbb{R})$: due to a classical argument on equivalence of norms in a finite dimensional space, there is a contant $c > 0$ such that, for all $f \in E_r$ and all $k \in \mathbb{Z}$,

$$\frac{1}{c} \sum_{i=0}^{r-1} \left(\left|f^{(i)}(k)\right|^2 + \left|f^{(i)}(k+1)\right|^2\right) \leqslant \int_k^{k+1} |f(x)|^2\, dx \leqslant c \sum_{i=0}^{r-1} \left(\left|f^{(i)}(k)\right|^2 + \left|f^{(i)}(k+1)\right|^2\right).$$

Thus

$$\frac{2}{c} \sum_{i=0}^{r-1} \sum_{k \in \mathbb{Z}} \left|f^{(i)}(k)\right|^2 \leqslant \int_{\mathbb{R}} |f(x)|^2\, dx \leqslant 2c \sum_{i=0}^{r-1} \sum_{k \in \mathbb{Z}} \left|f^{(i)}(k)\right|^2.$$

Consequently the family $\{V_n(r) = D^n(V_0(r)), n \in \mathbb{Z}\}$ forms a Multi-Resolution Analysis of multiplicity r.

Construction of Wavelets. Let $(V_n)_{n \in \mathbb{Z}}$ be a Multi-Resolution Analysis of multiplicity d. We denote by $W_n, n \in \mathbb{Z}$, the orthogonal complement of V_n in V_{n+1}. The definition of Multi-Resolution Analysis yields $\mathbb{L}^2(\mathbb{R}) = \oplus_{n\in\mathbb{Z}} W_n$. The following result is proved in [12].

THEOREM 2.1. *There exist d functions $\psi_1, \dots, \psi_d$ in W_0 such that*

1. *$\{\psi_i(\cdot - k), i = 1, \dots, d, k \in \mathbb{Z}\}$ forms an orthonormal basis for W_0.*
2. *$\{2^{j/2}\psi_i(2^j \cdot -k), i = 1, \dots, d, j, k \in \mathbb{Z}\}$ forms an orthonormal basis for $\mathbb{L}^2(\mathbb{R})$.*

In general, the scaling functions g_i and the wavelets ψ_i have the same regularity and the same localization (see [18]). The converse problem, that is to construct d scaling functions from d functions $\psi_1, \dots, \psi_d$ satisfying statement 2 of Theorem 2.1, is studied in [2, 19].

3. SCALING MATRIX FILTER

In this work, for $F = {}^t[f_1, \dots, f_d] \in L^2(\mathbb{R}, \mathbb{C}^d)$, we set $\hat{F} = {}^t[\hat{f}_1, \dots, \hat{f}_d]$. Let $\{V_n, n \in \mathbb{Z}\}$ be a Multi-Resolution Analysis of multiplicity d, and let $g_1, \dots, g_d$ be d scaling functions. From $V_0 \subset V_1$, it follows that there exists a family of complexes $\{m_{i,j}(n), i, j = 1, \dots, d; n \in \mathbb{Z}\}$ in $(l^2(\mathbb{Z}))^{d^2}$ such that

$$g_i(x) = \sum_{j=1}^{d} \sum_{k\in\mathbb{Z}} m_{i,j}(k) g_j(2x + k), \quad \forall i = 1, \dots, d. \tag{1}$$

For $k \in \mathbb{Z}$, define the $d \times d$ matrix: $M_k = (m_{i,j}(k))_{i,j=1,\dots,d}$. Equation (1) becomes

$$G(x) = \sum_{k\in\mathbb{Z}} M_k G(2x + k), \tag{2}$$

where $G = {}^t[g_1, \dots, g_2]$. Using the Fourier transform, this can be rewritten as

$$\hat{G}(\lambda) = M\left(\frac{\lambda}{2}\right) \hat{G}\left(\frac{\lambda}{2}\right), \tag{3}$$

where

$$M(\lambda) = \frac{1}{2} \sum_{k\in\mathbb{Z}} e^{2i\pi\lambda k} M_k.$$

It follows that

$$\hat{G}(\lambda) = M\left(\frac{\lambda}{2}\right) M\left(\frac{\lambda}{4}\right) \cdots M\left(\frac{\lambda}{2^n}\right) \hat{G}\left(\frac{\lambda}{2^n}\right), \quad \forall n \geq 1.$$

Suppose that $\hat{G}$ is continuous at 0, and that, for all $\lambda \in \mathbb{R}$, the matrix sequence $\{M(\lambda/2) \cdots M(\lambda/2^n), n \geq 1\}$ converges to a $d \times d$ matrix $M_\infty(\lambda)$. Then $\hat{G}(\lambda) = M_\infty(\lambda)\hat{G}(0)$. More generally, for all $\vec{x} \in \mathbb{C}^d$, the $\mathbb{C}^d$-valued function $M_\infty(\cdot)\vec{x}$ is solution of (3).

Conversely, let H be a continuous 1-periodic function, defined on $\mathbb{R}$ and taking values in $\mathcal{M}(d, \mathbb{C})$. Suppose that H satisfies the following properties:

(P1) For all $\lambda \in \mathbb{R}$, the matrix sequence $(H(\lambda/2) \cdots H(\lambda/2^n))_{n\geq 1}$ converges to a $d \times d$ matrix $\Pi_\infty(\lambda)$, and Π_∞ is continuous on $\mathbb{R}$.

(P2) There exists a vector $\vec{x} \in \mathbb{C}^d$ such that the d functions $\hat{\phi}_1, \dots, \hat{\phi}_d$, defined by

$$\hat{\Phi}(\lambda) = \begin{pmatrix} \hat{\phi}_1(\lambda) \\ \vdots \\ \hat{\phi}_d(\lambda) \end{pmatrix} = \Pi_\infty(\lambda)\vec{x}, \quad \lambda \in \mathbb{R}, \tag{4}$$

belong to $\mathbb{L}^2(\mathbb{R})$, and such that $\{\phi_i(\cdot - k), i = 1, \dots, d, k \in \mathbb{Z}\}$ forms a Riesz family, where $\phi_1, \dots, \phi_d$ are the inverse Fourier transforms of $\hat{\phi}_1, \dots, \hat{\phi}_d$.

Then, the space $V_0 = \overline{\text{linear span}\{\phi_i(\cdot - k), i = 1, \dots, d, k \in \mathbb{Z}\}}$ clearly verifies $V_0 \subset DV_0$, and $\{V_n = D^n V_0, n \in \mathbb{Z}\}$ satisfies axioms 2, 3, and 4 of the Multi-Resolution Analysis definition (with $g_i = \phi_i$ and $G = H$). As for axiom 1, note that it holds if the ϕ_i are continuous and sufficiently localized (see Section 7.1).

DEFINITION. If (P1) and (P2) hold, we say that H is a scaling $d \times d$ matrix filter, and the functions $\phi_1, \dots, \phi_d$ defined in (P2) are called the scaling functions (with respect to H).

Remarks. For the following remarks, we suppose that (P1) holds, and that $\hat{\phi}_1, \dots, \hat{\phi}_d$ given by (4) are in $\mathbb{L}^2(\mathbb{R})$.

(a) we can define, in $\mathbb{L}^1([0, 1])$, the functions

$$a_{i,j}(\lambda) = \sum_{k\in\mathbb{Z}} \hat{\phi}_i(\lambda + k) \overline{\hat{\phi}_j(\lambda + k)}, \quad i, j = 1, \dots, d,$$

and

$$\Theta_\Phi(\lambda) = (a_{i,j}(\lambda))_{i,j=1,\dots,d} = \sum_{k\in\mathbb{Z}} \hat{\Phi}(\lambda + k) \hat{\Phi}(\lambda + k)^*. \tag{5}$$

Observe that $\Theta_\Phi(\lambda)$ is a non-negative Hermitian matrix, and that Θ_Φ is 1-periodic. The following property is proved in [12]: *$\{\phi_i(\cdot - k), i = 1, \dots, d, k \in \mathbb{Z}\}$ forms a Riesz family if and only if there is a constant $c > 0$ such that*

$$\frac{1}{c} I_d \leq \Theta_\Phi(\lambda) \leq c I_d, \quad \text{for almost all } \lambda \in \mathbb{R}. \tag{6}$$

(b) We have

$$\int_0^1 a_{i,j}(\lambda) e^{-2i\pi\ell\lambda} d\lambda = \left\langle \phi_i, \phi_j(\cdot - \ell) \right\rangle_{L^2(\mathbb{R})\times L^2(\mathbb{R})}, \quad \forall \ell \in \mathbb{Z}.$$

If $a_{i,j} \in \mathbb{L}^2([0,1])$, then

$$a_{i,j}(\lambda) = \sum_{\ell \in \mathbb{Z}} \left\langle \phi_i, \phi_j(\cdot - \ell) \right\rangle_{L^2(R)\times L^2(R)} e^{2i\pi\ell\lambda} \quad \text{a.e.} \qquad (7)$$

(c) It follows that $\{\phi_i(\cdot - k), i = 1, \ldots, d, k \in \mathbb{Z}\}$ forms an orthonormal family if and only if $\Theta_\Phi(\lambda) = I_d$ almost everywhere.

(d) If $\{\phi_i(\cdot - k), i = 1, \ldots, d, k \in \mathbb{Z}\}$ is a Riesz family, then the integer translates of $\phi_1^0, \ldots, \phi_d^0$, defined by ${}^t[\hat{\phi}_1^0(\lambda), \ldots, \hat{\phi}_d^0(\lambda)] = [\Theta_\Phi(\lambda)]^{-1/2}\hat{\Phi}(\lambda)$, form an orthonormal family.

(e) From $\hat{\phi}(\frac{\lambda}{2}) = H(\frac{\lambda}{2})\hat{\phi}(\frac{\lambda}{2})$, it follows that

$$\Theta_\Phi(\lambda) = \sum_{k\in\mathbb{Z}} H\left(\frac{\lambda}{2} + \frac{k}{2}\right) \hat{\Phi}\left(\frac{\lambda}{2} + \frac{k}{2}\right) \times \hat{\Phi}\left(\frac{\lambda}{2} + \frac{k}{2}\right)^* H\left(\frac{\lambda}{2} + \frac{k}{2}\right)^*.$$

Separating the even and odd indeces in the sum, we obtain

$$\Theta_\Phi(\lambda) = H\left(\frac{\lambda}{2}\right) \Theta_\Phi\left(\frac{\lambda}{2}\right) H\left(\frac{\lambda}{2}\right)^* + H\left(\frac{\lambda}{2} + \frac{1}{2}\right) \Theta_\Phi\left(\frac{\lambda}{2} + \frac{1}{2}\right) H\left(\frac{\lambda}{2} + \frac{1}{2}\right)^*. \qquad (8)$$

Necessary Conditions for H to Be a Scaling Matrix Filter

Let H be a continuous scaling $d \times d$ matrix filter. For the two following lemmas, we assume that $\hat{\Phi}$ is defined by (4) with $\vec{x} = \vec{e}_1$, and that Θ_Φ given by (5) is continuous. In the scalar case, H necessarily satisfies $H(1/2) = 0$ and $|H(\cdot)|^2 + |H(\cdot + 1/2)|^2 > 0$. Let us generalize these two properties to $d \geq 2$.

LEMMA 3.1. *The non-negative Hermitian matrix*

$$H(\lambda)H(\lambda)^* + H\left(\lambda + \frac{1}{2}\right) H\left(\lambda + \frac{1}{2}\right)^* \quad \textit{is definite for all } \lambda \in \mathbb{R}. \qquad (9)$$

Assume, in addition, that $H(0) = \mathrm{diag}(1, \mu_2, \ldots, \mu_d)$ with $|\mu_i| \leq 1$. From $\hat{\Phi}(0) = \Pi_\infty(0)\vec{e}_1 = \vec{e}_1$, it follows that $\hat{\phi}_1(0) = 1$, and $\hat{\phi}_i(0) = 0$ for $i = 2, \ldots, d$. Moreover:

LEMMA 3.2. *We have* $H(1/2)^*\vec{e}_1 = \vec{0}, \hat{\phi}_1(k) = \delta_{0,k}$ *for all* $k \in \mathbb{Z}$, *and lastly* $H(1/2)^*\vec{e}_i \neq \vec{0}$ *and* $\mu_i \neq 1$ *for every* $i = 2, \ldots, d$.

Proof of Lemma 3.1. By (6), there is a constant $c > 0$ such that

$$\frac{1}{c}\|\vec{x}\|_2^2 \leq \langle \Theta_\Phi(\lambda)\vec{x}, \vec{x}\rangle \leq c\|\vec{x}\|_2^2, \quad \forall\lambda \in \mathbb{R}, \forall\vec{x} \in \mathbb{C}^d.$$

Consequently, by (8), we obtain

$$\left\langle \Theta_\Phi\left(\frac{\lambda}{2}\right) H\left(\frac{\lambda}{2}\right)^* \vec{x}, H\left(\frac{\lambda}{2}\right)^* \vec{x} \right\rangle + \left\langle \Theta_\Phi\left(\frac{\lambda}{2} + \frac{1}{2}\right) H\left(\frac{\lambda}{2} + \frac{1}{2}\right)^* \vec{x}, H\left(\frac{\lambda}{2} + \frac{1}{2}\right)^* \vec{x} \right\rangle \geq \frac{1}{c}\|\vec{x}\|_2^2,$$

hence $\|H(\lambda)^*\vec{x}\|_2^2 + \|H(\lambda + 1/2)^*\vec{x}\|_2^2 \geq (1/c^2)\|\vec{x}\|_2^2$ for all $\lambda \in \mathbb{R}$. ■

Proof of Lemma 3.2. Applying identity (8) with $\lambda = 0$, we have

$$\langle \Theta_\Phi(0)\vec{e}_1, \vec{e}_1\rangle = \langle H(0)\Theta_\Phi(0)H(0)^*\vec{e}_1, \vec{e}_1\rangle + \left\langle H\left(\frac{1}{2}\right)\Theta_\Phi\left(\frac{1}{2}\right)H\left(\frac{1}{2}\right)^*\vec{e}_1, \vec{e}_1\right\rangle.$$

Since $H(0)^*\vec{e}_1 = \vec{e}_1$, we have $\langle\Theta_\Phi(1/2)H(1/2)^*\vec{e}_1, H(1/2)^*\vec{e}_1\rangle = 0$, thus $H(1/2)^*\vec{e}_1 = \vec{0}$. For $k \in \mathbb{Z}, k \neq 0$, let us set $k = 2^p(2l+1)$, where $p \in \mathbb{N}$ and $l \in \mathbb{Z}$. By iteration of $\hat{\phi}(\lambda) = H(\frac{\lambda}{2})\hat{\phi}(\frac{\lambda}{2})$, we can deduce that $\hat{\Phi}(k) = (H(0))^pH(1/2)\hat{\Phi}(l + 1/2)$. Since $H(1/2)^*\vec{e}_1 = \vec{0}$, we obtain $\hat{\phi}_1(k) = 0$.

Now let $i \in \{2, \ldots, d\}$. If $\mu_i = 1$, then $H(1/2)^*\vec{e}_i = \vec{0}$. Suppose that $H(1/2)^*\vec{e}_i = \vec{0}$. Then the above argument applied to the index i implies that $\hat{\phi}_i(k) = 0$ for all integer $k \neq 0$. But we also have $\hat{\phi}_i(0) = 0$. It follows that the column of index i of $\Theta_\Phi(0)$ is equal to zero, which is impossible because of (6). Hence $H(1/2)^*\vec{e}_i \neq \vec{0}$ for every $i \in \{2, \ldots, d\}$, and $\mu_i \neq 1$. ■

4. OPERATOR P_H

4.1. Definitions

Let $\mathcal{H}(d, \mathbb{C})$ be the set of $d \times d$ complex Hermitian matrices. We write $\mathcal{M}_d$ (respectively $\mathcal{H}_d$) for the space of 1-periodic continuous functions defined on $[0, 1]$ and taking values in $\mathcal{M}(d, \mathbb{C})$ (respectively in $\mathcal{H}(d, \mathbb{C})$). Denoting by $|\cdot|_2$ the matrix norm associated to $\|\cdot\|_2$, we define, on $\mathcal{M}_d$ and $\mathcal{H}_d$, the norm

$$\|F\|_\infty = \sup_{\lambda\in[0,1]} |F(\lambda)|_2.$$

We denote by I the function of $\mathcal{M}_d$ defined by $I(\lambda) = I_d$ for all $\lambda \in [0, 1]$. For F, G in $\mathcal{H}_d$, we say that $F \leq G$ if $F(\lambda) \leq G(\lambda)$ for all $\lambda \in [0, 1]$. In particular, $F \in \mathcal{H}_d$ is said to be non-negative (respectively positive) if, for all $\lambda \in [0, 1], F(\lambda)$ is a non-negative Hermitian matrix (respectively a non-negative and definite Hermitian matrix). Then we write $F \geq 0$ (respectively $F > 0$).

DEFINITION. Let H be a matrix function of $\mathcal{M}_d$. We define, for all $F \in \mathcal{M}_d$,

$$P_H F(\lambda) = H\left(\frac{\lambda}{2}\right) F\left(\frac{\lambda}{2}\right) H\left(\frac{\lambda}{2}\right)^* + H\left(\frac{\lambda}{2}+\frac{1}{2}\right) F\left(\frac{\lambda}{2}+\frac{1}{2}\right) H\left(\frac{\lambda}{2}+\frac{1}{2}\right)^*. \quad (10)$$

P_H is a well-defined bounded operator on $\mathcal{M}_d$ and $\mathcal{H}_d$, and it is positive on $\mathcal{H}_d$ (if $F \geqslant 0$ then $P_H F \geqslant 0$). Note that, if Θ_Φ is continuous, then identity (8) implies that Θ_Φ is P_H-invariant.

4.2. Spectral Study of P_H

• *Case H is α-Holderian.* Let α be a real such that $0 < \alpha \leqslant 1$. Let $\mathcal{M}_d^\alpha$ (respectively $\mathcal{H}_d^\alpha$) be the subspace of $\mathcal{M}_d$ (respectively of $\mathcal{H}_d$) of functions satisfying the following condition:

$$m(F) = \sup\left\{\frac{|F(\lambda') - F(\lambda)|_2}{|\lambda' - \lambda|^\alpha}, \lambda, \lambda' \in [0,1], \lambda \neq \lambda'\right\} < +\infty.$$

The spaces $\mathcal{M}_d^\alpha$ and $\mathcal{H}_d^\alpha$ are equipped with the norm

$$|||F||| = m(F) + \|F\|_\infty .$$

Observe that F belongs to $\mathcal{M}_d^\alpha$ if the d^2 scalar functions given by the coefficients of F are α-Holderian on $\mathbb{R}$ (they belong to $\mathcal{M}_1^\alpha$). It is proved in [13, 17] that the operator P defined in section 1 has, on $\mathcal{H}_1^\alpha$, remarkable spectral properties. We generalyse this study to $d \geqslant 2$.

Let $H \in \mathcal{M}_d^\alpha$. It is clear that P_H is a well-defined bounded operator on $\mathcal{H}_d^\alpha$. If η is an eigenvalue of P_H on $\mathcal{H}_d^\alpha$, and if $\mathrm{Ker}(P_H - \eta)^i = \mathrm{Ker}(P_H - \eta)^{i+1}$ for some $i \in \mathbb{N}^*$, we denote by

$$\nu(\eta) = \inf\left\{i \in \mathbb{N}^* : \mathrm{Ker}\,(P_H - \eta)^i = \mathrm{Ker}\,(P_H - \eta)^{i+1}\right\}$$

the index of η.

THEOREM 4.1. *The spectral radius ρ of P_H on $\mathcal{H}_d^\alpha$ is given by*

$$\rho = \lim_{n\to+\infty} \left(\|P_H^n I\|_\infty\right)^{1/n}. \quad (11)$$

More precisely, ρ is an eigenvalue of P_H on $\mathcal{H}_d^\alpha$ admitting a finite index, $\nu(\rho)$, and there exists a non-negative function Γ in $\mathcal{H}_d^\alpha$ (not identically equal to zero) such that $P_H\Gamma = \rho\Gamma$. The spectral values η of modulus ρ are finite in number, and they are eigenvalues of P_H on $\mathcal{H}_d^\alpha$ such that $\nu(\eta) \leqslant \nu(\rho)$ and dim $\mathrm{Ker}(P_H - \eta)^{\nu(\eta)} < +\infty$. *Moreover, we have the decomposition*

$$\mathcal{H}_d^\alpha = \left(\oplus_{|\eta|=\rho} \mathrm{Ker}\,(P_H - \eta)^{\nu(\eta)}\right) \oplus \mathcal{F},$$

where $\mathcal{F}$ is a subspace in $\mathcal{H}_d^\alpha$, stable under P_H, and such that the spectral radius of $P_{H|\mathcal{F}}$ is $< \rho$.

Theorem 4.1 for $d = 1$ is proved in [13, 17]. Replacing $\mathbb{C}, \mathbb{R}, \mathbb{R}_+$ respectively with $\mathcal{M}(d,\mathbb{C}), \mathcal{H}(d,\mathbb{C})$, and the subset of $\mathcal{H}(d,\mathbb{C})$ of non-negative Hermitian matrices, the proof for $d \geqslant 2$ is similar, and we only sketch it:

First, since P_H is positive on $\mathcal{H}_d$, the number ρ defined by (11) is the spectral radius of P_H on $\mathcal{H}_d$. Let ρ_α be the spectral radius of P_H on $\mathcal{H}_d^\alpha$. From $|||I||| = 1$ and $||P_H^n I||_\infty \leqslant |||P_H^n I|||$, it follows that $\rho \leqslant \rho_\alpha$. By induction we easily show that, for all $n \geqslant 1$ and $F \in \mathcal{H}_d^\alpha$,

$$P_H^n F(\lambda) = \sum_{k=0}^{2^n-1} H\left(\frac{\lambda+k}{2}\right) \cdots H\left(\frac{\lambda+k}{2^n}\right) F\left(\frac{\lambda+k}{2^n}\right) \times H\left(\frac{\lambda+k}{2^n}\right)^* \cdots H\left(\frac{\lambda+k}{2}\right)^*,$$

and

$$|||P_H^n F||| \leqslant 2^{-n\alpha} \|P_H^n I\|_\infty\, |||F||| + R_n \|F\|_\infty, \quad (12)$$

where R_n is a positive constant that only depends on n and H. Inequality (12) and the fact that $\lim_{n\to+\infty} [2^{-n\alpha}||P_H^n I||_\infty]^{1/n} = 2^{-\alpha}\rho < \rho_\alpha$ imply that P_H is a quasi-compact operator on $\mathcal{H}_d^\alpha$ (see [13]). Most of statements in Theorem 4.1 result from this property. In particular there exists an eigenvalue η_0 of P_H on $\mathcal{H}_d^\alpha$ such that $|\eta_0| = \rho_\alpha$. Hence $\rho = \rho_\alpha$. Moreover the spectral values $\eta > 2^{-\alpha}\rho$ of P_H on $\mathcal{H}_d^\alpha$ are finite in number, and they are in fact eigenvalues such that $\nu(\eta) < +\infty$ and dim $\mathrm{Ker}(P_H - \eta)^{\nu(\eta)} < +\infty$. The existence of $\mathcal{F}$ is also guaranteed by the quasi-compacity of P_H. The other properties, which are proved below in the polynomial case, result from the positivity of P_H.

• *Case H is a trigonometric polynomial.* We assume here that

$$H(\lambda) = \sum_{k=p}^{q} e^{-2i\pi k\lambda} H_k, \quad (13)$$

where $p < q$ are two integers, and the H_k are matrices of $\mathcal{M}(d,\mathbb{C})$. All trigonometric polynomial F, with coefficients in $\mathcal{M}(d,\mathbb{C})$, can be expressed as

$$F(\lambda) = \sum_{k\in\mathbb{Z}} e^{4i\pi k\lambda} M(2k) + e^{2i\pi\lambda} \sum_{k\in\mathbb{Z}} e^{4i\pi k\lambda} M(2k+1) = F_0(2\lambda) + e^{2i\pi\lambda} F_1(2\lambda).$$

In particular, we have $H(\lambda) = H_0(2\lambda) + e^{2i\pi\lambda} H_1(2\lambda)$, and an easy computation gives

$$P_H F(\lambda) = 2\,[H_0(\lambda)F_0(\lambda)H_0(\lambda)^* + H_0(\lambda)F_1(\lambda)H_1(\lambda)^* + H_1(\lambda)F_0(\lambda)H_1(\lambda)^* + e^{2i\pi\lambda}H_1(\lambda)F_1(\lambda)H_0(\lambda)^*],$$

which proves that the set of trigonometric polynomials is stable under P_H.

More precisely, let $N = q - p$, and define, in $\mathscr{H}_d$, the subspace $\mathscr{T}_d^N$ of matrix functions F written as

$$F(\lambda) = \sum_{k=-N}^{N} e^{2i\pi\lambda k} M_k, \qquad M_k \in \mathcal{M}(d, \mathbb{C}).$$

We easily prove that $\mathscr{T}_d^N$ is stable under P_H. Define the operator

$$P_N = P_{H|\mathscr{T}_d^N}, \tag{14}$$

(which can be considered as a $L \times L$ matrix with $L = (2N+1)d(d+1)/2$). For every eigenvalue η of P_N, we denote by $\nu_N(\eta)$ the index of η—the smallest integer $l \geqslant 1$ such that $\mathrm{Ker}(P_N - \eta)^l = \mathrm{Ker}(P_N - \eta)^{l+1}$. Then

THEOREM 4.2. *The spectral radius ρ_N of P_N is equal to ρ, and it is the largest positive eigenvalue of P_N. For every eigenvalue η_0 of modulus ρ_N, we have $\nu_N(\eta_0) \leqslant \nu_N(\rho_N)$. Finally, there exists a matrix function $\Gamma \geqslant 0$ in $\mathscr{T}_d^N$ such that $P_N\Gamma = \rho_N\Gamma$.*

Proof. The space $\mathscr{T}_d^N$ being equipped with the norm $\|\cdot\|_\infty$, we denote by $|\cdot|_\infty$ the associated operator norm. Because $I \in \mathscr{T}_d^N$ and P, P_N are positive operators, we obtain $|P_N^n|_\infty = \|P_N^n I\|_\infty = \|P_H^n I\|_\infty$, hence $\rho_N = \rho$.

Let η_0 be an eigenvalue of P_N such that $|\eta_0| = \rho_N$, and consider a non-increasing sequence $(t_n)_{n\geqslant 1}$ of reals such that $\lim_{n\to+\infty} t_n = 1$. Define

$$\Gamma_n = (\rho_N t_n - P_N)^{-1} I.$$

Since $(\beta - P_N)^{-1} = \sum_{k\geqslant 0} \beta^{-(k+1)} P_N^k$ for $|\beta| > \rho_N$, it follows that

$$-\|G\|_\infty \Gamma_n(\lambda) \leqslant (\eta_0 t_n - P_N)^{-1} G(\lambda) \leqslant \|G\|_\infty \Gamma_n(\lambda), \qquad \forall \lambda \in [0,1], \quad \forall G \in \mathscr{T}_d^N.$$

This yields

$$\left\|(\eta_0 t_n - P_N)^{-1} G\right\|_\infty \leqslant \|G\|_\infty \|\Gamma_n\|_\infty.$$

We have $\lim_{n\to+\infty} |(\eta_0 t_n - P_N)^{-1}|_\infty = +\infty$, because η_0 is an eigenvalue. Hence $\lim_{n\to+\infty} \|\Gamma_n\|_\infty = +\infty$. Since the sequence $\{\|\Gamma_n\|_\infty^{-1}\Gamma_n, n \geqslant 1\}$ is uniformly bounded in $\mathscr{T}_d^N$, and $\dim \mathscr{T}_d^N < +\infty$, we may pass to the limit and obtain a function $\Gamma \in \mathscr{T}_d^N$ satisfying $\Gamma \geqslant 0$ and $\|\Gamma\|_\infty = 1$. From $(\rho_N t_n - P_N)(\|\Gamma_n\|_\infty^{-1}\Gamma_n) = \|\Gamma_n\|_\infty^{-1} I$, it follows that $P_N\Gamma = \rho_N\Gamma$.

The index $\nu(\eta_0)$ is also defined by the condition

$$\lim_{n\to+\infty} (t_n - 1)^\ell \left|(\eta_0 t_n - P_N)^{-1}\right|_\infty = +\infty, \qquad \forall \ell = 0, \ldots, \nu(\eta_0) - 1.$$

Using the above inequality, we obtain that $\lim_{n\to+\infty} (t_n - 1)^l |(\rho_N t_n - P_N)^{-1}|_\infty = +\infty$ for $l = 0, \ldots, \nu(\eta_0) - 1$. But this means that $\nu_N(\rho_N) \geqslant \nu(\eta_0)$. ■

5. CHARACTERIZATION OF SCALING MATRIX FILTERS

5.1. *Infinite Matrix Product*

The following lemma, which is proved in Appendix A, provides a simple and general condition for (P1).

LEMMA 5.1. *Let H be a function of $\mathcal{M}_d^\alpha$, and assume that there is an invertible $d \times d$ matrix M such that $M^{-1}H(0)M = \mathrm{diag}(1, \mu_2, \ldots, \mu_d)$ with $|\mu_i| < 1$ or $\mu_i = 1$. Then the sequence of matrix functions $\{H(\cdot/2)\cdots H(\cdot/2^n), n \geqslant 1\}$ converges uniformly on all compact set of $\mathbb{R}$ to a matrix function Π_∞ that is continuous on $\mathbb{R}$. Moreover, if $\mu_i \neq 1$, then $\Pi_\infty(\lambda)M\vec{e}_i = \vec{0}$.*

Remarks. Let us consider $H \in \mathcal{M}_d^\alpha$ satisfying the assumptions of Lemma 5.1. Let $\vec{x} \in \mathbb{R}^d$, and define, by (4), $\hat{\Phi} = {}^t[\hat{\phi}_1, \ldots, \hat{\phi}_d]$. Note that $\hat{\Phi}$ is continuous on $\mathbb{R}$. In addition:

(a) *the growth of $\hat{\phi}_1, \ldots, \hat{\phi}_d$ is at most polynomial on $\mathbb{R}$.* Indeed we can write

$$\|\hat{\Phi}(\lambda)\|_2 \leqslant \prod_{k=1}^{n} \left|H\left(\frac{\lambda}{2^k}\right)\right|_2 \left\|\hat{\Phi}\left(\frac{\lambda}{2^n}\right)\right\|_2, \qquad \forall \lambda \in \mathbb{R}, \forall n \geqslant 1.$$

Define $M = \|H\|_\infty = \sup_{\lambda\in[0,1]} |H(\lambda)|_2$ and $c = \sup_{\lambda\in[-1,1]} \|\hat{\Phi}(\lambda)\|_2$ ($c < +\infty$ because $\lambda \to \hat{\Phi}(\lambda)$ is continuous). For fixed λ, consider the smallest integer $l(\lambda)$ such that $|\lambda/2^{l(\lambda)}| \leqslant 1/2$. The above inequality applied with $n = l(\lambda) - 1$ shows that $\|\hat{\Phi}(\lambda)\|_2 \leqslant cM^{l(\lambda)-1} \leqslant c'|\lambda|^{\log_2 M}$.

(b) If H satisfies (13), then *the distributions $\phi_1, \ldots, \phi_d$, defined as the inverse Fourier transforms of $\hat{\phi}_1, \ldots, \hat{\phi}_d$, have compact support in $[p, q]$.* Indeed let T be the map defined on $L^2(\mathbb{R}, \mathbb{C}^d)$ by

$$TF(x) = 2\sum_{n=p}^{q} H_n F(2x + n).$$

We obtain $\widehat{TF}(\lambda) = H(\lambda/2)\hat{F}(\lambda/2)$, and $\widehat{T^nF}(\lambda) = H(\lambda/2)$

$\cdots H(\lambda/2^n)\hat{F}(\lambda/2^n)$. Assume, for convenience, that $\vec{x} = \vec{e}_1$ in (4), and consider a continuous function F in $L^2(\mathbb{R}, \mathbb{C}^d)$, whose support is in $[p,q]$, and such that $\hat{F}$ is continuous with $\hat{F}(0) = \vec{e}_1$. First it is clear that TF, and more generally $T^nF, n \geq 1$, are compactly supported in $[p,q]$. Otherwise we have $\lim_{n\to+\infty}\widehat{T^nF}(\lambda) = \hat{\Phi}(\lambda)$. Consequently the sequence $(T^nF)_{n\geq 1}$ converges, in the sense of distributions, to the vector-valued function $\Phi = {}^t[\phi_1, \ldots, \phi_d]$, which has compact support in $[p,q]$.

5.2. *Characterization of Holderian Scaling Matrix Filters*

Let us first define periodic points, and extend to the vector case the notions of orbit and trajectory developed in [7, 17]. We consider the maps S_0 and S_1 defined from [0, 1] to [0, 1] by

$$S_i : \lambda \to \frac{1}{2}(\lambda + i), \quad i = 0, 1.$$

DEFINITIONS. Let $m \in \mathbb{N}^*$. We say that a real λ in [0, 1] is an *m-periodic point* if there exists a sequence of m elements $\sigma_1, \ldots, \sigma_m$ in $\{S_0, S_1\}$, such that $\sigma_m \cdots \sigma_1\lambda = \lambda$, and if m is the smallest integer for which this equality holds. The family $\{\sigma_1, \ldots, \sigma_m\}$ is then unique. Define

$$\mathscr{C}_\lambda = \{\sigma_k \cdots \sigma_1\lambda, k = 1, \ldots, m\}.$$

Remarks. The following properties are proved in [17].

(1) Let $m \in \mathbb{N}^*$. The p-periodic points, such that $p \leq m$, are the reals $k/(2^p - 1)$, where $p \in \{1, \ldots, m\}$ and $k \in \{0, 1, \ldots, 2^p - 1\}$.

(2) If $\lambda \in [0,1]$, we let $\tilde{\lambda} = \{\lambda + 1/2\} = \lambda + 1/2 - [\lambda + 1/2]$. Then λ and $\tilde{\lambda}$ cannot be simultaneously periodic points.

(3) If λ is not periodic, the reals $\sigma_n \cdots \sigma_1\lambda$, where $n \in \mathbb{N}^*$ and $\sigma_1, \ldots, \sigma_n \in \{S_0, S_1\}$, are mutually distinct, and are not periodic.

DEFINITIONS. Let H be a function in $\mathscr{M}_d$ satisfying (9), and consider $\lambda \in [0,1], \vec{v} \in \mathbb{C}^d, \vec{v} \neq \vec{0}$. Any subset of the form $\{\sigma_n \cdots \sigma_1\lambda, n \geq 1, \sigma_n \in \{S_0, S_1\}\}$, where $(\sigma_n)_{n\geq 1}$ is such that $H(\sigma_n \cdots \sigma_1\lambda)^* \cdots H(\sigma_2\sigma_1\lambda)^* H(\sigma_1\lambda)^*\vec{v} \neq \vec{0}$ for all $n \geq 1$, is called a *trajectory of* λ, *with respect to* H *and* $\vec{v}$. The *orbit of* λ, with respect to H and $\vec{v}$, is the closure of the set of all the trajectories of λ with respect to H and $\vec{v}$.

HYPOTHESIS (Z). Let $H \in \mathscr{M}_d$ satisfying (9). We shall say that H verifies condition (Z) if, for all $\vec{v} \in \mathbb{C}^d, \vec{v} \neq \vec{0}$, and all $\lambda \in [0,1]$, the orbit of λ, with respect to H and $\vec{v}$, contains 0.

Using the above remarks (2) and (3), we easily prove that a sufficient condition for (Z) is that det $H(\cdot)$ has a finite number, Q, of zeros, and that every m-periodic point λ, with $\lambda \neq 0$ and $m \leq Q$, verifies the following assumption:

$$\exists y \in \mathscr{C}_\lambda \text{ such that } \det H\left(y + \frac{1}{2}\right) \neq 0. \tag{15}$$

Resolution of (P2). Let α be a real such that $0 < \alpha \leq 1$. We assume here that $H \in \mathscr{M}_d^\alpha$, and that there exists an invertible $d \times d$ matrix M such that

$$M^{-1}H(0)M = \operatorname{diag}(1, \mu_2, \ldots, \mu_d), \quad \text{where } |\mu_i| < 1, \forall i \in \{2, \ldots, d\}, \tag{16}$$

and

$$H\left(\frac{1}{2}\right)^* (M^{-1})^* \vec{e}_1 = \vec{0}. \tag{17}$$

Note that the assumptions of Lemma 5.1 hold, and that (17) is a necessary condition for H to be a scaling $d \times d$ matrix filter (see Lemma 3.2, in which we assumed $M = I_d$). Let us recall that the operator P_H is defined by (10), and that its spectral radius ρ, given by (11), is an eigenvalue on $\mathscr{H}_d^\alpha$ which admits a finite index, $\nu(\rho)$.

THEOREM 5.2. *If $\rho = 1$ and $\nu(\rho) = 1$, then the functions $\hat{\phi}_1, \ldots, \hat{\phi}_d$ defined by* (4) *with $\vec{x} = M\vec{e}_1$, belong to $\mathbb{L}^2(\mathbb{R})$. Moreover, if Θ_Φ, defined by* (5), *is continuous, and if H verifies* (9) *and* (Z), *then $\{\phi_i(\cdot - k), k \in \mathbb{Z}, i = 1, \ldots, d\}$ is a Riesz family if, and only if,* det $\Theta_\Phi(0) \neq 0$.

Proof. Without loss of generality, we may suppose $M = I_d$ in (16) and (17). Let $F \in \mathscr{M}_d$. It is straightforward to check that

$$\int_0^1 (P_H F)(\lambda)\, d\lambda = 2\int_0^1 H(\lambda)F(\lambda)H(\lambda)^*\, d\lambda,$$

and by induction that

$$\int_0^1 (P_H^n F)(\lambda)\, d\lambda = 2^n \int_0^1 H(2^{n-1}\lambda) \cdots \times H(\lambda)F(\lambda)H(\lambda)^* \cdots H(2^{n-1}\lambda)^*\, d\lambda, \quad \forall n \geq 1.$$

Because H and F are periodic, we may replace $\int_0^1$ with $\int_{-1/2}^{1/2}$, and we conclude that

$$\int_{-2^{n-1}}^{2^{n-1}} \Pi_n(\lambda)F\left(\frac{\lambda}{2^n}\right)\Pi_n(\lambda)^*\, d\lambda = \int_0^1 (P_H^n F)(\lambda)\, d\lambda, \quad \forall n \geq 1, \tag{18}$$

where $\Pi_n(\lambda) = H(\lambda/2)\cdots H(\lambda/2^n)$. Since $\rho = 1$ and $\nu(\rho) = 1$, it results from theorem 4.1 that $M = \sup_{n\geq 1}\|P_H^n I\|_\infty < +\infty$. Taking $F = I$ in (18), we have

$$\int_{-2^{n-1}}^{2^{n-1}} \langle \Pi_n(\lambda)^* \vec{e}_i, \Pi_n(\lambda)^* \vec{e}_i\rangle\, d\lambda \leq M, \quad \forall i = 1,\dots,d,$$

and by Fatou's lemma,

$$\int_{\mathbb{R}} \langle \Pi_\infty(\lambda)^* \vec{e}_i, \Pi_\infty(\lambda)^* \vec{e}_i\rangle\, d\lambda \leq M,$$

where $\Pi_\infty(\lambda) = \lim_{n\to+\infty}\Pi_n(\lambda)$. Lemma 5.1 and (16) imply that $\Pi_\infty(\lambda)^*\vec{e}_i = \hat{\phi}_i(\lambda)\vec{e}_1$ for every $i \in \{1,\dots,d\}$. Thus $\hat{\phi}_1,\dots,\hat{\phi}_d \in \mathbb{L}^2(\mathbb{R})$.

Now let us prove the second statement. We know that Θ_Φ is a non-negative matrix function that is P_H-invariant, and we have to check that $\Theta_\Phi > 0$ (that is, $\Theta_\Phi(\lambda)$ is definite for all λ). Suppose that there exist $\lambda \in]0,1[$ and $\vec{v} \in \mathbb{C}^d, \vec{v} \neq \vec{0}$, such that $\Theta_\Phi(\lambda)\vec{v} = \vec{0}$. Then it results from (8) that $\det\Theta_\Phi(\cdot)$ vanishes on the orbit of λ with respect to H and $\vec{v}$. From (Z), we conclude that $\det\Theta_\Phi(0) = 0$. ■

Remarks. (a) We investigate in Section 7 the regularity of scaling functions.

(b) By Theorem 4.1, if $\rho = 1$ and $\nu(\rho) = 1$, then $M = \sup_{n\geq 1}\|P_H^n I\|_\infty < +\infty$. Conversely, suppose $M < +\infty$. Then we have $\rho \leq 1$. From (16) and (17), it follows that, for all $F \in \mathcal{M}_d, \langle P_H^n F(0)\vec{x}_1, \vec{x}_1\rangle = \langle F(0)\vec{x}_1, \vec{x}_1\rangle$, where $\vec{x}_1 = (M^{-1})^*\vec{e}_1$. Thus we have $\rho = 1$, and $\nu(\rho) = 1$. Consequently, if (16) and (17) hold, the conditions $[\rho = 1, \nu(\rho) = 1]$ and $\sup_{n\geq 1}\|P_H^n I\|_\infty < +\infty$ are equivalent.

(c) In particular, if H *satisfies*

$$H(\lambda)H(\lambda)^* + H\left(\lambda + \frac{1}{2}\right)H\left(\lambda + \frac{1}{2}\right)^* \leq Id, \qquad \forall\lambda \in \left[0, \frac{1}{2}\right], \tag{19}$$

then $\hat{\phi}_1,\dots,\hat{\phi}_d \in \mathbb{L}^2(\mathbb{R})$.

(d) Suppose that H is a scaling matrix filter. By (6), we have $\Theta_\Phi(\lambda) \geq cI_d$ almost everywhere, with $c > 0$. From (8), it follows that $(P_H^n I)(\lambda) \leq 1/c\Theta_\Phi(\lambda)$ a.e. Thus $\sup_{n\geq 1}\|P_H^n I\|_\infty < +\infty$. In particular, if (16) and (17) hold, the conditions $\rho = 1$ and $\nu(\rho) = 1$ are necessary for H to be a scaling matrix filter.

(e) Let $\hat{\phi}_1,\dots,\hat{\phi}_d$ be defined by (4). Suppose that $\hat{\phi}_1,\dots,\hat{\phi}_d \in \mathbb{L}^2(\mathbb{R})$, and that their inverse Fourier transforms $\phi_1\dots,\phi_d$ are such that

$$\sum_{j=0}^{d} |\phi_j(x)| \leq C(1 + |x|)^{-1-\varepsilon}, \quad \forall x \in \mathbb{R}, \tag{20}$$

where $c > 0$ and $\epsilon > 0$ are independent of x. Then the Fourier series

$$b_{i,j}(\lambda) = \sum_{\ell\in\mathbb{Z}} \left\langle \phi_i, \phi_j(\cdot - \ell)\right\rangle e^{2i\pi\ell\lambda}, \quad i,j = 1,\dots,d$$

are absolutely convergent, and from Poisson's formula, it follows that

$$\sum_{\ell\in\mathbb{Z}} \left\langle \phi_i, \phi_j(\cdot - \ell)\right\rangle e^{2i\pi\ell\lambda} = \sum_{k\in\mathbb{Z}} \hat{\phi}_i(\lambda + k)\overline{\hat{\phi}_j(\lambda + k)} \quad \text{a.e.}$$

Define $\Theta_0(\lambda) = [b_{i,j}(\lambda)]_{i,j=1,\dots,d}$. We have $\Theta_0 \in \mathcal{M}_d$, and $\Theta_0(\lambda) = \Theta_\Phi(\lambda)$ almost everywhere. Thus $\Theta_0 \geq 0$, and by (8), $P_H\Theta_0 = \Theta_0$. The Riesz family property in (P2) holds if and only if (6) is satisfied with Θ_0 instead of Θ_Φ. In particular, if the ϕ_i satify (20), then Theorem 5.2 holds with Θ_0 instead of Θ_Φ. Furthermore, we have $\Theta_\Phi(0) = \Theta_0(0)$; that is,

$$b_{i,j}(0) = \sum_{\ell\in\mathbb{Z}} \left\langle \phi_i, \phi_j(\cdot - \ell)\right\rangle = \sum_{k\in\mathbb{Z}} \hat{\phi}_i(k)\overline{\hat{\phi}_j(k)}, \qquad i,j = 1,\dots,d. \tag{21}$$

To see (21), observe that the functions $h_m(x) = \sum_{\ell\in\mathbb{Z}} \phi_m(x+\ell)$ are periodic, continuous, and that $b_{i,j}(0) = \int_0^1 h_i(x)\overline{h_j(x)}dx$. Since $(\hat{\phi}_m(k))_{k\in\mathbb{Z}}$ is the sequence of Fourier coefficients of h_m, we conclude by Parseval's identity.

5.3. *Scaling Matrix Filters of Finite Length*

We assume here that H is of the form $H(\lambda) = \sum_{k=p}^{q} e^{-2i\pi k\lambda} H_k$, where $p, q \in \mathbb{Z}, p < q$, and $H_k \in \mathcal{M}(d,\mathbb{C})$. Let $N = q - p$.

Let us denote by Q the number of zeros of $\det H(\cdot)$. We assume that every m-periodic point λ, with $\lambda \neq 0$ and $m \leq Q$, verifies (15), and that there exists a $d \times d$ invertible matrix M satisfying (16) and (17). Recall that P_N denotes the restriction of P_H to $\mathcal{T}_d^N$, that ρ_N is the largest positive eigenvalue of P_N, and $\nu_N = \nu_N(\rho_N)$ is the index of ρ_N (see Theorem 4.2).

THEOREM 5.3. *Let* $\hat{\phi}_1,\dots,\hat{\phi}_d$ *be defined by* (4) *with* $\vec{x} = M\vec{e}_1$. *A necessary and sufficient condition for H to be a scaling matrix filter is that* $\rho_N = 1, \nu_N = 1, H$ *verifies* (9) *and* $\det\Theta_\Phi(0) \neq 0$. *Then, the functions* $\phi_1,\dots,\phi_d$ *are compactly supported in* $[p,q]$.

Proof. First, observe that, if $\hat{\phi}_1,\dots,\hat{\phi}_d \in \mathbb{L}^2(\mathbb{R})$, then their inverse Fourier transforms $\phi_1,\dots,\phi_d$ are compactly supported in $[p,q]$ (see Section 5.1). From the above remark, it follows that $\Theta_0 = \Theta_\Phi$ almost everywhere, and that $\Theta_0(0) = \Theta_\Phi(0)$. By checking supports, we obtain $\langle\phi_i, \phi_j(\cdot - \ell)\rangle = 0$ if $|\ell| \geq N$. Thus $\Theta_0 \in \mathcal{T}_d^N$. Now let us prove Theorem 5.3:

If H is a scaling matrix filter, then we have (9), and $\Theta_0 \geqslant cI$, where $c > 0$. Thus $\det \Theta_\Phi(0) = \det \Theta_0(0) \neq 0$. By (8), we obtain $P_N\Theta_0 = \Theta_0$. We conclude that $\sup_{n\geqslant 1}||P_N^n I||_\infty < +\infty$, and using (16) and (17), that $\rho_N = 1, \nu_N = 1$ (see Remarks (d) and (b) above).

Conversely, if $\rho_N = 1$ and $\nu_N = 1$, then, applying Theorem 4.2 and the arguments used in the previous theorem (with P_N and Θ_0 instead of P_H and Θ_Φ), we prove that $\hat\phi_1, \ldots, \hat\phi_d \in \mathbb{L}^2(\mathbb{R})$, and that $\Theta_0 > 0$. ■

5.4. Examples

EXAMPLE 3. Let $b, c \in \mathbb{C}$ and $e, f \in \mathbb{R}$ such that $|e + f| < 1, \bar c + 2bf = 0, |b| \leqslant 1/2$ and $e^2 + f^2 \leqslant 1/2$. Let

$$H(\lambda) = \begin{pmatrix} \cos^2 \pi\lambda & b \sin 2\pi\lambda \\ c \sin 2\pi\lambda & e + f \cos 2\pi\lambda \end{pmatrix}.$$

H satisfies (16) and (17), with $M = I_d$, and (19) (by using the above assumptions on b, c, e, f). Therefore the functions $\hat\phi_1, \hat\phi_2$ given by (4) with $\vec x = \vec e_1$ are in $\mathbb{L}^2(\mathbb{R})$. For instance if $b = 1/2, c = 1/4, e = 1/2$, and $f = -1/4$, we have $\det H(\lambda) = 1/4\cos^2\pi\lambda$. Thus H satisfies (9) and (15). Otherwise we can show, by an approximation, that $\hat\phi_2(1) \neq 0$. Using Lemma 3.2, we conclude that $\det\Theta_\Phi(0) \neq 0$. It results from Theorem 5.3 that H is a scaling matrix filter.

Let $(\mathcal{V}_n)_{n\in\mathbb{Z}}$ be a Multi-Resolution Analysis of multiplicity 2 (we choose $d = 2$ for convenience). It is worth noticing that $\mathcal{V}_0$ may also constitute the set V_1 of a Multi-Resolution Analysis $(V_n)_{n\in\mathbb{Z}}$ of multiplicity 1. For instance, if we take in the previous example $b = c = 1/2$ and $e = -f = 1/2$, then the space $\mathcal{V}_0$, spanned by the integer translates of ϕ_1, ϕ_2, is the set V_1 of the scalar Multi-Resolution Analysis with respect to the Haar basis. More precisely, let $\phi_0 = 1_{[0,1]}$ and $\psi_0 = 1_{[0,1/2]} - 1_{[1/2,1]}$. The wavelet ψ_0 generates the Haar basis and ϕ_0 is the associated scaling function (the integer translates of these two functions form an orthonormal basis for V_1). Then an easy computation yields

$$\begin{pmatrix} \hat\phi_1(\lambda) \\ \hat\phi_2(\lambda) \end{pmatrix} = e^{-i\pi\lambda} R(\lambda) \begin{pmatrix} \hat\phi_0(\lambda) \\ \hat\psi_0(\lambda) \end{pmatrix},$$

where $R(\lambda)$ is the rotation by $\pi\lambda$. The matrix function Θ_Φ associated to ϕ_1, ϕ_2 verifies $\Theta_\Phi(\lambda) = R(\lambda)R(\lambda)^* = I_d$. Therefore the integer translates of ϕ_1 and ϕ_2 form an orthonormal basis for V_1.

EXAMPLE 4. Hermite Interpolation, $r = 2$ (See Example 2 of Section 2). For $i = 0, 1$, we obtain

$$\phi_i(x) = (-1)^i r_i(-x) 1_{[-1,0]}(x) + r_i(x) 1_{[0,1]}(x),$$

where $r_0(x) = (x-1)^2(2x+1)$ and $r_1(x) = (x-1)^2 x$. The Fourier transforms

$$\hat\phi_0(\lambda) = -12\left[(2\pi\lambda)^{-3}\sin 2\pi\lambda + 2(2\pi\lambda)^{-4}(\cos 2\pi\lambda - 1)\right]$$
$$\hat\phi_1(\lambda) = 4i\left[(2\pi\lambda)^{-3}(2 + \cos 2\pi\lambda) - 3(2\pi\lambda)^{-4}\sin 2\pi\lambda\right],$$

satisfy $\hat\phi(\lambda) = H(\frac{\lambda}{2})\hat\phi(\frac{\lambda}{2})$ with

$$H(\lambda) = \begin{pmatrix} \cos^2 \pi\lambda & -\frac{3}{4} i \sin 2\pi\lambda \\ \frac{i}{8} \sin 2\pi\lambda & \frac{1}{4} - \frac{1}{8}\cos 2\pi\lambda \end{pmatrix}.$$

Note that $\det H(\lambda) = 1/8\cos^4\pi\lambda$. Thus H verifies (9) and (15). Consequently, using Theorem 5.3, we again show that the integer translates of ϕ_0 and ϕ_1 form a Riesz basis for $V_0(2)$.

EXAMPLE 5. Hermite Interpolation, $r = 3$. We have, for $i = 0, 1, 2$,

$$\phi_i(x) = p_i(x) 1_{[0,1]}(x) + (-1)^i p_i(-x) 1_{[-1,0]}(x),$$

where

$$p_0(x) = -6x^5 + 15x^4 - 10x^3 + 1,$$
$$p_1(x) = -3x^5 + 8x^4 - 6x^3 + x,$$
$$p_2(x) = -\frac{1}{2}x^5 + \frac{3}{2}x^4 - \frac{3}{2}x^3 + \frac{1}{2}x^2.$$

The Fourier transforms $\hat\phi_0, \hat\phi_1, \hat\phi_2$ satisfy $\hat\phi(\lambda) = H(\frac{\lambda}{2})\hat\phi(\frac{\lambda}{2})$ with

$$H(\lambda) = \begin{pmatrix} \cos^2 \pi\lambda & -\frac{15}{16} i \sin 2\pi\lambda & 0 \\ \frac{5i}{32}\sin 2\pi\lambda & \frac{1}{4} - \frac{7}{32}\cos 2\pi\lambda & -\frac{3}{8} i \sin 2\pi\lambda \\ \frac{1}{64}\cos 2\pi\lambda & -\frac{i}{64}\sin 2\pi\lambda & \frac{1}{8} - \frac{1}{16}\cos 2\pi\lambda \end{pmatrix}.$$

Since $\det H(\lambda) = 5 \cdot 2^{-9}\cos^6\pi\lambda$, H verifies (9) and (15). Otherwise, we obtain

$$\Theta_\Phi(0) = \begin{pmatrix} 1 & 0 & \frac{1}{60} \\ 0 & a & 0 \\ \frac{1}{60} & 0 & \frac{144}{9!} \end{pmatrix}$$

with $a > 0$, hence $\det\Theta_\Phi(0) \neq 0$. Therefore we find again the fact that ϕ_0, ϕ_1, and ϕ_2 generate by integer translates a Riesz basis for $V_0(3)$.

6. ORDER *r* DYADIC INTERPOLATION

Consider a family $(G_s)_{s\in\mathbb{N}}$ of discrete subgroups of $\mathbb{R}^n, n \geqslant 1$, such that $G_s \subset G_{s+1}$ and $G_\infty = \cup_{s\geqslant 0} G_s$ is dense in $\mathbb{R}^n$. Let f be a real-valued function defined on G_0. The interpolating subdivision scheme allows to extend f, by iterative rule, to $G_1, G_2, \ldots, G_n, \ldots$: we obtain, therefore, an interpolating function defined on G_∞. One of the important questions is to characterize the schemes, called continuous, such that every interpolating function has a continuous extension defined on $\mathbb{R}^n$ (see [10]). The notion of interpolating subdivision schemes, which is prior to the development of wavelet theory, arises in several fields of pure and applied mathematics [4, 11, 9, 23].

Let $r \in \mathbb{N}^*$. The order r interpolating schemes [21] are a natural generalization of the previous ones: starting from real-valued functions a_α, defined on G_0 and indexed by the multi-indexes α, $|\alpha| \leqslant r$, one wishes to construct a function f on G_∞, which admits an extension $\tilde{f}$ of class $\mathscr{C}^r$ defined on $\mathbb{R}^n$ such that every $\partial^\alpha \tilde{f}/\partial x^\alpha$ coincides with a_α on G_0.

In this paper, we consider $n = 1$ and $G_s = 2^{-s}\mathbb{Z}, s \in \mathbb{N}$, that is, dyadic interpolating schemes. Let us recall that, in this case, a continuous interpolating scheme (i.e., $r = 0$) yields a Multi-Resolution Analysis of multiplicity 1. We start by giving definitions and simple properties relative to order r interpolating schemes. Then we study the connection with the Multi-Resolution Analyses of multiplicity $r + 1$.

6.1. *Definitions*

Let $r \in \mathbb{N}^*$. We denote by $\vec{e}_0, \ldots, \vec{e}_r$ the canonical basis for $\mathbb{R}^{r+1}$, and by D the set of all dyadic reals. Provided we use vector notations, the results of this section can be proved as in the case $r = 0$. Let us start by defining the dyadic vector interpolation scheme.

DEFINITION. Consider $p, q \in \mathbb{Z}$, with $p < q$, and a family $\{C(s, k), s \in \mathbb{N}, k \in \mathbb{Z}\}$ of matrices in $\mathscr{M}(r + 1, \mathbb{R})$ such that $C(s, k) = 0$ if $k \notin [p, q]$. The associated dyadic vector interpolation scheme $(\mathscr{D})$ is defined as follows: given any sequence $\{\vec{A}(n), n \in \mathbb{Z}\}$ of vectors in $\mathbb{R}^{r+1}$, we construct the vector function $\vec{F}$ defined on D by the iterative process:

- $\vec{F}(n) = \vec{A}(n)$, if $n \in \mathbb{Z}$,
- $\vec{F}(2^{-s}n + 2^{-(s+1)}) = \sum_{k\in\mathbb{Z}} C(s, n-k)\vec{F}(2^{-s}k), n \in \mathbb{Z}, s = 0, 1, 2, \ldots.$

$\vec{F}$ is called the vector interpolating function (by $(\mathscr{D})$ and from $(\vec{A}(n))_{n\in\mathbb{Z}}$), and we write $\vec{F} = \mathscr{D}(\vec{A})$. Let δ_0 be the sequence defined by $\delta_0(0) = 1$ and $\delta_0(n) = 0$ if $n \in \mathbb{Z}, n \neq 0$. For $i = 0, \ldots, r$, we easily prove that the $r + 1$ vector functions $\mathscr{D}(\delta_0\vec{e}_i)$ have bounded support in $D \cap [2p + 1, 2q + 1]$, and that every vector interpolating function $\vec{F} = {}^t[f_0, \ldots, f_r]$ can be expressed as

$$\vec{F}(x) = \sum_{i=1}^{r} \sum_{k\in\mathbb{Z}} f_i(k) \mathscr{D}(\delta_0\vec{e}_i)(x - k), \quad x \in D.$$

In order to define the order r dyadic interpolation scheme, we use the notation

$$\overrightarrow{\Delta f} = \begin{pmatrix} f \\ f' \\ \vdots \\ f^{(r)} \end{pmatrix}.$$

DEFINITIONS. We say that $(\mathscr{D})$ is of order r if there exist $r + 1$ real-valued functions $\phi_0, \ldots, \phi_r$, at least of class $\mathscr{C}^r$ on $\mathbb{R}$, such that, for every integer $i = 0, \ldots, r$ and all $x \in D$, the equality $\mathscr{D}(\delta_0\vec{e}_i)(x) = \overrightarrow{\Delta\phi_i}(x)$ holds. The functions ϕ_i, called *fundamental interpolating functions with respect to* $(\mathscr{D})$, are compactly supported in $[2p + 1, 2q + 1]$, and verify

$$(\phi_i)^{(\ell)}(n) = \delta_{0,n} \cdot \delta_{i,\ell}, \quad \forall n \in \mathbb{Z}, \forall \ell = 0, \ldots, r. \tag{22}$$

Let

$$\Phi_C = \begin{pmatrix} \phi_0 \\ \vdots \\ \phi_r \end{pmatrix}, \quad \text{and} \quad \hat{\Phi}_C = \begin{pmatrix} \hat{\phi}_0 \\ \vdots \\ \hat{\phi}_r \end{pmatrix}.$$

If $(\mathscr{D})$ is of order r, then, for all vector interpolating function $\vec{F}$, there exists a real-valued function f of class $\mathscr{C}^r$ on $\mathbb{R}$ such that $\overrightarrow{\Delta f}(x) = \vec{F}(x)$ for all $x \in D$, and

$$f(x) = \sum_{i=0}^{r} \sum_{k\in\mathbb{Z}} f^{(i)}(k)\phi_i(x - k), \forall x \in \mathbb{R},$$

the successive derivatives of f (up to r) being computed by termwise differenciation. For convenience f is still called interpolating function with respect to $(\mathscr{D})$. Note that f only depends on the values of its r first derivatives on $\mathbb{Z}$.

Due to the dyadic character of $(\mathscr{D})$, it is natural to require that, if f is an interpolating function, the same holds for the function $x \to f(x/2)$. We easily check that a necessary and sufficient condition for this requirement is that

$$C(s, k) = A_r^{-s}C(0, k)A_r^s, \quad \forall k = p, \ldots, q, \forall s \in \mathbb{N}, \tag{23}$$

where $A_r = \mathrm{diag}(1, 2^{-1}, \ldots, 2^{-r})$.

6.2. *Connection with Scaling Matrix Filters*

Let us consider a family $\{C(s, k), s \in \mathbb{N}, k = p, \ldots, q\}$ of matrices in $\mathscr{M}(r + 1, \mathbb{R})$, satisfying (23), and let $(\mathscr{D})$ be the associated dyadic interpolation scheme. We define

$$H_C(\lambda) = \frac{1}{2}A_r + \frac{1}{2}\sum_{k=p}^{q} e^{-2i\pi(2k+1)\lambda}C(k)^* A_r.$$

THEOREM 6.1. *If $(\mathscr{D})$ is of order r, then the vector function $\hat{\Phi}_C$ satisfies the equation $\hat{\Phi}_C(\lambda) = H_C(\lambda/2)\hat{\Phi}_C(\lambda/2)$. Conversely if H_C is a scaling matrix filter and if its $r + 1$ scaling functions $\tau_0, \ldots, \tau_r$ are at least of class $\mathscr{C}^r$, and such that*

$$(\tau_i)^{(\ell)}(n) = \delta_{0,n} \cdot \delta_{i,\ell}, \quad \forall n \in \mathbb{Z}, \forall \ell = 0, \ldots, r, \tag{24}$$

then $(\mathcal{D})$ *is of order* r, *and its fundamental interpolating functions are* $\tau_0, \ldots, \tau_r$.

Proof. Let us set, for all $n \in \mathbb{Z}, H_{2n} = \delta_{0,n} A_r, H_{2n+1} = C(n)^* A_r$, and $H_n = [h_n(i,j)]_{i,j=1,\ldots,d}$. Suppose that $(\mathcal{D})$ is of order r. We show the equality

$$\Phi_C\left(\frac{x}{2}\right) = \sum_{n\in\mathbb{Z}} H_n \Phi_C(x-n), \quad x \in \mathbb{R}, \tag{25}$$

which, by Fourier transform, is equivalent to the equation of the theorem. It suffices to check that

$$A_r \overrightarrow{\Delta\phi}_i\left(\frac{x}{2}\right) = \sum_{k\in\mathbb{Z}} \sum_{j=0}^{r} h_k(i,j)\, \overrightarrow{\Delta\phi}_j(x-k), \quad \forall i = 0, \ldots, r. \tag{26}$$

Indeed the $(r+1)$ equalities given by every first line of the vector identities in (26) give (25). Let $i \in \{0, \ldots, r\}$, and denote by $\vec{F}_i(x)$ and $\vec{G}_i(x), x \in \mathbb{R}$, respectively, the left and the right term of (26). Using the definition of a vector interpolating scheme, we easily prove that $\vec{F}_i = \vec{G}_i$ on $\mathbb{Z}$. Since $\vec{F}_i$ and $\vec{G}_i$ are both vector interpolating functions, $\vec{F}_i$ and $\vec{G}_i$ are equal on D, and by a continuity argument, equal on $\mathbb{R}$.

Conversely, suppose that H_C is a scaling matrix filter such that its $r+1$ scaling functions $\tau_0, \ldots, \tau_r$ are of class $\mathscr{C}^r$ and satisfy (24). Then we obtain

$$2^{-\ell s} \tau_i^{(\ell)}\left(2^{-s}n + 2^{-(s+1)}\right) = \sum_{j=0}^{r} \sum_{k\in\mathbb{Z}} 2^{-js} \tau_j^{(\ell)}\left(\frac{2(n-k)+1}{2}\right) \tau_i^{(j)}(2^{-s}k), \quad \forall s \in \mathbb{N}.$$

Denoting by $W(x)$ the $(r+1)\times(r+1)$ matrix, whose column of index j, for $j = 0, \ldots, r$, is given by $\overrightarrow{\Delta\tau}_j(x)$, the previous identities are equivalent to the following vector equality:

$$A_r^s \overrightarrow{\Delta\tau}_i\left(2^{-s}n + 2^{-(s+1)}\right) = \sum_{k\in\mathbb{Z}} W\left(\frac{2(n-k)+1}{2}\right) A_r^s \overrightarrow{\Delta\tau}_i(2^{-s}k), \quad i = 0, \ldots, r.$$

Since $\tau_0, \ldots, \tau_r$ verify equation (1) with respect to $m_{i,j}(k) = h_k(i,j)$, it follows that $W(x/2)^* A_r = \sum_{k\in\mathbb{Z}} H_k W(x-k)^*$, hence, by (24), $W((2l+1)/2)^* A_r = H_{2l+1} = C(l)^* A_r$. This ensures that $\tau_0, \ldots, \tau_r$ are the fundamental interpolating functions associated to the scheme $(\mathcal{D})$, which is, therefore, of order r. ∎

Remark. Let $I \in \mathcal{M}_{r+1}$, given by $I(\lambda) = I_{r+1}$, and let Q_C be the operator defined on $\mathcal{M}_{r+1}$ by

$$Q_C F(\lambda) = \left(H_C\left(\frac{\lambda}{2}\right) F\left(\frac{\lambda}{2}\right) + H_C\left(\frac{\lambda}{2} + \frac{1}{2}\right) F\left(\frac{\lambda}{2} + \frac{1}{2}\right)\right) A_r^{-1}.$$

Suppose that H_C is a scaling matrix filter such that its scaling functions $\tau_0, \ldots, \tau_r$ are at least of class $\mathscr{C}^r$, ans suppose that I is the unique Q_C-invariant function in $\mathcal{M}_{r+1}$ (modulo a complex factor). Then the τ_i satisfy (24).

Indeed, we have $Q_C I = I$, and using Poisson's formula, we easily check that the matrix function Γ given by

$$\Gamma(\lambda) = \begin{pmatrix} \sum_{n\in Z} \tau_0(n) e^{2i\pi n\lambda} & \cdots & \sum_{n\in Z} \tau_0^{(r)}(n) e^{2i\pi n\lambda} \\ \vdots & & \vdots \\ \sum_{n\in Z} \tau_r(n) e^{2i\pi n\lambda} & \cdots & \sum_{n\in Z} \tau_r^{(r)}(n) e^{2i\pi n\lambda} \end{pmatrix}, \quad \lambda \in [0,1],$$

is invariant by Q_C. More precisely, since Γ is of finite length, Q_C acts on a finite dimensional space, and the action of Q_C in this space can be represented by a matrix Q_0. Consequently, in order to prove that $\Gamma = I$, it suffices to check that $\dim\mathrm{Ker}(Q_0 - Id) = 1$.

6.3. Examples

EXAMPLE 6. The following family of order 1 interpolating schemes is drawn from [21]. For all real μ, we set $\mu' = (1-\mu)/2$, and we consider the family $\{C_\mu(s,k), s \in \mathbb{N}, k \in \mathbb{Z}\}$ of 2×2 matrices defined by (23) with $C_\mu(0,k) = 0$ for $k \neq -1, 0$, and

$$C_\mu(0,-1) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{8} \\ \mu & \mu' \end{pmatrix}$$
$$C_\mu(0,0) = \begin{pmatrix} \frac{1}{2} & \frac{1}{8} \\ -\mu & \mu' \end{pmatrix}.$$

We denote by $(\mathcal{D}_\mu)$ the associated vector dyadic interpolating scheme. It is shown in [21] that, if $|2-\mu| < 1$, then $(\mathcal{D}_\mu)$ is of order 1. More precisely, in this case, the fundamental interpolating functions $\phi_{0,\mu}$ and $\phi_{1,\mu}$ are of class $\mathscr{C}^\beta$ for all real $\beta < 2$ (the derivatives of $\phi_{0,\mu}$ and $\phi_{1,\mu}$ are $(\beta-1)$-Holderian). Examples of interpolating functions are given in appendix B. The Fourier transforms $\hat{\phi}_{0,\mu}$ and $\hat{\phi}_{1,\mu}$ satisfy the scaling matrix equation with

$$H_\mu(\lambda) = \begin{pmatrix} \cos^2 \pi\lambda & -\frac{i\mu}{2}\sin 2\pi\lambda \\ \frac{i}{8}\sin 2\pi\lambda & \frac{1}{2}\left(\frac{1}{2} + \mu'\cos 2\pi\lambda\right) \end{pmatrix}.$$

If $\mu = 3/2$, we find again example 2 of section 2. The value $\mu = 2$ corresponds to the quadratic splines (see Example 1 in Sect. 2). For these two cases the integer translates of the fundamental interpolating functions form a Riesz family.

This property generalizes as follows: If $|2-\mu|<1$, *then* $\phi_{0,\mu}$ *and* $\phi_{1,\mu}$ *are the scaling functions of a Multi-Resolution Analysis of multiplicity* 2 (see [14]).

EXAMPLE 7. Let us return to Example 5 of Section 5.4: the scaling functions ϕ_0, ϕ_1, ϕ_2 also constitute the fundamental interpolating functions of the order 2 dyadic interpolation scheme ($\mathscr{D}$) associated to the family $\{C(s,k), s \in \mathbb{N}, k \in \mathbb{Z}\}$ of 3×3 matrices defined by (23) with $C(0,k)=0$ for $k \neq -1, 0$, and

$$C(0,0)=\begin{pmatrix} \frac{1}{2} & \frac{5}{32} & \frac{1}{64} \\ -\frac{15}{8} & -\frac{7}{16} & -\frac{1}{32} \\ 0 & -\frac{3}{2} & -\frac{1}{4} \end{pmatrix}$$

$$C(0,-1)=\begin{pmatrix} \frac{1}{2} & -\frac{5}{32} & \frac{1}{64} \\ \frac{15}{8} & -\frac{7}{16} & \frac{1}{32} \\ 0 & \frac{3}{2} & -\frac{1}{4} \end{pmatrix}.$$

Remark. We may consider, more generally, the p-adic interpolating schemes with $p \geqslant 3$. For example, in the triadic case, the scheme is given by the two following iterative formulae:

$$\begin{cases} F\left(3^{-r}n+3^{-(r+1)}\right) &= \sum_{k\in\mathbb{Z}} C_1(r,n-k)F(3^{-r}k) \\ F\left(3^{-r}n+2\cdot3^{-(r+1)}\right) &= \sum_{k\in\mathbb{Z}} C_2(r,n-k)F(3^{-r}k). \end{cases}$$

The Multi-Resolution Analysis of multiplicity 1, associated to the cubic splines, yields this type of interpolation. ■

7. CONCLUSION

We conclude this work by dealing with some additional questions on Multi-Resolution Analysis of multiplicity $d \geqslant 2$. In particular we show that the wavelet bases (for $d \geqslant 2$) also provide unconditional bases for many other spaces than $\mathbb{L}^2(\mathbb{R})$, and we present a simple computation of Sobolev (integer) coefficients of the scaling functions associated to scaling matrix filters of finite length.

7.1. Asymptotic Conditions for Multi-Resolution Analyses

Let H be a scaling $d\times d$ matrix filter, and $\phi_1,\ldots,\phi_d$ the associated scaling functions. We set, as usual,

$$V_0=\overline{span\{\phi_1(\cdot-k),\ldots,\phi_d(\cdot-k), k\in\mathbb{Z}\}},$$

and $V_n = D^nV_0$ for every $n\in\mathbb{Z}$. As it was mentioned in remark (d) of section 3, we may suppose that $\{\phi_1(\cdot - k),\ldots,\phi_d(\cdot-k), k\in\mathbb{Z}\}$ is an orthonormal basis for V_0. The family $(V_n)_{n\in\mathbb{Z}}$ satisfies the statements 2, 3, 4 of the Multi-Resolution Analysis definition. Condition 1 can be rewritten in this way:

$$\lim_{n\to-\infty} R_nf=0 \quad\text{and}\quad \lim_{n\to+\infty} R_nf=f \quad \text{in } \mathbb{L}^2(\mathbb{R}), \forall f\in\mathbb{L}^2(\mathbb{R}), \tag{27}$$

where R_n is the orthogonal projection on V_n. It is straightforward to check that the kernel of R_n is: $A_n(x,y)=2^n\sum_{i=1}^d\sum_{k\in\mathbb{Z}}\phi_i(2^nx-k)\overline{\phi_i(2^ny-k)}$. By using classical arguments on approximations of identity, it is proved in [22] that (27) holds if the functions ϕ_j are continuous on $\mathbb{R}$, and verifies (20). In this case $(V_n)_{n\in\mathbb{Z}}$ forms a Multi-Resolution Analysis of multiplicity d. If the functions ϕ_i are supposed more regular and localized, then (27) remains valid in others spaces (Sobolev spaces for instance) (see [22]).

7.2. Wavelets Basis Properties

Let $(V_n)_{n\in\mathbb{Z}}$ be a Multi-Resolution Analysis of multiplicity d. We denote by $\psi_1,\ldots,\psi_d$ (simply ψ if $d=1$) the associated wavelets (see Section 2). It is shown in [18] that, in most of cases, the wavelets ψ_i have the same regularity and the same localization as the ϕ_i. From Theorem 2.1, it follows that

$$f(x)=\lim_{n\to+\infty}\sum_{j,k=-n}^{n}\sum_{i=1}^{d}2^j\langle f,\psi_i(2^j\cdot-k)\rangle\,\psi_i(2^jx-k) \quad \text{in } \mathbb{L}^2(\mathbb{R}), \forall f\in\mathbb{L}^2(\mathbb{R}).$$

When $d=1$, this convergence, and more generally, the unconditional basis property, extend to a lot of spaces [22] (Sobolev spaces, Holderian functions spaces, ...). In order to prove these statements, one considers the kernel K_n of the operator defined by the above sum, that is, $K_n(x,y)=\sum_{j,k=-n}^n 2^j\psi(2^jx-k)\overline{\psi(2^jy-k)}$, and one uses the theory of Zygmund–Calderon's operators, which rests on the following properties of K_n :

$$|K_n(x,y)|\leqslant C\,|x-y|^{-1},$$
$$\left|\frac{\partial K_n(x,y)}{\partial x}\right|+\left|\frac{\partial K_n(x,y)}{\partial y}\right|\leqslant C\,|x-y|^{-2}, \quad x\neq y.$$

Recall that these inequalities hold from the moment that the wavelet ψ is sufficiently regular and localized. It is straightforward to check that these tools remain valid for $d\geqslant 2$ if every ψ_i satisfies the same conditions as ψ. Consequently the properties of unconditional wavelet basis generalize to $d\geqslant 2$.

7.3. Algebraic Properties for Scaling Matrix Filters

Let H be a 1-periodic continuous complex-valued function. An algebraic assumption is a necessary condition for H to be a scaling filter such that the associated scaling function ϕ satisfies a given property. For example the Riesz family property in (P2) implies that $H(1/2) = 0$, and $|H(\cdot)|^2 + |H(\cdot + 1/2)|^2 > 0$ (see Section 3). If we want $\{\phi(\cdot - k), k \in \mathbb{Z}\}$ to constitute an orthonormal family, then H must verify the QMF condition: $|H(\cdot)|^2 + |H(\cdot + 1/2)|^2 = 1$. On the same way, if it is required that $\phi(n) = \delta_{0,n}$ for every integer n (interpolating condition), then H must be chosen such that $H(\cdot) + H(\cdot + 1/2) = 1$.

The converse problem is to check if the previous conditions on H yield the desired requirements on ϕ. This is done in [5, 7] for the QMF case, in [15, 24] for the Riesz basis property, and in [25, 16] for the interpolating condition. For instance, suppose that H is a regular QMF, and consider the operator P defined in Section 1. We have $P1 = 1$. It is proved in [7] that a necessary and sufficient condition for $\{\phi(\cdot - k), k \in \mathbb{Z}\}$ to constitute an orthonormal family is that the constant functions are the only 1-periodic continuous P-invariant functions. Moreover this statement is equivalent to a simple condition on the set Z of zeros of H (if Z is finite in number, this condition is given by (15) with $d = 1$).

Let us consider the corresponding question for $d \geqslant 2$. If H is a scaling matrix filter of $\mathcal{M}_d$ such that the scaling functions $\phi_1, \ldots, \phi_d$ generate by integer translates an orthonormal family, then we have $\Theta_\Phi(\cdot) = I_d$. Using (8), this implies the necessary condition

$$H(\cdot)H(\cdot)^* + H\left(\cdot + \frac{1}{2}\right)H\left(\cdot + \frac{1}{2}\right)^* = I_d. \tag{28}$$

We have the following converse result: let $H \in \mathcal{M}_d^\alpha$ satisfying (28), (16), and (17) with $M = I_d$. Consider the functions $\hat{\phi}_1, \ldots, \hat{\phi}_d$ defined by (4) with $\vec{x} = \vec{e}_1$. That $\hat{\phi}_1, \ldots, \hat{\phi}_d$ are in $\mathbb{L}^2(\mathbb{R})$ results from (19). If, in addition, $\det H(\cdot)$ has a finite number, Q, of zeros, and if H satisfies (15), then $\{\phi_i(\cdot - k), i = 1, \ldots, d, k \in \mathbb{Z}\}$ constitutes an orthonormal family if and only if $\Theta_\Phi(0) = I_d$.

To see that, it suffices to prove that, if F, G are P_H-invariant and such that $F(0) = G(0)$, then $F = G$ (see [14, Sect. 5.2]).

7.4. Sobolev Integer Coefficients for Scaling Functions

Let us first focus on the case $d = 1$ by considering a scaling filter H and the scaling function ϕ defined by $\hat{\phi}(\lambda) = \Pi_{k\geqslant 1} H(\lambda/2^k)$. The study of regularity of ϕ is based on the condition $H(1/2) = 0$. This implies that H is of the form $H(\lambda) = (1 + e^{2i\pi\lambda})/2)^r v(\lambda)$, where $r \in \mathbb{N}^*$ and $v(1/2) \neq 0$, and that $\hat{\phi}(\lambda) = e^{i\pi r\lambda}(\sin\pi\lambda)/\pi\lambda)^r \Pi_{k\geqslant 1} v(\lambda/2^k)$. Therefore, the problem amounts to studying the growth of this infinite product (see [8, 24, 15]).

Now let H be a scaling $d \times d$ matrix filter of the form (13), satisfying (16) and (17), and assume that the scaling functions $\phi_1, \ldots, \phi_d$ are defined by (4) with $\vec{x} = \vec{e}_1$. Of course, since the matrix product in $\mathcal{M}(d, \mathbb{C})$ is not commutative, the above statements don't extend to $d \geqslant 2$. However we show that a more precise spectral study of P_H provides conditions for $\phi_1, \ldots, \phi_d \in H^p, p \in \mathbb{N}^*$, and where H^p denotes the usual Sobolev space defined by the condition $(1 + |\lambda|^p)\hat{f}(\lambda) \in \mathbb{L}^2(\mathbb{R})$. Let us recall that P_N is the restriction of P_H to the finite-dimensional space $\mathcal{T}_d^N$.

We proved in section 5 that the matrix function Θ_Φ, defined by (5), is almost everywhere equal to $\Theta_0 \in \mathcal{T}_d^N$, which is a positive P_N-invariant function. We generalize this remark by considering the matrix-valued function $\tilde{\Theta}_p$ formally defined, for $p \in \mathbb{N}^*$, by

$$\tilde{\Theta}_p(\lambda) = \sum_{k\in\mathbb{Z}} |\lambda + k|^{2p}\, \hat{\Phi}(\lambda + k)\hat{\Phi}(\lambda + k)^*.$$

If $\phi_1, \ldots, \phi_d \in H^p$, then $\tilde{\Theta}_p(\lambda)$ is well defined for almost all $\lambda \in \mathbb{R}$. By Poisson's formula, $\tilde{\Theta}_p$ is almost everywhere equal to a function of $\mathcal{T}_d^N$, which we denote by Θ_p. Moreover, we obtain by using the matrix scaling equation

$$\tilde{\Theta}_p(\lambda) = \sum_{k\in\mathbb{Z}} \left|\frac{\lambda}{2} + \frac{k}{2}\right|^{2p} H\left(\frac{\lambda}{2} + \frac{k}{2}\right) \times \hat{\Phi}\left(\frac{\lambda}{2} + \frac{k}{2}\right)\hat{\Phi}\left(\frac{\lambda}{2} + \frac{k}{2}\right)^* H\left(\frac{\lambda}{2} + \frac{k}{2}\right)^*.$$

Separating the even and odd indeces, it follows that $P_N\Theta_p = 2^{-2p}\Theta_p$. We have the following converse result.

PROPOSITION 7.1. *Let $p \in \mathbb{N}^*$. Suppose that there exists a non-negative function Γ of $\mathcal{T}_d^N$ satisfying $P_N\Gamma = 2^{-2p}\Gamma$ and $\Gamma(\lambda) \geqslant C|\lambda|^{2p}I_d$ for all λ in some neighborhood of 0, C being a positive constant independant of λ. Then, $\phi_1, \ldots, \phi_d \in H^p$.*

Proof. From (18) applied with $F = \Gamma$, it follows that

$$\int_{-2^{n-1}}^{2^{n-1}} \Pi_n(\lambda)\Gamma\left(\frac{\lambda}{2^n}\right)\Pi_n(\lambda)^*\, d\lambda = 2^{-2np}\int_0^1 \Gamma(\lambda)\, d\lambda = 2^{-2np}M,$$

hence, for each $i \in \{1, \ldots, d\}$,

$$\int_{-2^{n-1}}^{2^{n-1}} |\lambda|^{2p} \left\langle \left(\left|\frac{2^n}{\lambda}\right|^{2p} \Gamma\left(\frac{\lambda}{2^n}\right)\right)\Pi_n(\lambda)^*\vec{e}_i, \Pi_n(\lambda)^*\vec{e}_i \right\rangle d\lambda = M,$$

Using Fatou's lemma, the assumption on Γ, and the fact that $\lim_{n\to+\infty}\Pi_n(\lambda)^*\vec{e}_i = \hat{\phi}_i(\lambda)\vec{e}_1$ (lemma 5.1), it follows that $\int_{\mathbb{R}} |\lambda|^{2p}|\hat{\phi}_i(\lambda)|^2 d\lambda \leqslant C^{-1}M$. ∎

Remark. Suppose $d = 2$. If $\phi_1, \phi_2 \in H^p, p \in \mathbb{N}^*$, then there exists an integer $\ell \geqslant p$ such that $\Theta_p(\lambda) \geqslant C|\lambda|^{2l} I_d$ for all λ in some neighborhood of $0, C$ being a positive constant.

Indeed, using Lemma 3.2, we obtain

$$\Theta_p(0) = \tilde{\Theta}_p(0) = \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix}$$

with $\beta = \sum_{k \neq 0} |k|^{2p} |\hat{\phi}_2(k)|^2 > 0$ (if $\beta = 0$, then det $\Theta_\Phi(0) = 0$, but this is forbidden by the Riesz basis property). For $\lambda \in [0, 1]$, we define $a(\lambda)$ and $b(\lambda)$, respectively, as the smallest and the largest eigenvalue of the non-negative Hermitian matrix $\Theta_p(\lambda)$. We have $a(\lambda)b(\lambda) = \det\Theta_p(\lambda)$, hence $a(\lambda) \approx \beta^{-1}\det\Theta_p(\lambda)$ at 0. Note that det $\Theta_p(\cdot)$ is a $\mathbb{R}^+$-valued trigonometric polynomial equal to 0 at 0. Thus there exist $C > 0$ and $\ell \in \mathbb{N}^*$ such that det $\Theta_p(\lambda) \approx C\lambda^{2\ell}$ at 0.

Suppose $\ell < p$. Then the above identity applied with Θ_p instead of Γ implies that

$$\int_{-2^{n-1}}^{2^{n-1}} |\lambda|^{2\ell} \left\langle \left(\left| \frac{2^n}{\lambda} \right|^{2\ell} \Theta_p\left(\frac{\lambda}{2^n}\right) \right) \Pi_n(\lambda)^* \vec{e}_i, \Pi_n(\lambda)^* \vec{e}_i \right\rangle \times d\lambda = 2^{-2n(p-\ell)} N,$$

N being a positive constant. We conclude that $\int_{\mathbb{R}} |\lambda|^{2\ell} |\hat{\phi}_i(\lambda)|^2 d\lambda = 0$ for $i = 1, 2$. This is, of course, impossible. Thus $\ell \geqslant p$.

APPENDIX A: PROOF OF LEMMA 5.1

Define $G(\lambda) = M^{-1}H(\lambda)M$. We have $\Pi_{k=1}^n H(\lambda/2^k) = M[\Pi_{k=1}^n G(\lambda/2^k)]M^{-1}$. It is, therefore, sufficient to prove the lemma with $M = I_d$. We let $\mu_1 = 1$.

Note that $|H(0)|_2 = 1$, and $||H(\lambda)|_2 - 1| \leqslant C|\lambda|^\alpha$. Consider a real number $A > 0$. We obtain for all $\lambda \in [-A, A]$,

$$\left| \ln \left| H\left(\frac{\lambda}{2^k}\right) \right|_2 \right| \underset{k \to +\infty}{\sim} \left| \left| H\left(\frac{\lambda}{2^k}\right) \right|_2 - 1 \right| \leqslant C \frac{A^\alpha}{2^{\alpha k}}.$$

This implies the uniform convergence on $[-A, A]$ of the series $\sum_{k \geqslant 1} |\ln |H(\lambda/2^k)|_2|$, and thus of the sequence $\{\Pi_{k=1}^n |H(\lambda/2^k)|_2, n \geqslant 1\}$ which is, therefore, uniformly bounded on $[-A, A]$ by a constant $D > 0$. Define, for $n \geqslant 1$ and $i, j = 1, \ldots, d$,

$$\alpha_{i,j}^n(\lambda) = \left\langle \prod_{k=1}^n H\left(\frac{\lambda}{2^k}\right) \vec{e}_j, \vec{e}_i \right\rangle.$$

For $q > p$, we write

$$\alpha_{i,j}^q(\lambda) = \left\langle \prod_{k=p+1}^q H\left(\frac{\lambda}{2^k}\right) \vec{e}_j, \left(\prod_{k=1}^p H\left(\frac{\lambda}{2^k}\right)\right)^* \vec{e}_i \right\rangle,$$

and

$$\alpha_{i,j}^p(\lambda) = \left\langle \vec{e}_j, \left(\prod_{k=1}^p H\left(\frac{\lambda}{2^k}\right)\right)^* \vec{e}_i \right\rangle.$$

From Cauchy-Schwarz's inequality in $\mathbb{C}^d$, it follows that

$$\begin{aligned} |\alpha_{i,j}^q(\lambda) - \mu_j^{q-p}\alpha_{i,j}^p(\lambda)| &\leqslant \left\| \left(\prod_{k=1}^p H\left(\frac{\lambda}{2^k}\right)\right)^* \vec{e}_i \right\|_2 \\ &\quad \times \left\| \prod_{k=p+1}^q H\left(\frac{\lambda}{2^k}\right) \vec{e}_j - \mu_j^{q-p}\vec{e}_j \right\|_2 \\ &\leqslant D \left\| \prod_{k=p+1}^q H\left(\frac{\lambda}{2^k}\right) \vec{e}_j - \mu_j^{q-p}\vec{e}_j \right\|_2. \end{aligned}$$

We have

$$\begin{aligned} \prod_{k=p+1}^q H\left(\frac{\lambda}{2^k}\right) \vec{e}_j - \mu_j^{q-p}\vec{e}_j &= \prod_{k=p+1}^q H\left(\frac{\lambda}{2^k}\right) \vec{e}_j \\ &\quad - \mu_j \prod_{k=p+1}^{q-1} H\left(\frac{\lambda}{2^k}\right) \vec{e}_j \\ &\quad + \mu_j \left(\prod_{k=p+1}^{q-1} H\left(\frac{\lambda}{2^k}\right) \vec{e}_j - \mu_j \right. \\ &\quad \left. \times \prod_{k=p+1}^{q-2} H\left(\frac{\lambda}{2^k}\right) \vec{e}_j \right) \\ &\quad \vdots \\ &\quad + \mu_j^{q-p-2} \\ &\quad \times \left(H\left(\frac{\lambda}{2^{p+1}}\right) H\left(\frac{\lambda}{2^{p+2}}\right) \vec{e}_j \right. \\ &\quad \left. - \mu_j H\left(\frac{\lambda}{2^{p+1}}\right) \vec{e}_j \right) \\ &\quad + \mu_j^{q-p-1} \left(H\left(\frac{\lambda}{2^{p+1}}\right) \vec{e}_j - \mu_j \vec{e}_j \right) \\ &= \left(\prod_{k=p+1}^{q-1} H\left(\frac{\lambda}{2^k}\right) \right) \\ &\quad \times \left(H\left(\frac{\lambda}{2^q}\right) \vec{e}_j - \mu_j \vec{e}_j \right) \\ &\quad + \mu_j \left(\prod_{k=p+1}^{q-2} H\left(\frac{\lambda}{2^k}\right) \right) \\ &\quad \times \left(H\left(\frac{\lambda}{2^{q-1}}\right) \vec{e}_j - \mu_j \vec{e}_j \right) \\ &\quad \vdots \\ &\quad + \mu_j^{q-p-2} H\left(\frac{\lambda}{2^{p+1}}\right) \end{aligned}$$

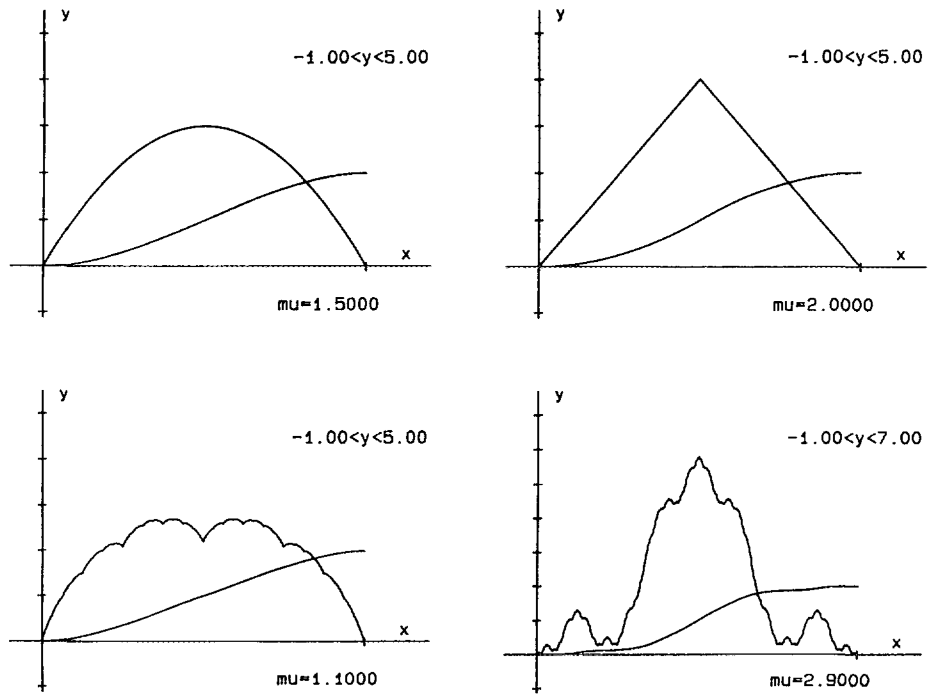

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FIG. 1.

$$\times \left(H\left(\frac{\lambda}{2^{p+2}}\right)\vec{e}_j - \mu_j \vec{e}_j \right) + \mu_j^{q-p-1}\left(H\left(\frac{\lambda}{2^{p+1}}\right)\vec{e}_j - \mu_j \vec{e}_j \right).$$

Let $H(\lambda) = [h_{i,j}(\lambda)]_{i,j=1,\ldots,d}$. Then

$$\left\| \prod_{k=p+1}^{q} H\left(\frac{\lambda}{2^k}\right)\vec{e}_j - \mu_j^{q-p}\vec{e}_j \right\|_2 \leqslant D \sum_{k=p+1}^{q} \left\| H\left(\frac{\lambda}{2^k}\right)\vec{e}_j - \mu_j \vec{e}_j \right\|_2 \leqslant D \sum_{k=p+1}^{q} \sqrt{ \left| h_{j,j}\left(\frac{\lambda}{2^k}\right) - \mu_j \right|^2 + \sum_{\substack{i=1\\ i\neq j}}^{d} \left| h_{i,j}\left(\frac{\lambda}{2^k}\right) \right|^2 }.$$

We know that $h_{i,j}(0) = \mu_j \delta_{i,j}$, and that $h_{i,j}$ is α-Holderian. Thus,

$$\left| h_{j,j}\left(\frac{\lambda}{2^k}\right) - \mu_j \right|^2 + \sum_{\substack{i=1\\ i\neq j}}^{d} \left| h_{i,j}\left(\frac{\lambda}{2^k}\right) \right|^2 \leqslant \frac{CA^{2\alpha}}{2^{2\alpha k}}, \qquad \forall \lambda \in [-A, A].$$

Consequently, for all real $\epsilon > 0$, we may choose a sufficiently large integer N so that

$$\forall q > p \geqslant N, \forall \lambda \in [-A, A], \qquad \left| \alpha_{i,j}^{q}(\lambda) - \mu_j^{q-p}\alpha_{i,j}^{p}(\lambda) \right| \leqslant \varepsilon.$$

If $\mu_j = 1$, then Cauchy's property in $\mathbb{C}$ involves that $(\alpha_{i,j}^n)_{n\geqslant 1}$ converges uniformly on $[-A, A]$. Because the sequence $(\alpha_{i,j}^n)_{n\geqslant 1}$ is uniformly bounded on $[-A, A]$, if $|\mu_j| < 1$, then $(\alpha_{i,j}^n)_{n\geqslant 1}$ converges uniformly to 0 on $[-A, A]$ (for every $i = 1, \ldots, d$), which ends the proof of the lemma.

APPENDIX B: ILLUSTRATIONS OF ORDER 1 INTERPOLATING SCHEMES

In Fig. 1 we give, for the values $\mu = 1.5, 2, 1.1$, and 2.9 of Example 6 (see Section 6.3), the interpolating function and its derivative on [0, 1], obtained from the initial conditions on $\mathbb{Z}$: $f(0) = f'(0) = 0, f(1) = 2, f'(1) = 0$, and $f(n) = f'(n) = 0$ if $n \neq 0, n \neq 1$. These examples are derived from [21].

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