STABLE LAWS

AND PRODUCTS OF POSITIVE RANDOM MATRICES

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Summary

Let S be the multiplicative semigroup of $q \times q$ matrices with positive entries such that every row and every column contains a strictly positive element. Denote by $(X_n)_{n\geq 1}$ a sequence of independent identically distributed random variables in S and by $X^{(n)} = X_n \cdots X_1,\ n \geq 1$, the associated left random walk on S. We assume that $(X_n)_{n\geq 1}$ verifies the contraction property

$$I\!\!P\Big(\bigcup_{n\geq 1}[X^{(n)} \in S^\circ]\Big) > 0,$$

where S° is the subset of all matrices which have strictly positive entries. We state conditions on the distribution of the random matrix X_1 which ensure that the logarithms of the entries, of the norm, and of the spectral radius of the products $X^{(n)}$, $n \geq 1$, are in the domain of attraction of a stable law.

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I. STATEMENT OF THE RESULT

Let S be the multiplicative semigroup of $q \times q$ matrices with real non negative entries such that every row and every column contains a strictly positive element. The subset of S composed of matrices with strictly positive entries is a subsemigroup of S denoted by S°.
Let $(e_i)_{i=1,\ldots,q}$ be the canonical basis of the linear space $I\!\!R^q$. Then a $q \times q$ matrix is identified with an endomorphism of $I\!\!R^q$. We denote by $\langle \cdot, \cdot \rangle$ the canonical scalar product on $I\!\!R^q$, and we define the cones C and $\overline{C}$ by

$$C = \{x : x \in I\!\!R^q, \forall i = 1, \ldots, q,\ \langle x, e_i \rangle > 0\}, \quad \overline{C} = \{x : x \in I\!\!R^q, \forall i = 1, \ldots, q,\ \langle x, e_i \rangle \geq 0\}.$$

If g is a $q \times q$ matrix, g^* will stand for its adjoint. We have $g \in S$ [resp. $g \in S^\circ$] if and only if $g(C) \subset C$ and $g^*(C) \subset C$ [resp. either $g(\overline{C} \backslash \{0\}) \subset C$ or $g^*(\overline{C} \backslash \{0\}) \subset C$].

The product of g and g' in S is denoted by gg', and for $x \in \overline{C}$, gx is the image of x under g. Finally $I\!\!R^q$ is endowed with the norm $\| \cdot \|$ defined by

$$x \in I\!\!R^q, \quad \|x\| = \sum_{i=1}^{q} |\langle x, e_i \rangle|.$$

Let $(X_n)_{n \geq 1}$ be a sequence of independent identically distributed (i.i.d) random variables (r.v) in S defined on a probability space $(\Omega, \mathcal{F}, I\!\!P)$. We consider the left random walk

$$X^{(n)},\ n \geq 1, \quad X^{(1)}(\omega) = X_1(\omega), \quad X^{(n+1)}(\omega) = X_{n+1}(\omega) X^{(n)}(\omega).$$

Our basic assumption is that $(X_n)_{n \geq 1}$ verifies the **contraction property**

$$(\mathcal{C}) \qquad I\!\!P\Big(\bigcup_{n \geq 1} [X^{(n)} \in S^\circ]\Big) > 0.$$

The subsemigroup S° is in fact an ideal of S, that is : if $g \in S^\circ$ and $g' \in S$, then $g'g$ and $gg' \in S^\circ$. Consequently S° is stochastically closed for the random walk $(X^{(n)})_{n \geq 1}$. We set

$$T(\omega) = \inf\{n : n \geq 1,\ X^{(n)}(\omega) \in S^\circ\}.$$

It is easily shown, Lemma II.1, that : $(\mathcal{C}) \ \Leftrightarrow\ I\!\!P[T < +\infty] = 1 \ \Leftrightarrow\ I\!\!P(\cup_{n \geq 1} [X^{(n)} \in S^\circ]) = 1$.

Our aim is to present conditions on X_1 ensuring the distributional convergence to a stable law for the sequences of real random variables

$$(1_{[T \leq n]} \ln \langle y, X^{(n)} x \rangle)_{n \geq 1}, \quad x, y \in \overline{C} \backslash \{0\}.$$

Denoting by $\vec{\mathbf{1}}$ the vector in $I\!\!R^q$ whose all entries equal 1, we point out that the scalar products $\langle y, X^{(n)} x \rangle$, $x, y \in \overline{C} \backslash \{0\}$, include :

- the matrix entries : $\langle e_i, X^{(n)} e_j \rangle$, $\ i, j = 1, \ldots, q$,
- the norm of the image under $X^{(n)}$ of any $x \in \overline{C} \backslash \{0\}$: $\ \|X^{(n)} x\| = \langle \vec{\mathbf{1}}, X^{(n)} x \rangle$,
- the norm $|||X^{(n)}||| = \langle \vec{\mathbf{1}}, X^{(n)} \vec{\mathbf{1}} \rangle$ of $X^{(n)}$.

Closely related to these quantities is the spectral radius Λ_n of the matrix $X^{(n)}$. Actually the Perron-Frobenius Theorem yields $\Lambda_n > 0$, and we shall see that the above mentioned distributional convergences also concern the sequence $(\ln \Lambda_n)_{n \geq 1}$.

To state our result, one needs the two following real r.v :

$$N_1 = |||X_1||| = \sum_{i,j=1}^{q} \langle e_i, X_1 e_j\rangle, \quad \text{and} \quad V_1 = \min_{i=1,\dots,q} \sum_{j=1}^{q} \langle e_i, X_1 e_j\rangle.$$

N_1 takes in account the size of the matrix X_1 while V_1 measures the smallness of its lines.

Theorem I. *Assume that $(\mathcal{C})$ holds and that there exist a real number α, $0 < \alpha \leq 2$, a slowly varying function L which is unbounded in case $\alpha = 2$, and finally some positive constants c_+ and c_- with $c_+ + c_- > 0$ such that*

(i) $\displaystyle\lim_{u\to+\infty} \frac{u^\alpha}{L(u)}\, \mathbb{P}[N_1 > e^u] = c_+, \qquad \lim_{u\to+\infty} \frac{u^\alpha}{L(u)}\, \mathbb{P}[N_1 \leq e^{-u}] = c_-,$

(ii) $\displaystyle\limsup_{u\to+\infty} \frac{u^\alpha}{L(u)}\, \mathbb{P}[V_1 \leq e^{-u}] < +\infty.$

Then there exist a sequence $(a_n)_{n\geq 1}$ in $\mathbb{R}_+^*$ with $\lim_n a_n = +\infty$ and a sequence $(b_n)_{n\geq 1}$ in $\mathbb{R}$ such that, for any sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ of unit vectors of $\overline{C}$, the random sequences

$$\left(\frac{1}{a_n}\Big(1_{[T\leq n]} \ln\langle y_n, X^{(n)} x_n\rangle - b_n\Big)\right)_{n\geq 1} \qquad \textit{and} \qquad \left(\frac{1}{a_n}\Big(\ln \Lambda_n - b_n\Big)\right)_{n\geq 1}$$

converge in distribution to a stable law of index α.

Observe that Hypothesis *(i)* means that the real r.v. $\ln N_1$ belongs to the domain of attraction of a stable law of index α, $0 < \alpha \leq 2$, the standard Gaussian case being here excluded since L is assumed to be unbounded when $\alpha = 2$. As it will be seen later on, the hypotheses of Theorem I imply that the above considered sequences of random variables have the same distributional behaviour that a sum of i.i.d random variables (See § IV). However it is worth noticing that this is not true when $\alpha = 2$ and L is bounded. In fact, to complete the Gaussian case, recall it is proved in [10] that, if Conditions *(i)* and *(ii)* are replaced by the moment condition $\mathbb{E}[|\ln N_1|^2] + \mathbb{E}[|\ln V_1|^2] < +\infty$, then the random sequences of Theorem I converge to a normal law. The method used in [10] is based on martingale techniques, and the central limit theorem proved there is also valid when $(X_n)_n$ is supposed stationary and satisfies suitable mixing conditions. By the way, recall that, in some cases, the unnormalized random products $(X^{(n)})_{n\geq 1}$ converge in distribution, see [10] [15] [16].

Consider the case $q = 1$. Then Theorem I corresponds to the well-known statement of convergence to stable laws for i.i.d random variables : we have $S = \mathbb{R}_+^*$, Condition $(\mathcal{C})$ holds, and Condition *(i)* states that $\ln X_1$ is in the above described domain of attraction, *(ii)* is a consequence of *(i)*. So Theorem I gives the expected conclusion.

The proof of Theorem I is based on the spectral method that was introduced by Nagaev [17], [18] and later developped by several authors, see [11]. Although this method has been essentially used to prove Central Limit Theorems and their refinements, we mention that Nagaev himself [17] has considered the convergence to stable laws, and that his method has been extended to the context of dynamical systems, See e.g [8] [2] [3] [4] [9].

Section II summarizes some statements of [10], based on Condition $(\mathcal{C})$ and related to the projective action defined by $g \cdot x = \frac{gx}{\|gx\|}$ for $g \in S$ and unit vector x in $\overline{C}$. In Section III, denoting by Y_k the adjoint matrix of X_k, and setting $\xi(g, x) = \ln \|gx\|$, we show that the distributional convergences of Theorem I are valid if, for any unit vector y in $\overline{C}$, the same holds for the random variables $\xi(Y_k, (Y_{k-1} \dots Y_1) \cdot y)$. Since these r.v may be seen as a functional of the Markov chain

$(Y_k, (Y_{k-1} \dots Y_1) \cdot y)_k$, Nagaev's method applies. In fact, it shall be applied to the transition probability P of the simpler Markov chain $(Y_k \dots Y_1 \cdot y)_k$, and we shall prove, by using the contractivity properties stated in Section II, that P satisfies a strong ergodicity condition on a certain Lipschitz function space, and finally, by applying the perturbation theory, that the Fourier kernels P_t associated to P and ξ inherit near $t = 0$ the spectral properties of P.
As usual in Nagaev's method, the previous preparation will show that the desired distributional convergence is based on the behaviour of the power of the dominating eigenvalue $\lambda(t)$ of P_t. Actually, one of the main arguments is Proposition III.1 which links $\lambda(t)$ with the characteristic function of the r.v ξ under the stationary distribution of $(Y_k, (Y_{k-1} \dots Y_1) \cdot y)_k$. So everything shall turn out as in the i.i.d case, provided that ξ belongs to the already mentioned domain of attraction. We shall see in Section IV that this requirement holds under Conditions *(i)* and *(ii)*.

The above relation between the dominating perturbed eigenvalue of the Fourier kernels and the characteristic function of the functional under invariant distribution has been already exploited in [9] [4], and mentioned under a different form in [11] (Lem. IV.4'). It is worth noticing that such a relation holds whenever the spectral method applies, and that it greatly makes easier the use of Nagaev's method when dealing with stable laws excluding the standard Gaussian case ; for instance it yields a significant simplification of some proofs in [2] [3].

II. CONTRACTIVITY

II.1. Preliminaries. We set

$$B = C \cap \{x : x \in I\!\!R^q, \|x\| = 1\}, \quad \text{and} \quad \overline{B} = \overline{C} \cap \{x : x \in I\!\!R^q, \|x\| = 1\},$$

and we define the adjoint random walk $(Y^{(n)})_{n\geq 1}$ of $(X^{(n)})_{n\geq 1}$ by

$$Y_n = X_n^*, \qquad Y^{(n)} = {X^{(n)}}^* = Y_1 \cdots Y_n, \qquad n \geq 1.$$

Lemma II.1. $(\mathcal{C})$ *is equivalent to* $I\!\!P[T < +\infty] = 1$. *Let* ω *be such that* $T(\omega) < +\infty$. *Then*

(i) for $n \geq T(\omega)$, $X^{(n)}(\omega) \in S^\circ$,

(ii) setting, for $n \geq 1$, $D_n(\omega) = \sup\Big\{ \Big| 1_{[T\leq n]}(\omega) \ln\langle y, X^{(n)}(\omega)x\rangle - \ln \|Y^{(n)}(\omega)y\| \Big| \ : \ x, y \in \overline{B} \Big\}$, *we have* $\sup_{n\geq 1} D_n(\omega) < +\infty$,

(iii) setting $\chi = \frac{1}{q}\vec{\mathbf{1}}$ *and, for* $n \geq 1$, $\widetilde{D}_n(\omega) = \Big| \ln \Lambda_n(\omega) - \ln \|Y^{(n)}(\omega)\chi\| \Big|$, *we have* $\sup_{n\geq 1} \widetilde{D}_n(\omega) < +\infty$.

Proof. Suppose $(\mathcal{C})$ holds. Then there exists $k \in I\!\!N^*$ such that $p = I\!\!P[X^{(k)} \in S^\circ] > 0$. The r.v.

$$T' = \inf\{n : n \geq 1, X_{nk} \dots X_{(n-1)k+1} \in S^\circ\}$$

has a geometric distribution with parameter p. Since S° is an ideal, we have $T \leq kT'$, hence $I\!\!P[T < +\infty] = 1$. The converse implication is obvious.
Now let any fixed $\omega \in \Omega$ be such that $T(\omega) < +\infty$. Assertion (i) follows from the fact S° is an ideal. To prove (ii), it suffices to establish that $\sup_{n\geq T(\omega)} D_n(\omega) < +\infty$. In the following inequalities, one considers any fixed integer n such that $n \geq T(\omega)$. For convenience, ω will be omitted in most of the next computations. Let a, b be two strictly positive real numbers such that, for $i, j = 1, \dots, q$,

we have $a \leq \langle e_i, X^{(T)} e_j \rangle \leq b$, and let x and y be any elements of $\overline{B}$. Using $\|x\| = \|y\| = 1$, we obtain for $i = 1, \ldots, q$

$$a \leq \langle e_i, X^{(T)} x \rangle \leq b.$$

(That is, $a\vec{\mathbf{1}} \leq X^{(T)}x \leq b\vec{\mathbf{1}}$ for the coordinatewise order relation on $\mathbb{R}^q$.) Moreover, using the formula $\langle y, X^{(n)}x \rangle = \langle Y_{T+1} \cdots Y_n y, X^{(T)}x \rangle$, one gets successively

$$a \, \langle Y_{T+1} \ldots Y_n y, \vec{\mathbf{1}} \rangle \leq \langle y, X^{(n)} x \rangle \leq b \, \langle Y_{T+1} \cdots Y_n y, \vec{\mathbf{1}} \rangle,$$

$$| \ln \langle y, X^{(n)} x \rangle - \ln \|Y_{T+1} \cdots Y_n y\| \, | \leq \max\{| \ln a|, | \ln b|\}.$$

In particular, with $x = \vec{\mathbf{1}}$, this gives $| \ln \|Y^{(n)}y\| - \ln \|Y_{T+1} \cdots Y_n y\| \, | \leq \max\{| \ln a|, | \ln b|\}$. These two inequalities imply $\sup\limits_{n \geq T(\omega)} D_n(\omega) < +\infty$.

To prove (iii), again consider $\omega \in \Omega$ such that $T(\omega) < +\infty$, and recall that, from the Perron-Frobenius Theorem, there exists $R_n(\omega) \in \overline{B}$ such that $X^{(n)}(\omega) R_n(\omega) = \Lambda_n(\omega) R_n(\omega)$. With $x = R_n$ and $y = \chi = \frac{1}{q}\vec{\mathbf{1}}$, Assertion (ii) yields

$$\left| 1_{[T \leq n]} \ln \frac{1}{q} \langle \vec{\mathbf{1}}, X^{(n)} R_n \rangle - \ln \|Y^{(n)} \chi\| \right| \leq D_n.$$

From $\langle \vec{\mathbf{1}}, X^{(n)} R_n \rangle = \Lambda_n \|R_n\| = \Lambda_n$, it follows that

$$\widetilde{D}_n(\omega) \leq D_n(\omega) + 1_{[T > n]}(\omega) \, | \ln \Lambda_n(\omega)| \; + 1_{[T \leq n]}(\omega) \ln q.$$

This proves assertion *(iii).* $\square$

We deduce from the above lemma that, for any sequence $(a_n)_{n \geq 1}$ in $\mathbb{R}_+^*$ such that $\lim_n a_n = +\infty$, we have $\lim\limits_n \dfrac{1}{a_n} D_n = 0$ and $\lim\limits_n \dfrac{1}{a_n} \widetilde{D}_n = 0$ a.s.

Consequently, the conclusion in Theorem I for $(1_{[T \leq n]} \ln \langle y_n, X^{(n)} x_n \rangle)_n$ *and* $(\ln \Lambda_n)_n$ *will hold if the same is valid for* $(\ln \|Y^{(n)} y_n\|)_n$ *for any sequence* $(y_n)_{n \geq 1}$ *of vectors of* $\overline{B}$.

II.2. Projective action of positive matrices. It is well known that the projective action of matrices plays a key part in the study of the asymptotic behaviour of random invertible matrix products, cf. [6] for example. As shown in [10], this is also true in the case of positive matrices. About this action, we now recall the facts that we shall use throughout, refering to [10] for more details and for the proofs.

Consider the subset $\widetilde{\overline{C}}$ of the q-dimensional projective space associated with the cone $\overline{C}$. In other words, $\widetilde{\overline{C}}$ is the set of lines through 0 and some point in $\overline{C} \backslash \{0\}$. These may be represented by points of the closed polygon $\overline{B}$. An element $g \in S$ maps a line in $\widetilde{\overline{C}}$ onto a line in $\widetilde{\overline{C}}$, and this defines its projective action on $\widetilde{\overline{C}}$. As $\widetilde{\overline{C}}$ is represented by $\overline{B}$, the projective action of g moves to the action on $\overline{B}$ defined by

$$g \cdot x = \frac{gx}{\|gx\|}.$$

(Recall that gx is the image of x under the linear action of g.) The projective action has the following basic properties : if e stands for the identity matrix and $g, g' \in S$, $x \in \overline{B}$, we have

$$e \cdot x = x, \qquad (gg') \cdot x = g \cdot (g' \cdot x).$$

It is well known [5] that, when B is equipped with the Hilbert distance d_H, the elements of S have a contractive action on B, and that this contractive action is strict for elements of S°. However, because the Hilbert distance is unbounded and only defined on B, it is more convenient for our purposes to use a bounded distance d on $\overline{B}$ which have similar properties. This distance, already used in [10], is defined as follows. For $x = (x_1, \dots, x_q)$ and $y = (y_1, \dots, y_q)$ in $\overline{B}$, we write

$$m(x, y) = \sup\{\lambda : \lambda \in I\!R_+, \forall i = 1, \dots, q, \lambda y_i \le x_i\} = \min\{y_i^{-1} x_i : i = 1, \dots, q, y_i > 0\}.$$

Besides let φ be the one-to-one function on $[0, 1]$ defined by $\varphi(s) = \dfrac{1-s}{1+s}$. Then, if $x, y \in \overline{B}$, one has $\sum_{i=1}^q x_i = \sum_{i=1}^q y_i = 1$, thus $0 \le m(x, y) \le 1$, so one may define

$$d(x, y) = \varphi(m(x, y)m(y, x)).$$

Proposition II.1. (cf. [10], § 10) *The map d defines a distance on $\overline{B}$ having the following properties*
(i) $\sup\{d(x, y) : x, y \in \overline{B}\} = 1$ **(ii)** *if $x, y \in \overline{B}$, $\|x - y\| \le 2d(x, y)$*
(iii) *the topology of (B, d) is the topology induced on B by the standart topology of $I\!R^q$.*
Moreover, for $g \in S$, there exists $c(g)$ such that
(iv) *if $x, y \in \overline{B}$, $d(g \cdot x, g \cdot y) \le c(g)d(x, y) \le c(g)$* **(v)** *$c(g) \le 1$, and $c(g) < 1$ if and only if $g \in S^\circ$*
(vi) *if $g' \in S$, $c(gg') \le c(g)c(g')$* **(vii)** *$c(g^*) = c(g)$.*

For any $x \in \overline{B} \backslash B$ and any $y \in B$, we have $m(x, y) = 0$, so that $d(x, y) = 1$. Thus

$$\overline{B} \backslash B = \cup_{x \in \overline{B} \backslash B} \{y : y \in \overline{B}, d(x, y) < 1/2\}.$$

is an open subset of $(\overline{B}, d)$. It follows that the topology of $(\overline{B}, d)$ and the topology induced by $I\!R^q$ on $\overline{B}$ do not coincide ; from *(ii)* the former is finer than the latter. In the sequel, unless otherwise stated, when we appeal to topological properties of $\overline{B}$ and B, we shall assume that these sets are endowed with the topologies induced by $I\!R^q$; the distance d will be only used to express contractivity.

II.3. Stochastic contractivity. Denote by μ the probability distribution of $Y_1 = X_1^*$ and by $\mu^{(n)}$ the distribution of $Y^{(n)} = Y_1 \dots Y_n$, $n \ge 1$. For $n \ge 1$, we set

$$c(\mu^{(n)}) = \sup\Big\{ \int_S \frac{d(g \cdot y, g \cdot y')}{d(y, y')} d\mu^{(n)}(g) : y, y' \in \overline{B}, y \ne y' \Big\}.$$

Since $c(\cdot) \le 1$, we have $c(\mu^{(n)}) \le 1$. Furthermore, the sequence $(c(\mu^{(n)}))_{n \ge 1}$ is clearly submultiplicative, so we can define

$$\kappa(\mu) = \lim_n c(\mu^{(n)})^{\frac{1}{n}} = \inf_{n \ge 1} c(\mu^{(n)})^{\frac{1}{n}}.$$

Using Assertion (v) in Proposition II.1, it is easily shown that $(\mathcal{C})$ is equivalent to $\kappa(\mu) < 1$.

Theorem II.1. *Under Condition $(\mathcal{C})$, there exists a r.v. Z_1 taking values in B such that $(Y^{(n)} \cdot x)_n$ converges a.s to Z_1, the convergence being uniform for $x \in \overline{B}$. The probability distribution ν of*

Z_1 verifies $\nu(B) = 1$. It is the unique μ-invariant probability distribution on $\overline{B}$, i.e. the unique probability distribution on $\overline{B}$ such that, for any bounded continuous function f on $\overline{B}$, we have

$$\int_{\overline{B}} \Big(\int_G f(g \cdot x) d\mu(g) \Big) d\nu(x) = \int_{\overline{B}} f(x) d\nu(x).$$

Proof. Using the contractivity properties of $c(\cdot)$, we see that the sequence of positive r.v. $(c(Y^{(n)}))_{n \geq 1}$ decreases and hence converges almost surely. Under $(\mathcal{C})$, there exists an integer $b \in \mathbb{N}^*$ such that $\mathbb{E}[c(Y^{(b)})] < 1$. The independence then yields $\limsup_k \mathbb{E}[c(Y^{(kb)})] \leq \lim_k (\mathbb{E}[c(Y^{(b)})])^k = 0$. It follows from these two facts that $\lim_n c(Y^{(n)}) = 0$ a.s. Notice that, by means of the subadditive ergodic theorem, we can get, more precisely, $\lim_n (c(Y^{(n)}))^{\frac{1}{n}} = \kappa$ a.s.

Set $\Omega_1 = \{\omega : \lim_n c(Y^{(n)}(\omega)) = 0\}$. Let $\omega \in \Omega_1$. For $n \geq T(\omega)$, the polygons $K_n(\omega) = Y^{(n)}(\omega) \cdot (\overline{B})$ form a decreasing sequence of compact subsets of B, so that $K(\omega) = \cap_{n \geq 1} K_n(\omega) \neq \emptyset$. Moreover, for the distance d, the diameter $\Delta(\omega)$ of $K(\omega)$ is equal to 0. Indeed we have for $n \geq T(\omega)$,

$$\Delta(\omega) \leq \Delta_n(\omega) = \sup\{d(Y^{(n)}(\omega) \cdot x, Y^{(n)}(\omega) \cdot y) : x, y \in \overline{B}\} \leq c(Y^{(n)}(\omega)).$$

Define $Z_1(\omega)$ by setting $K(\omega) = \{Z_1(\omega)\}$. Then $Z_1 \in K_n(\omega)$ implies $d(Y^{(n)}(\omega) \cdot x, Z_1(\omega)) \leq c(Y^{(n)}(\omega))$, and (ii) in Proposition II.1 yields the desired convergence.

Now denote by ν the law of Z_1. Set $Z_2 = \lim_n (Y_2 \dots Y_n) \cdot x$ a.s. Clearly Z_2 has the distribution ν, and we have $Y_1 \cdot Z_2 = Z_1$ a.s. This gives the μ-invariance of ν. Let ν' be any μ-invariant distribution on $\overline{B}$. Then, for any continuous bounded function f on $\overline{B}$ and $n \geq 1$, we have $\int_{\overline{B}} \mathbb{E}[f(Y^{(n)} \cdot x)] d\nu'(x) = \int_{\overline{B}} f(x) d\nu'(x)$. Thus $\mathbb{E}[f(Z_1)] = \int_{\overline{B}} f(x) d\nu'(x)$. Hence $\nu = \nu'$. □

III. FOURIER KERNELS

III.1. Definition and link with our distributional problem. Recall that our aim is to study the distributional behaviour of the sequences $(\ln \|Y_1 \cdots Y_n y\|)_{n \geq 1}$, $y \in \overline{B}$ (cf. the end of § II.1). However, since $((Y_n \dots Y_1 \, y))_{n \geq 1}$ is a Markov chain and $Y_n \dots Y_1 y$ has the same distribution as $Y_1 \cdots Y_n y$, it is more convenient to consider $(\ln \|Y_n \dots Y_1 y\|)_{n \geq 1}$. So we introduce the new left random walk on S,

$$\widetilde{Y}^{(n)} = Y_n \dots Y_1, \quad n \geq 1, \quad \widetilde{Y}^{(0)} = e.$$

For $y_0 \in \overline{B}$, consider the sequence of r.v. in $\overline{B}$ defined by $(\widetilde{Y}^{(n)} \cdot y_0)_{n \geq 0}$. It is easily checked that it is a Markov chain on $\overline{B}$ starting at y_0 and associated with the transition probability P defined by

$$Pf(x) = \int_S f(g \cdot x) d\mu(g),$$

where $x \in \overline{B}$ and f is a bounded measurable function on $\overline{B}$. Theorem II.1 shows that ν is the unique P-invariant distribution. Finally, for $g \in S$ and $x \in \overline{B}$, define

$$\xi(g, x) = \ln \|gx\|.$$

The function ξ is connected with the projective action of S on $\overline{B}$ by the additive cocycle property

$$\xi(gg', x) = \xi(g, g' \cdot x) + \xi(g', x) \quad (g, g' \in S, \, x \in \overline{B}).$$

This property shows that, for any $y \in \overline{B}$ and $n \geq 1$, we have

$$(\star) \qquad \ln \|\widetilde{Y}^{(n)} y\| = \xi(\widetilde{Y}^{(n)}, y) = \sum_{k=1}^{n} \xi(Y_k, \widetilde{Y}^{(k-1)} \cdot y).$$

With the function ξ and the transition probability P, we associate the Fourier kernels P_t, $t \in I\!R$,

$$x \in \overline{B}, \quad P_t f(x) = \int_S e^{it\xi(g,x)} f(g \cdot x) d\mu(g),$$

with f as above. The Markov property implies that for $n \geq 1$, $y \in \overline{B}$ and $t \in I\!R$ (see e.g [11])

$$(\star\star) \qquad I\!E[e^{it \ln \|\widetilde{Y}^{(n)} y\|}] = {P_t}^n \mathbf{1}(y), \quad \text{where} \quad \mathbf{1} = 1_{\overline{B}}.$$

This basic relation shows that limit theorems for the sequence $(\xi(\widetilde{Y}^{(n)}, x))_{n \geq 1}$ may be deduced from the asymptotic behaviour of the iterates of the operators P_t acting on a suitable Banach space. This is the main idea of the spectral method. In Sections III.2-4 below, we shall prove that P satisfies a strong ergodicity property on the usual space of Lipschitz functions on $\overline{B}$, and we shall apply the standard operator perturbation theorem to the Fourier kernels.

III.2. A strong ergodicity property for P. We denote by $\mathcal{L}$ the space of all complex-valued functions f on $\overline{B}$ such that

$$m(f) = \sup\Big\{\frac{|f(x) - f(x')|}{d(x, x')} : x, x' \in \overline{B},\, x \neq x'\Big\} < +\infty.$$

Since the distance d is bounded, the elements of $\mathcal{L}$ are bounded, so we can equip $\mathcal{L}$ with the norm

$$f \in \mathcal{L}, \quad \|f\|_{\mathcal{L}} = \|f\|_u + m(f), \quad \text{with} \quad \|f\|_u = \sup\{|f(x)| : x \in \overline{B}\}.$$

Then $(\mathcal{L}, \|\cdot\|_{\mathcal{L}})$ is a Banach space. Notice that the functions in $\mathcal{L}$ may be discontinuous on $\overline{B}$ w.r.t the induced topology of $I\!R^q$, see the remark following Proposition II.1. We still denote by $\|\cdot\|_{\mathcal{L}}$ the operator norm on $\mathcal{L}$, and Π stands for the rank one projection on $\mathcal{L}$ defined by : $\Pi f = \nu(f)\mathbf{1}$.

Theorem III.1. *Under Condition $(\mathcal{C})$, for any $\kappa_0 \in]\kappa(\mu), 1[$, there exists $C > 0$ such that, for all $n \geq 1$, we have $\|P^n - \Pi\|_{\mathcal{L}} \leq C\kappa_0^n$.*

Proof. We follow [11]. For $x, x' \in M, x \neq x'$, we have

$$\frac{|P^n f(x) - P^n f(x')|}{d(x, x')} \ \leq \ \int \frac{|f(g \cdot x) - f(g \cdot x')|}{d(g \cdot x, g \cdot x')} \frac{d(g \cdot x, g \cdot x')}{d(x, x')} d\mu^{(n)}(g) \ \leq \ m(f)\, c(\mu^{(n)}),$$

so $P^n f \in \mathcal{L}$ and $m(P^n f) \leq m(f)\, c(\mu^{(n)})$. Since $\|Pf\|_u \leq \|f\|_u$, P acts continuously on $\mathcal{L}$.
Now set $H = \mathrm{Ker}(\nu) \cap \mathcal{L}$. Since ν is P-invariant and defines a continuous linear functional on $\mathcal{L}$, H is a closed P-invariant subspace in $\mathcal{L}$. Moreover, when restricted to H, the semi-norm m is equivalent to the norm $\|\cdot\|_{\mathcal{L}}$: more precisely, if $h \in H$, we have

$$m(h) \leq \|h\|_{\mathcal{L}} \leq (2 \sup\{d(y, y') : y, y' \in \overline{B}\} + 1) m(h) \leq 3m(h),$$

the second inequality being deduced from the fact that, if $\nu(f) = 0$, there exist $x_1, x_2 \in \overline{B}$ such that Re $f(x_1) =$ Im $f(x_2) = 0$. Let $f \in \mathcal{L}$. Since $f - \Pi(f) \in H$, we have $P^n(f - \Pi(f)) \in H$ for all $n \geq 1$. Hence

$$\|P^n(f - \Pi(f))\|_{\mathcal{L}} \leq 3\, m(P^n(f - \Pi(f))) \leq 3\, c(\mu^{(n)}) m(f - \Pi(f)) = 3\, c(\mu^{(n)}) m(f) \leq 3\, c(\mu^{(n)}) \|f\|_{\mathcal{L}}.$$

Finally, under $(\mathcal{C})$, we have $\lim_n c(\mu^{(n)})^{\frac{1}{n}} = \kappa(\mu)$ (§ II.3). This gives the desired statement. □

III.3. The Fourier kernels near 0. To apply the perturbation theory near $t = 0$ to the Fourier kernels P_t, we have to show that P_t is a bounded operator of $\mathcal{L}$, and to study $\|P_t - P\|_{\mathcal{L}}$ when $t \to 0$. For that, we shall need the following notations. For $g \in S$, define

$$\|g\| = \sup\{\|gx\| : x \in \overline{B}\}, \quad v(g) = \inf\{\|gx\| : x \in \overline{B}\}, \quad \text{and } \ell(g) = |\ln \|g\|\,| + |\ln v(g)|.$$

(Notice that $v(g) > 0$.) Finally set $\varepsilon(t) = \int_S \min\{|t|\ell(g),\, 2\} d\mu(g)$, and observe that $\lim_{t \to 0} \varepsilon(t) = 0$.

Theorem III.2. *For $t \in \mathbb{R}$, P_t defines a bounded operator of $\mathcal{L}$, and* $\|P_t - P\|_{\mathcal{L}} = O(\varepsilon(t) + |t|)$.

Proof. Recall P_t is associated to P and $\xi(g, x) = \ln \|gx\|$ ($g \in S$, $x \in \overline{B}$).

Lemma III.1. *For $g \in S$ and $z, x, y \in \overline{B}$ such that $d(x, y) < 1$, we have*

$$|\xi(g, z)| \leq \ell(g), \qquad |\xi(g, x) - \xi(g, y)| \leq 2 \ln \frac{1}{1 - d(x, y)}.$$

Proof. The first inequality is obvious. The second one is Assertion *(ii)* of Lemma 5.3 in [10], for completeness we reproduce the proof here. Let $x = (x_1, \ldots, x_q)$, $y = (y_1, \ldots, y_q) \in \overline{B}$ and $g = [g_{ij}]_{i,j=1,\ldots,q}$. Then $\|gx\| = \sum_{i=1}^{q} \sum_{j=1}^{q} g_{ij} x_j \geq m(x, y) \sum_{i=1}^{q} \sum_{j=1}^{q} g_{ij} y_j = m(x, y) \|gy\|$. As $d(x, y) < 1$, the number $m(x, y)$ and $m(y, x)$ are in $]0, 1]$. Consequently, the symmetry in x and y yields $m(x, y) \leq \frac{\|gx\|}{\|gy\|} \leq \frac{1}{m(y, x)}$, and

$$\begin{aligned} |\xi(g, x) - \xi(g, y)| \leq \max\{-\ln m(y, x), -\ln m(x, y)\} &\leq -\ln m(y, x) - \ln m(x, y) \\ &= -\ln \varphi^{-1}(d(x, y)) = \ln \frac{1 + d(x, y)}{1 - d(x, y)}. \end{aligned}$$

For $t \in [0, 1[$, $2 \ln \frac{1}{1 - t} - \ln \frac{1 + t}{1 - t} = \ln \frac{1}{1 - t^2} \geq 0$, thus *(ii)* follows. □

Set $\Delta_t = P_t - P$. For $f \in \mathcal{L}$ and $x \in \overline{B}$, we have $\Delta_t f(x) = \int (e^{it\xi(g,x)} - 1) f(g \cdot x) d\mu(g)$.
Before we proceed, notice the inequality : $\forall u, v \in \mathbb{R}$, $|e^{iu} - e^{iv}| \leq \min\{|u - v|, 2\}$. Thus

$$|\Delta_t f(x)| \leq \int \min\{|t|\ell(g),\, 2\}\, |f(g \cdot x)| d\mu(g) \leq \varepsilon(t) \|f\|_u.$$

So $\|\Delta_t f\|_u \leq \varepsilon(t)\, \|f\|_u$. Now for $x, y \in \overline{B}$, write $\frac{\Delta_t f(x) - \Delta_t f(y)}{d(x, y)} = A(x, y) + B(x, y)$, with

$$A(x, y) = \int \frac{e^{it\xi(g,x)} - e^{it\xi(g,y)}}{d(x, y)} f(g \cdot x) d\mu(g) \quad \text{and} \quad B(x, y) = \int (e^{it\xi(g,y)} - 1) \frac{f(g \cdot x) - f(g \cdot y)}{d(x, y)} d\mu(g).$$

If $d(x,y) > 1/2$, we have

$$|e^{it\xi(g,x)} - e^{it\xi(g,y)}| \le \min\{|t||\xi(g,x)-\xi(g,y)|, 2\}\,(2d(x,y)) \le 4\min\{|t|\,\ell(g), 1\}\,d(x,y),$$

while, for $d(x,y) \le 1/2$, the inequality of Lemma III.1 gives

$$|e^{it\xi(g,x)} - e^{it\xi(g,y)}| \le 2|t|\ln\frac{1}{1-d(x,y)} \le 2C|t|d(x,y),$$

with $C = \sup\{\frac{1}{u}\ln\frac{1}{1-u} : 0 < u \le 1/2\} < +\infty$. From that, we obtain $|A(x,y)| \le (4\varepsilon(t)+2C|t|)\,\|f\|_u$. Otherwise, since $c(g) \le 1$,

$$|B(x,y)| \le \int |e^{it\xi(g,x)} - 1|\,|\frac{f(g\cdot x) - f(g\cdot y)}{d(g\cdot x, g\cdot y)}|\;\frac{d(g\cdot x, g\cdot y)}{d(x,y)}\,d\mu(g) \le m(f)\,\varepsilon(t).$$

So $m(\Delta_t f) \le (4\varepsilon(t) + 2C|t|)\,\|f\|_u + \varepsilon(t)\,m(f)$, therefore $\|\Delta_t\|_{\mathcal{L}} \le 4\varepsilon(t) + 2C|t|$. □

III.4. Spectral properties of P_t near $t = 0$. The following perturbation theorem extends the spectral conclusion of Theorem III.1 to P_t for t near 0. Let κ_0 be chosen as in Theorem III.1.

Theorem III.3. *We assume that Condition ($\mathcal{C}$) holds. Let $\kappa \in]\kappa_0, 1[$. There exists an open interval I centered at $t = 0$ such that, for $t \in I$, P_t admits a dominating eigenvalue $\lambda(t) \in \mathbb{C}$, with a corresponding rank-one eigenprojection $\Pi(t)$, satisfying the following properties :*

$$\lim_{t\to 0}\lambda(t) = 1, \quad \|\Pi(t) - \Pi\|_{\mathcal{L}} = O(\|P_t - P\|_{\mathcal{L}}) \;\text{ and }\; \sup_{t\in I}\|P_t^n - \lambda(t)^n\,\Pi(t)\|_{\mathcal{L}} = O(\kappa^n).$$

Proof. We only sketch the proof, refering to [7] for the details and using standard notations. It follows from Theorem III.1 that the spectrum $\sigma(P)$ of P is contained in $\{1\} \cup \overline{D(0,\kappa_0)}$. Since $t \mapsto P_t$ is continuous (Th. III.2), there exists $t_0 > 0$ such that, for $|t| \le t_0$, we have $\sigma(P_t) \subset D(1, \frac{1-\kappa}{2}) \cup D(0,\kappa)$, and $\sigma(P_t) \cap D(1, \frac{1-\kappa}{2}) = \{\lambda(t)\}$, where $\lambda(t)$ is a simple eigenvalue of P_t with a corresponding rank-one eigenprojectioon $\Pi(t)$ depending continuously on t. Let Γ be the oriented circle $\mathcal{C}(0,\kappa)$. Since $(z,t) \mapsto (z - P_t)^{-1}$ is continuous on the compact set $\Gamma \times [-t_0, t_0]$, the formula $P_t^n - \lambda(t)^n\Pi(t) = \frac{1}{2i\pi}\int_\Gamma z^n(z - P_t)^{-1}dz$ leads to the last estimate of Theorem. □

The next proposition states a simple expansion for the perturbed eigenvalue $\lambda(t)$.

Proposition III.1. *For $t \in I$, we have $\lambda(t) = \mu\otimes\nu(e^{it\xi}) + O(\|P_t - P\|_{\mathcal{L}}^2)$.*

Proof. Since ν defines a continuous linear functional on $\mathcal{L}$ and $\|P_t - P\|_{\mathcal{L}} \to 0$ when $t \to 0$, the rank-one eigenprojection $\Pi(t)$, defined in Theorem III.3, is such that $\nu(\Pi(t)\mathbf{1}) \to \nu(\Pi\mathbf{1}) = 1$. So one may assume that $\nu(\Pi(t)\mathbf{1}) \neq 0$ for any $t \in I$, with I possibly reduced. For $t \in I$, set $v(t) = (\nu(\Pi(t)\mathbf{1}))^{-1}\,\Pi(t)\mathbf{1}$. Then we have $\lambda(t)v(t) = P_t v(t)$ and $\nu(v(t)) = 1$, therefore

$$\lambda(t) = \nu(P_t v(t)) = \nu(P_t\mathbf{1}) + \nu(P_t(v(t) - \mathbf{1})) = \mu\otimes\nu(e^{it\xi}) + \nu((P_t - P)(v(t) - \mathbf{1})),$$

the last equality following from $\nu(P(v(t) - \mathbf{1})) = \nu(v(t) - \mathbf{1}) = 0$ since ν is P-invariant. We conclude by observing that $\|v(t) - \mathbf{1}\|_{\mathcal{L}} = \|v(t) - v(0)\|_{\mathcal{L}} = O(\|\Pi(t) - \Pi\|_{\mathcal{L}}) = O(\|P_t - P\|_{\mathcal{L}})$. □

IV. PROOF OF THEOREM I

Let us point out how the results of the previous sections will be used to establish Theorem I. We have to study the distributional behaviour of

$$\ln \|\widetilde{Y}^{(n)} y_n\| = \sum_{k=1}^{n} \xi(Y_k, \widetilde{Y}^{(k-1)} \cdot y_n)$$

for any sequence $(y_n)_{n\geq 1}$ of vectors of $\overline{B}$. Let $y \in \overline{B}$, from Theorem II.1 and the independence of Y_k and $\widetilde{Y}^{(k-1)}$, the sequence $(\xi(Y_k, \widetilde{Y}^{(k-1)} \cdot y))_k$ converges in distribution to $\xi(Y_1, Z_2)$, where Z_2 is independent of Y_1 and has distribution ν. From this we may guess that $(\ln \|\widetilde{Y}^{(n)} y_n\|)_{n\geq 0}$ has the same asymptotical behaviour that a sequence of sums of stationary random variables distributed as $\xi(Y_1, Z_2)$. This is confirmed by a look at the characteristic functions. In fact, the characteristic function of $\ln \|\widetilde{Y}^{(n)} y_n\|$ is $P(t)^n 1(y_n)$ whose asymptotic behaviour is, as shown by the spectral decomposition of Theorem III.3, essentially ruled by $\lambda(t)^n$ which doesn't depend on y_n or on any initial distribution. Notice that this can be used as will be done in the sequel to deduce a limit theorem from the expansion of $\lambda(t)$ at 0, but also conversely to get an expansion of $\lambda(t)$ at 0 from a known limit theorem, see [12] [13]. Now observe that the characteristic function of $\xi(Y_1, Z_2)$ is $\mu \otimes \nu(e^{it\xi})$, which is precisely the first term in the expansion of $\lambda(t)$ in Proposition III.1. So we see that if, for a sequence $(a_n)_n$ of positive real numbers, we have

$$(\star\star\star) \qquad \|P_{\frac{t}{a_n}} - P\|^2_{\mathcal{L}} = o(\frac{1}{n}),$$

then $(\mu \otimes \nu(e^{i\frac{t}{a_n}\xi}))^n$ is the principal part of the expansion of $(\lambda(\frac{t}{a_n}))^n$, so that $(\frac{1}{a_n} \ln \|\widetilde{Y}^{(n)} y_n\|)_{n\geq 0}$ will behave as $(\frac{1}{a_n} \sum_{k=1}^{n} \Xi_k)_{n\geq 1}$, where $(\Xi_k)_{k\geq 1}$ is a sequence of independent random variables distributed as $\xi(Y_1, Z_2)$.

Actually we shall show that, under the hypotheses of Theorem I, the law of $\xi(Y_1, Z_2)$ is in the domain of attraction of a stable law and that $(\star\star\star)$ is verified with the corresponding scaling sequence $(a_n)_n$, these two facts lead to the claimed result. Finally observe that Condition $(\star\star\star)$ is not fulfilled with $a_n = \sqrt{n}$ in the standard gaussian case since it is known that in this case the variance of the limit law is not $\sigma^2(\xi(Y_1, Z_2))$ [11].

Proposition IV.1. *Suppose that Conditions* $(\mathcal{C})$ *and (i)-(ii) hold. Let* F *be the distribution function of* $\xi(Y_1, Z_2)$ *:*

$$F(u) = \mu \otimes \nu\{(g, x) : \xi(g, x) \leq u\} \quad (u \in I\!R).$$

Then there exist positive functions ρ_+ *and* ρ_- *defined on* $I\!R_+^*$ *such that, for* $u > 0$,

$$1 - F(u) = \frac{\rho_+(u)L(u)}{u^\alpha}, \quad F(-u) = \frac{\rho_-(u)L(u)}{u^\alpha}, \quad \text{with} \quad \lim_{u\to+\infty} \rho_+(u) = c_+ \quad \text{and} \quad \lim_{u\to+\infty} \rho_-(u) = c_-.$$

Proof. Since $\xi(g, x) = \ln \|gx\|$ and μ is the law of Y_1, we have $F(u) = \int_{\overline{B}} I\!P[\|Y_1 x\| \leq e^u] d\nu(x)$.

For $x \in B$, we set $N_1^x = \|Y_1 x\| = \sum_{i,j=1}^{q} < e_i, X_1 e_j > x_i$. As $\nu(B) = 1$, one gets for $u > 0$

$$1 - F(u) = \int_B I\!P[N_1^x > e^u] d\nu(x) \quad \text{and} \quad F(-u) = \int_B I\!P[N_1^x \leq e^{-u}] d\nu(x).$$

To proceed, we have to compare the tails of N_1^x, for $x \in B$, with that of N_1. Let u_0 be such that, for $u \geq u_0$, we have $L(u) > 0$. For $u > u_0$, and for $x \in B$, we set

$$
\begin{aligned}
c_+(u) = \frac{u^\alpha}{L(u)} I\!P[N_1 > e^u], \quad c_-(u) = \frac{u^\alpha}{L(u)} I\!P[N_1 \le e^{-u}], \\
c_+(x,u) = \frac{u^\alpha}{L(u)} I\!P[N_1^x > e^u], \quad c_-(x,u) = \frac{u^\alpha}{L(u)} I\!P[N_1^x \le e^{-u}].
\end{aligned}
$$

Setting $m(x) = \min_{i=1,\dots,q} x_i$, we have the following obvious inequalities $m(x)N_1 \le N_1^x \le N_1$, and

$$
I\!P[m(x)N_1 > e^u] \le I\!P[N_1^x > e^u] \le I\!P[N_1 > e^u]
$$

$$
I\!P[N_1 \le e^{-u}] \le I\!P[N_1^x \le e^{-u}] \le I\!P[m(x)N_1 \le e^{-u}].
$$

Now let ε, $0 < \varepsilon < 1$. Suppose that $u > 0$ is such that $e^{-\varepsilon u} \le m(x)$, we get

$$
\begin{aligned}
\frac{u^\alpha}{L(u)} c_+((1+\varepsilon)u) \frac{L((1+\varepsilon)u)}{((1+\varepsilon)u)^\alpha} \le c_+(x,u) \le c_+(u), \\
c_-(u) \le c_-(x,u) \le \frac{u^\alpha}{L(u)} c_-((1-\varepsilon)u) \frac{L((1-\varepsilon)u)}{((1-\varepsilon)u)^\alpha}.
\end{aligned}
$$

Since by hypothesis, $c_+(v) \to c_+$ and $c_-(v) \to c_-$ when $v \to +\infty$, it follows that

$$
\begin{aligned}
\frac{c_+}{(1+\varepsilon)^\alpha} \le \liminf_{u\to+\infty} c_+(x,u) \le \limsup_{u\to+\infty} c_+(x,u) \le c_+, \\
c_- \le \liminf_{u\to+\infty} c_-(x,u) \le \limsup_{u\to+\infty} c_-(x,u) \le \frac{c_-}{(1-\varepsilon)^\alpha}.
\end{aligned}
$$

Thus $\lim_{u\to+\infty} c_+(x,u) = c_+$ and $\lim_{u\to+\infty} c_-(x,u) = c_-$.

Lastly $N_1^x \le N_1$ yields $I\!P[N_1^x > e^u] \le I\!P[N_1 > e^u]$, while $N_1^x = \|Y_1 x\| = \sum_{j=1}^q \|Y_1 e_j\| x_j \ge V_1$ gives $I\!P[N_1^x \le e^{-u}] \le I\!P[V_1 \le e^{-u}]$. Therefore, for any $x \in B$ and $u > 0$, we have

$$
c_+(x,u) \le c_+(u) \quad \text{and} \quad c_-(x,u) \le \frac{u^\alpha}{L(u)} I\!P[V_1 \le e^{-u}],
$$

and by (i)-(ii), the functions of the variable u in each right term of these inequalities are bounded on $I\!R_+$. Now one may conclude. We have

$$
\frac{u^\alpha}{L(u)}(1 - F(u)) = \int_B c_+(x,u)d\nu(x) \quad \text{and} \quad \frac{u^\alpha}{L(u)} F(-u) = \int_B c_-(x,u)d\nu(x),
$$

and Lebesgue's Theorem implies that these integrals converge to c_+ and c_- respectively as $u \to +\infty$. □

Proposition IV.1 means that $\xi(Y_1, Z_2)$ belongs to the domain of attraction of a stable law with order α, $0 < \alpha \le 2$, the standard Gaussian case being excluded since, for $\alpha = 2$, L is supposed to be unbounded.

Let $(\Xi_k)_k$ be an independent sequence of real r.v distributed as $\xi(Y_1, Z_2)$. From Proposition IV.I, there exist sequences $(a_n)_n$ in $I\!R_+^*$ and $(b_n)_n$ in $I\!R$ such that $(\frac{\Xi_1 + \ldots + \Xi_n - b_n}{a_n})_n$ converges in distribution to a stable law of order α, see [14]. It is known that $\lim_n a_n = +\infty$ and that $(a_n)_{n\ge1}$ may be chosen such that $\frac{n}{a_n^\alpha} L(a_n) = 1$.

Proposition IV.2. *Suppose that Conditions* $(\mathcal{C})$ *and (i)-(ii) hold. For any fixed real* t, *we have* $\|P_{\frac{t}{a_n}} - P\|_{\mathcal{L}}^2 = o(\frac{1}{n})$ *when* $n \to +\infty$.

Proof. The notations $\|g\|$, $v(g)$ and $\ell(g)$ below have been introduced for Theorem III.2. We have $V_1 = v(Y_1)$: indeed, by definition, $V_1 = \min_{i=1,\dots,q} \|Y_1 e_i\|$, so $e_i \in \overline{B}$ implies $v(Y_1) \leq V_1$, and if $x \in \overline{B}$, one has $\|Y_1 x\| = \sum_{j=1}^q \|Y_1 e_j\| x_j \geq V_1$, thus $v(Y_1) \geq V_1$. Besides $|||g||| = \langle \vec{\mathbf{1}}, g\vec{\mathbf{1}}\rangle$ is a norm for $q \times q$-matrices, while, for $g \in S$, the quantity $\|g\|$ corresponds to the matrix norm associated to the norm $\|\cdot\|$ on $\mathbb{R}^q$. Since the two previous norms are equivalent, and $N_1 = |||X_1|||$, $\|X_1\| = \|Y_1\|$, there exists a constant $C \geq 1$ such that $C^{-1}N_1 \leq \|Y_1\| \leq CN_1$.

Now let $0 < \beta < \alpha$. Hypotheses (i)-(ii) show that $\mathbb{E}[(\ln^+ N_1)^\beta] < +\infty$ and $\mathbb{E}[(\ln^- V_1)^\beta] < +\infty$. From the previous remarks, we deduce that $\int_S \ell(g)^\beta d\mu(g) < +\infty$ (recall μ is the law of Y_1). Denote by m_β the previous integral. If $\beta \leq 1$, then we have $\min\{|t|\ell(g), 2\} \leq 2|t|^\beta \ell(g)^\beta$, so that $\varepsilon(t) \leq 2|t|^\beta m_\beta$. If $\beta > 1$, then $\varepsilon(t) \leq 2|t| \int \ell(g) d\mu(g) \leq 2|t| m_\beta^{\frac{1}{\beta}}$. Thus Theorem III.2 gives $\|P_t - P\|_{\mathcal{L}} = O(|t|^\beta)$ if $0 < \alpha \leq 1$, and $\|P_t - P\|_{\mathcal{L}} = O(t)$ if $1 < \alpha \leq 2$. Finally, using $\dfrac{n}{a_n^\alpha} L(a_n) = 1$ and the fact that L is unbounded in the case $\alpha = 2$, this easily yields the desired statement. $\square$

Proof of Theorem I. Let $(y_n)_{n\geq 1}$ be any sequences of vectors in $\overline{B}$, and let ϕ_n [resp. ψ_n] be the characteristic function of $\frac{\ln\|\widetilde{Y}^{(n)} y_n\| - b_n}{a_n}$ [resp. of $\frac{\Xi_1 + \dots + \Xi_n - b_n}{a_n}$]. Let $\phi(t) = \mu \otimes \nu(e^{it\xi})$ be the characteristic function of $\xi(Y_1, Z_2)$. Let $t \in \mathbb{R}$ and $n \in \mathbb{N}^*$ be such that $\frac{t}{a_n} \in I$, and set $\ell_n(t) = \Pi(\frac{t}{a_n})\mathbf{1}(y_n)$ (cf. Th. III.3). By Theorems III.2-3, one gets $\lim_n \ell_n(t) = 1$. Furthermore we have

$$\begin{aligned}
\phi_n(t) = e^{-i\frac{b_n t}{a_n}} \mathbb{E}[e^{i\frac{t}{a_n}\ln\|\widetilde{Y}^{(n)} y_n\|}] &= e^{-it\frac{b_n}{a_n}} P_{\frac{t}{a_n}} \mathbf{1}(y_n) \quad \text{(by } (\star\star) \text{ in § III.1)} \\
&= e^{-it\frac{b_n}{a_n}} \lambda(\frac{t}{a_n})^n \ell_n(t) + O(\kappa^n) \quad \text{(by Th. III.3)} \\
&= e^{-it\frac{b_n}{a_n}} [\phi(\frac{t}{a_n}) + o(\frac{1}{n})]^n \ell_n(t) + O(\kappa^n) \quad \text{(Prop. III.1, IV.2).}
\end{aligned}$$

Finally, since $\psi_n(t) = e^{-it\frac{b_n}{a_n}} \phi(\frac{t}{a_n})^n$, one gets $\phi_n(t) = \psi_n(t)\,[1 + o(\frac{1}{n})]^n \ell_n(t) + O(\kappa^n)$, therefore $\lim_n \phi_n(t) = \lim_n \psi_n(t)$. Since $(\frac{\Xi_1 + \dots + \Xi_n - b_n}{a_n})_n$ converges in distribution to a stable law of order α, the same holds for $(\frac{\ln\|\widetilde{Y}^{(n)} y_n\| - b_n}{a_n})_n$. $\square$

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